ON THE C^1 NON-INTEGRABILITY OF THE AUTONOMOUS DIFFERENTIAL SYSTEMS

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Abstract. In the study of the dynamics of the autonomous differential systems to know the existence or non-existence of first integrals is a relevant fact. These last decades the meromorphic non-integrability of the autonomous differential systems have been studied intensively using the Ziglin's and the Morales-Ramis' theories. Here we study the C^1 non-integrability of the autonomous differential systems, these studies goes back to Poincaré.

It is known that the semiclassical Jayne-Cummings differential system of dimension five has only two independent meromorphic first integrals, namely H and F, and of course any meromorphic function in the variables H and F. Here we illustrate how to study the C^1 non-integrability of the autonomous differential systems showing that the semiclassical Jayne-Cummings differential system of dimension five has only two independent C^1 first integrals H and F, and of course any C^1 function in the variables H and F.

1. Introduction

Ziglin's theory on the meromorphic non-integrability was inspired in the studies of the integrability of the rigid body done by Kovalevskaya. This theory study the non-integrability of an autonomous differential system using the monodromy group of the variational equation associated to some non-equilibrium solution of the analysed differential system. Ziglin's theory was improved by the Morales-Ramis theory that considers the Galois differential group instead of the monodromy group of the variational equation, and in general the Galois group is easier to study, see [12]. But both theories only allow to study the non-existence of meromorphic first integrals.

As Arnold said in [1] the mentioned both theories of non-integrability, in a beginning inspired in Kovalevskaya's ideas, go back to Poincaré, because Poincaré for studying the non-integrability of the autonomous differential systems already used the multipliers of the monodromy group of the variational equations associated to periodic orbits. Apparently the mathematical community forgot the results of Poincaré until that some Russian mathematicians published on them, see [1, 5]. In what follows we shall summarize the results of Poincaré.

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Let U be an open subset of $\mathbb{R}^n$ and let $f\colon U \to \mathbb{R}^n$ be C^2 function. We shall work with an autonomous differential system of the form

$$\frac{dx}{dt} = \dot{x} = f(x). \tag{1}$$

As usual we denote by $\phi(t, x_0)$ the solution of system (1) such that $\phi(0, x_0) = x_0 \in U$, and where t varies in the maximal interval of definition of this solution.

When $\phi(T, x_0) = x_0$ and $\phi(t, x_0) \neq x_0$ for $t \in (0, T)$ we say that $\phi(t, x_0)$ is a *periodic solution* of period $T > 0$, or simply a *T-periodic solution*. The set $\gamma = \{\phi(t, x_0) : t \in [0, T]\} \subset U$ is the *T-periodic orbit* associated to the T-periodic solution $\phi(t, x_0)$.

As usual the differential equation

$$\dot{M} = \left(\frac{\partial f(x)}{\partial x}\Big|_{x=\phi(t,x_0)} \right) M, \tag{2}$$

where M is an $n \times n$ matrix, is the *variational equation* associated to the T-periodic solution $\phi(t, x_0)$. Here $\partial f(x)/\partial x$ denotes the Jacobian matrix of f. Let $M(t, x_0)$ be the solution of the differential equation (2) such that $M(0, x_0)$ is the identity matrix. Then the matrix $M(T, x_0)$ is the *monodromy matrix* associated to the T-periodic solution $\phi(t, x_0)$. And the *multipliers* of the periodic solution $\phi(t, x_0)$ are the eigenvalues of the monodromy matrix $M(T, x_0)$.

A non-constant function $F\colon U \to \mathbb{R}$ of class C^1 satisfying that

$$\nabla F(x) \cdot f(x) = 0,$$

where $\nabla F(x)$ is the gradient vector of the function F and the $\cdot$ indicates the usual inner product of $\mathbb{R}^n$, is a *first integral* of the differential system (1), because F is constant on the solutions of system (1).

Let $F\colon U \to \mathbb{R}$ and $G\colon U \to \mathbb{R}$ be two first integrals. They are *independent* if their gradients are independent in all the points of the open set U except perhaps in a set of zero Lebesgue measure.

