# A CLASSIFICATION OF BRAID TYPES FOR DIFFEOMORPHISMS OF SURFACES OF GENUS ZERO WITH TOPOLOGICAL ENTROPY ZERO

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ABSTRACT

We classify the braid types for all finite unions of periodic orbits of diffeomorphisms of surfaces of genus zero, both orientation-preserving and -reversing, having zero topological entropy.

## 1. *Introduction*

The topological entropy $h(f)$ of a continuous map $f$ of a compact metric space to itself is a measure of its dynamical complexity (see **[28]** for an exposition). In this paper we shall study diffeomorphisms of surfaces of genus zero (such as the sphere $S^2$, disk $D^2$ and annulus $A$), both orientation-preserving and -reversing, with zero topological entropy. We shall find all the possible structures of the set of periodic orbits that can occur for such diffeomorphisms, in the sense of braid type, which we shall now define.

Let $X$ be a surface, $f: X \to X$ a diffeomorphism, and $\mathscr{P}$ a finite union of periodic orbits of $f$. Define a surface $X_{\mathscr{P}}$ by removing $\mathscr{P}$ from $X$ and recompactifying by adding a circle to each end of $X \backslash \mathscr{P}$. Extend $f | X \backslash \mathscr{P}$ to a homeomorphism $f_{\mathscr{P}}: X_{\mathscr{P}} \to X_{\mathscr{P}}$ as follows **[3]**. If $S_x$ is the circle replacing $x \in \mathscr{P}$, then obtain $f_{\mathscr{P}} | S_x$ from the linear map $Df_x: T_x X \to T_{f_x} X$ by radially projecting the image of the unit circle in $T_x X$ onto the unit circle in $T_{f_x} X$. Since **[28]**

$$h(f_{\mathscr{P}} | \bigcup_{x \in \mathscr{P}} S_x) = 0,$$

if $h(f) = 0$ then $h(f_{\mathscr{P}}) = 0$ (see **[11]**). We call this operation *blowing up* $\mathscr{P}$.

Two homeomorphisms $f, g$ of a space $X$ are said to be *isotopic* if there is a continuous map $\phi: X \times [0,1] \hookleftarrow$ such that $\phi(\cdot, 0) = f$, $\phi(\cdot, 1) = g$ and $\phi(\cdot, t)$ is a homeomorphism of $X$ for each $t \in [0,1]$. We shall write o-p for orientation-preserving and o-r for orientation-reversing. We say two maps $f: X \to X$, $g: Y \to Y$ are *o-p conjugate*, written $f \simeq g$, if there exists an o-p homeomorphism $k: X \to Y$ such that $kf = gk$. Note that the restriction that $k$ be o-p is not necessary if $f$ (or $g$) is o-r since if $k$ is an o-r conjugacy then $kf$ (respectively, $gk$) is an o-p one. But it is a restriction if both $f$ and $g$ are o-p.

Given two diffeomorphisms $f, g: X \to X$ of an orientable surface $X$ and finite unions $\mathscr{P}, Q$ of periodic orbits for $f, g$ respectively, we say that $(\mathscr{P}, f)$ and $(Q, g)$ have

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