Ergod. Th. & Dynam. Sys. (1991), **11**, 115–128
Printed in Great Britain

Rotation vectors and entropy for homeomorphisms of the torus isotopic to the identity†

J. LLIBRE

Departament de Matemàtiques, Universitat Autonoma de Barcelona, Bellaterra, 08193 Barcelona, Spain

R. S. MACKAY

Nonlinear Systems Laboratory, Mathematics Institute, University of Warwick, Coventry CV4 7AL, England

(*Received* 8 *November* 1988 *and revised* 13 *May* 1989)

Abstract. We show that if a homeomorphism f of the torus, isotopic to the identity, has three or more periodic orbits with non-collinear rotation vectors, then it has positive topological entropy. Furthermore, for each point ρ of the convex hull Δ of their rotation vectors, there is an orbit of rotation vector ρ, for each rational point p/q, $p \in \mathbb{Z}^2$, $q \in \mathbb{N}$, in the interior of Δ, there is a periodic orbit of rotation vector p/q, and for every compact connected subset C of Δ there is an orbit whose rotation set is C. Finally, we prove that f has 'toroidal chaos'.

1. *Introduction and statement of results*

There is growing interest in maps of the torus homotopic to the identity, both as models for physical situations with coupled oscillations, and for their intrinsic mathematical fascination.

If f is an endomorphism of the circle homotopic to the identity and has two periodic orbits with different rotation numbers ρ_1, ρ_2, then it has positive topological entropy and periodic orbits of all rational rotation numbers in the interval $I = [\rho_1, \rho_2]$[**BGMY**]. Furthermore it has orbits of all rotation numbers in I[**CGT, Mis**], and for every closed subinterval J of I there is an orbit whose rotation set is J[**BMPT**]. In fact, there is an invariant subset Y on which f is semi-conjugate to a quotient of a one-sided subshift, containing periodic orbits of all rational rotation vectors in I with distinct symbol sequences (this is implicit in [**Boy**]).

† This work was made possible by an invitation from the Nonlinear Systems Laboratory of the Mathematics Insitute of the University of Warwick, supported by SERC, to the first author, who is partially supported by a CICYT grant, number PB86-0351.