The following result goes back to Poincaré, see [13]. For a detailed proof see Corollary 3 of [9].

Theorem 1. *Consider the C^2 differential system* (1). *If there is a periodic orbit γ having only $s+1$ multipliers equal to* 1, *then system* (1) *has at most $F_1, \ldots, F_s$ C^1 linearly independent first integrals defined in a neighborhood of γ if the $s+1$ vectors $\nabla F_1(x), \ldots, \nabla F_s(x)$ and $f(x)$ are linearly independent on the points $x \in \gamma$.*

Theorem 1 provides a tool for studying the C^1 non-integrability of an autonomous differential system (1) in a neighborhood of the periodic orbit γ, and consequently in the domain of definition U of the differential system (1).

Probably the main reason that Theorem 1 has almost not used for studying the C^1 non-integrability of an autonomous differential system (1) is that to find a periodic orbit γ of that differential system for which we can compute its multipliers and verify that vectors $\nabla F_1(x), \ldots, \nabla F_s(x)$ of the known first integrals of the differential system and $f(x)$ are linearly independent on the points $x \in \gamma$ is not in general an easy task. Now the recent developments of the averaging theory for computing periodic solutions facilitates this task.

Maciejewski and Szumiński in [10] proved that the semiclassical Jaynes-Cummings system (see [2, 4, 6, 7, 11]) defined by the autonomous differential system

$$\dot{x} = -y, \quad \dot{y} = x + zu, \quad \dot{z} = -yu, \quad \dot{u} = v, \quad \dot{v} = \alpha x - \mu^2 u, \tag{3}$$

only has the two meromorphic first integrals, namely

$$H = \alpha(z - x)u + \frac{\mu^2}{2}u^2 + \frac{1}{2}v^2, \quad \text{and} \quad F = x^2 + y^2 + z^2, \tag{4}$$

or meromorphic functions in the variables H and F.

Our main result is the following one.

Theorem 2. *The only C^1 first integrals of the semiclassical Jaynes-Cummings system* (3) *are the first integrals H, F and the C^1 functions in the variables H and F.*

Theorem 2 is proved in section 3.

The tools that we shall use for proving Theorem 2 can be used for studying the C^1 non-integrability in arbitrary autonomous differential systems.

The key steps in the proof of Theorem 2 are:

1. To write the differential system (3) in the normal form for applying the averaging theory for computing periodic orbits, see section 2 for a summary of this averaging theory.
2. Compute a good analytical approximation of a periodic solution γ of the differential system (3).
3. Compute the multipliers of the periodic solution γ and verify the assumptions of Theorem 1.

2. The averaging theory for computing periodic orbits

Since the semiclassical Jaynes-Cummings differential system (3) is of class C^2 we can apply the classical averaging theory for computing periodic solutions, see [14]. If the differential system is only continuous, or a piecewise discontinuous differential system we can use the results on the averaging theory of [3] or [8] for computing periodic solutions, respectively.

In Theorem 11.5 of Verhulst [14] is it proved the following result which gives a first order approximation for the periodic solutions of a periodic differential system.

Consider first the differential system

$$\dot{x} = \varepsilon F(t, x) + \varepsilon^2 R(t, x, \varepsilon), \qquad x(0) = x_0, \tag{5}$$

with $x \in U$ where U is an open subset of $\mathbb{R}^n$, and $t \geq 0$. We suppose that the function $F(t, \mathrm{x})$ is T periodic in t. Consider now in U the averaged differential system

$$\dot{y} = \varepsilon f(y), \qquad y(0) = x_0, \tag{6}$$

where

$$f(y) = \frac{1}{T}\int_0^T F(t, y)dt. \tag{7}$$

In the next theorem we shall see, under convenient assumptions, that the equilibrium solutions of the averaged system (6) provide T–periodic solutions of the differential system (5).

Theorem 3. *Consider the differential systems* (5) *and* (6)*, and assume the following three conditions:*

(i) *The function F, its Jacobian matrix $\partial F/\partial x$, its Hessian matrix $\partial^2 F/\partial x^2$ are continuous and bounded by an independent constant ε in $[0,\ \infty) \times U$ and $\varepsilon \in (0,\ \varepsilon_0]$.*

(ii) *The funciton $F(t, x)$ is T–periodic in t, and its period T does not depend on ε.*

(iii) *The curve $y(t)$ is in U for all $t \in [0, 1/\varepsilon]$*

Then for every equilibrium point p of the averaged equation (6) *satisfying*

$$\det\left(\frac{\partial f}{\partial y}\right)\Big|_{\mathrm{y}=p} \neq 0, \tag{8}$$

there is a T–periodic solution $x(t, \varepsilon)$ of the differential system (5) *such that $x(0, \varepsilon) \to p$ as $\varepsilon \to 0$.*

3. Proof of Theorem 2

It is easy to verify that the differential system (3) has two straight lines filled up with equilibrium points, namely $(0, 0, z, 0, 0)$ for all $z \in \mathbb{R}$ and $(\mu^2 u/\alpha, 0, -\mu^2/\alpha, u, 0)$ for all $u \in \mathbb{R}$.

3.1. **Writting the differential system** (3) **into the normal form** (5) **of the averaging theory.** First we translate the equilibrium point $(0, 0, -\mu^2/\alpha, 0, 0)$ to the origin of coordinates doing the change of variables $z \to \sigma$ doing $z = \sigma - \mu^2/\alpha$, in the new variables (x, y, σ, u, v) system (3) writes

$$\dot{x} = -y, \quad \dot{y} = x + u\left(\sigma - \frac{\mu^2}{\alpha}\right), \quad \dot{\sigma} = -uy, \quad \dot{u} = v, \quad \dot{v} = \alpha x - \mu^2 u, \tag{9}$$

and the two first integrals H and F given in (4) now become

$$H = \alpha\left(\sigma - xu - \frac{\mu^2}{\alpha}\right) + \frac{\mu^2}{2}u^2 + \frac{1}{2}v^2, \quad \text{and} \quad F = x^2 + y^2 + \left(\sigma - \frac{\mu^2}{\alpha}\right)^2.$$

The linear part of the differential system (9) at the origin of coordinates is

$$\begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -\mu^2/\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & -\mu^2 & 0 \end{pmatrix}. \tag{10}$$

So the eigenvalues of the equilibrium point localized at the origin of coordinates of system (9) are $0, 0, 0, \pm\sqrt{\mu^2+1}\, i$.

Doing the change of variables $(x, y, \sigma, u, v =) \to (X, Y, Z, U, V)$ given by

$$\begin{pmatrix} X \\ Y \\ Z \\ U \\ V \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{\alpha}{\mu^2} & 0 & 0 & 1 \\ -\alpha\sqrt{\mu^2+1}/\mu^2 & 0 & 0 & \sqrt{\mu^2+1} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \alpha\varepsilon & 0 & 0 & \varepsilon & 0 \\ 0 & -\alpha & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ \sigma \\ u \\ v \end{pmatrix}, \tag{11}$$

we write the matrix (10) in its real Jordan normal form

$$\begin{pmatrix} 0 & -\sqrt{\mu^2+1} & 0 & 0 & 0 \\ \sqrt{\mu^2+1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\varepsilon > 0$ will be the small parameter which appears in the normal form of the differential system (5) for applying the averaging theory. And the differential system (9) in the new variables (X, Y, Z, U, V) becomes

$$\begin{aligned} \dot{X} &= \frac{\alpha\sqrt{\mu^2+1}UZ - \varepsilon\mu^2 Y\left(\left(\mu^2+1\right)^2 - \alpha Z\right)}{\varepsilon\mu^2\left(\mu^2+1\right)^{3/2}}, \\ \dot{Y} &= \sqrt{\mu^2+1}X, \\ \dot{Z} &= \frac{\mu^2(V-X)\left(\sqrt{\mu^2+1}U + \varepsilon\mu^2 Y\right)}{\varepsilon\alpha\left(\mu^2+1\right)^{5/2}}, \\ \dot{U} &= \varepsilon V, \\ \dot{V} &= -\frac{\alpha Z\left(\sqrt{\mu^2+1}U + \varepsilon\mu^2 Y\right)}{\varepsilon\left(\mu^2+1\right)^{3/2}}. \end{aligned} \tag{12}$$

Now the first integrals H and F for the system (12) are

$$\begin{aligned} \tilde{H} = {} & \varepsilon^2 H = -\frac{\mu^2}{2(\mu^2+1)^2}U^2 + \varepsilon\frac{\mu^2}{(\mu^2+1)^{5/2}}UY + \varepsilon^2\frac{1}{2(\mu^2+1)^3}\Big((\mu^2+1)V^2 \\ & +2\mu^2(\mu^2+1)VX + \mu^2\big(-2\mu^6 + \mu^4(X^2+Y^2-6) \\ & +\mu^2(X^2+2Y^2-6) - 2\big) + 2\alpha(\mu^2+1)^3 Z\Big), \\ \tilde{F} = {} & \varepsilon^2 F = \frac{\mu^4}{\alpha^2(\mu^2+1)^2}U^2 - \varepsilon\frac{2\mu^4}{\alpha^2\left(\mu^2+1\right)^{5/2}}UY \\ & +\varepsilon^2\left(\frac{\mu^4(V-X)^2}{\alpha^2\left(\mu^2+1\right)^2} + \frac{\mu^4 Y^2}{\alpha^2\left(\mu^2+1\right)^3} + \left(Z - \frac{\mu^2}{\alpha}\right)^2\right). \end{aligned}$$

Since the normal form of the differential system (5) for applying the averaging theory needs the small parameter ε in front of all the equations of the differential system, first we do the change of variables $(X, Y) = (r\cos\theta, r\sin\theta)$ and after the rescaling $(r, Z, U, V) = (\varepsilon^2 R, \varepsilon^2 S, \varepsilon^2 Q, \varepsilon^2 W)$, so in the new variables (R, θ, S, Q, W)

the differential system (12) becomes

$$
\begin{aligned}
\dot{R} &= \varepsilon \frac{\alpha Q S \cos\theta}{\mu^4+\mu^2} + \varepsilon^2 \frac{\alpha R S \sin\theta \cos\theta}{\left(\mu^2+1\right)^{3/2}},\\
\dot{\theta} &= \sqrt{\mu^2+1} - \varepsilon \frac{\alpha Q S \sin\theta}{\left(\mu^4+\mu^2\right) R} - \varepsilon^2 \frac{\alpha S \sin^2\theta}{\left(\mu^2+1\right)^{3/2}},\\
\dot{S} &= \varepsilon \frac{\mu^2 Q (W - R\cos\theta)}{\alpha \left(\mu^2+1\right)^2} - \varepsilon^2 \frac{\mu^4 R \sin\theta (R\cos\theta - W)}{\alpha \left(\mu^2+1\right)^{5/2}},\\
\dot{Q} &= \varepsilon W,\\
\dot{W} &= -\varepsilon \frac{\alpha Q S}{\mu^2+1} - \frac{\alpha \mu^2 R S \sin\theta}{\left(\mu^2+1\right)^{3/2}},
\end{aligned}
\tag{13}
$$

and the two first integrals for the system (13) become

$$
\begin{aligned}
H = &\; -\varepsilon^2\mu^2 + \varepsilon^4 \left(\alpha S - \frac{\mu^2 Q^2}{2\left(\mu^2+1\right)^2}\right) + \varepsilon^5 \frac{\mu^2 Q R \sin\theta}{\left(\mu^2+1\right)^{5/2}} + \varepsilon^6 \frac{1}{2\left(\mu^2+1\right)^3}\\
&\left(-\frac{1}{2}\mu^4 R^2 \cos(2\theta) + \mu^6 R^2 + \frac{3\mu^4 R^2}{2} + 2\left(\mu^2+1\right)\mu^2 R W \cos\theta + \mu^2 W^2 + W^2\right),\\
F = &\; \varepsilon^2 \frac{\mu^4}{\alpha^2} + \varepsilon^4 \frac{\mu^2}{\alpha^2}\left(\frac{\mu^2 Q^2}{\left(\mu^2+1\right)^2} - 2\alpha S\right) - \varepsilon^5 \frac{2\mu^4 Q R \sin\theta}{\alpha^2\left(\mu^2+1\right)^{5/2}} + \varepsilon^6 \frac{1}{\alpha^2\left(\mu^2+1\right)^3}\\
&\big(\mu^4 R^2 \sin^2\theta + (\mu^2+1)\mu^4 R^2 \cos^2\theta - 2(\mu^2+1)\mu^4 R W \cos\theta\\
&+ \left(\mu^2+1\right)\left(\alpha^2\left(\mu^2+1\right)^2 S^2 + \mu^4 W^2\right)\big).
\end{aligned}
$$

We shall restrict the differential system (13) to the invariant levels

$$
H = -\varepsilon^2\mu^2 + \varepsilon^4 h, \quad \text{and} \quad F = \varepsilon^2\frac{\mu^4}{\alpha^2} - \varepsilon^4\frac{2h\mu^2}{\alpha^2} - \varepsilon^5\frac{f}{\alpha^2(\mu^2+1)^{5/2}},
$$

where h and f are constants. Then from this first equation we obtain

$$
S = \frac{2h\left(\mu^2+1\right)^2 + \mu^2 Q^2}{2\alpha\left(\mu^2+1\right)^2} + O(\varepsilon),
\tag{14}
$$

and from the second one we get

$$
R = \frac{f}{2\mu^4 Q \sin\theta} + O(\varepsilon).
\tag{15}
$$

Additionally taking the variable θ as the new independent variable the differential system (13) becomes a 2π-periodic differential system in the normal form (5) for applying the averaging theory, i.e.

$$
\begin{aligned}
\frac{dQ}{d\theta} &= \varepsilon \frac{W}{\sqrt{\mu^2+1}} + O(\varepsilon^2)\\
\frac{dW}{d\theta} &= -\varepsilon \frac{Q\left(2h\left(\mu^2+1\right)^2 + \mu^2 Q^2\right)}{2\left(\mu^2+1\right)^{7/2}} + O(\varepsilon^2).
\end{aligned}
\tag{16}
$$

3.2. **Computing an explicit periodic solution of the differential system (16).** Now the averaged function (7) for the differential system (16) is $f(Q,W) =$

$(f_1(Q,W), f_2(Q,W))$ where

$$f_1(Q,W) = \frac{W}{\sqrt{\mu^2+1}},$$
$$f_2(Q,W) = -\frac{Q\left(2h\left(\mu^2+1\right)^2+\mu^2Q^2\right)}{2\left(\mu^2+1\right)^{7/2}},$$

Hence the averaged differential system (6) in our case have the equilibrium points

$$p = (0,0), \quad p_\pm = \left(\pm\frac{\mu^2+1}{\mu}\sqrt{-2h}, 0\right).$$

Of course the equilibria $p_\pm$ only exist if $h<0$. Note that the Jacobian (8) at the points p and $p_\pm$ takes the nonzero values $h/(\mu^2+1)^2$ and $-2h/(\mu^2+1)^2$ if $h \neq 0$. Then from Theorem 3 we know that the differential system (16) has a 2π-periodic solution of the form

$$(Q(\theta), W(\theta)) = \left(\frac{\mu^2+1}{\mu}\sqrt{-2h}, 0\right) + O(\varepsilon),$$

associated to the equilibrium point p_+. We have such a periodic solution for every value of the constants $h<0$ and f. Then we fix one of these periodic solutions for a value of $f \neq 0$, and denote this periodic orbit by γ.

From (14) and (15) the periodic orbit γ of the differential system (16) in the differential system (13) becomes $(R(t), \theta(t), S(t), Q(t), W(t))$ equal to

$$\left(\frac{f}{2\sqrt{-2h}\mu^3\left(\mu^2+1\right)\sin\theta}, t\sqrt{\mu^2+1}, 0, \frac{\mu^2+1}{\mu}\sqrt{-2h}, 0\right) + O(\varepsilon),$$

and has period $2\pi/\sqrt{\mu^2+1}$. This periodic orbit in the coordinates of the differential system (12) is $(X(t), Y(t), Z(t), U(t), V(t))$ equal to

$$\left(\varepsilon^2\frac{f\cot\left(t\sqrt{\mu^2+1}\right)}{2\sqrt{-2h}\mu^3\left(\mu^2+1\right)}, \varepsilon^2\frac{f}{2\sqrt{-2h}\mu^3\left(\mu^2+1\right)}, 0, \varepsilon^2\frac{\mu^2+1}{\mu}\sqrt{-2h}, 0\right) + O(\varepsilon^3).$$

Using the change of variables (11) this periodic orbit for the differetnial system (9) becomes $(x(t), y(t), \sigma(t), u(t), v(t))$ equal to

$$\left(\varepsilon\tfrac{\sqrt{-2h}\mu}{\alpha} - \varepsilon^2\frac{f}{2\alpha\sqrt{-2h}\mu\left(\mu^2+1\right)^{5/2}}, \varepsilon^2\frac{f\cot\left(t\sqrt{\mu^2+1}\right)}{2\alpha\sqrt{-2h}\mu\left(\mu^2+1\right)^2}, 0,\right.$$
$$\left.\varepsilon\frac{\sqrt{-2h}}{\mu} + \varepsilon^2\frac{f}{2\sqrt{-2h}\mu\left(\mu^2+1\right)^{5/2}}, \varepsilon^2\frac{f\cot\left(t\sqrt{\mu^2+1}\right)}{2\sqrt{-2h}\mu\left(\mu^2+1\right)^2}\right) + O(\varepsilon^3).$$

Finally the periodic orbit γ of period $2\pi/(\sqrt{\mu^2+1})$ in the coordinates of the differential system (3) is $(x(t), y(t), z(t), u(t), v(t))$ equal to

$$\left(\varepsilon\tfrac{\sqrt{-2h}\mu}{\alpha} - \varepsilon^2\frac{f}{2\alpha\sqrt{-2h}\mu\left(\mu^2+1\right)^{5/2}}, \varepsilon^2\frac{f\cot\left(t\sqrt{\mu^2+1}\right)}{2\alpha\sqrt{-2h}\mu\left(\mu^2+1\right)^2}, -\frac{\mu^2}{\alpha},\right.$$
$$\left.\varepsilon\frac{\sqrt{-2h}}{\mu} + \varepsilon^2\frac{f}{2\sqrt{-2h}\mu\left(\mu^2+1\right)^{5/2}}, \varepsilon^2\frac{f\cot\left(t\sqrt{\mu^2+1}\right)}{2\sqrt{-2h}\mu\left(\mu^2+1\right)^2}\right) + O(\varepsilon^3). \tag{17}$$

3.3. **Computing the multipliers of the periodic solution γ.** The variational equation (2) on the periodic orbit (17) of the differential system (3) restricted to the dominant terms of order ε is

$$\dot{Y} = MY, \tag{18}$$

where

$$\dot{Y}(t) = \begin{pmatrix} y_1'(t) & y_2'(t) & y_3'(t) & y_4'(t) & y_5'(t) \\ y_6'(t) & y_7'(t) & y_8'(t) & y_9'(t) & y_{10}'(t) \\ y_{11}'(t) & y_{12}'(t) & y_{13}'(t) & y_{14}'(t) & y_{15}'(t) \\ y_{16}'(t) & y_{17}'(t) & y_{18}'(t) & y_{19}'(t) & y_{20}'(t) \\ y_{21}'(t) & y_{22}'(t) & y_{23}'(t) & y_{24}'(t) & y_{25}'(t) \end{pmatrix},$$

$$M = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & \varepsilon\dfrac{\sqrt{-2h}}{\mu} & -\dfrac{\mu^2}{\alpha} & 0 \\ 0 & -\varepsilon\dfrac{\sqrt{-2h}}{\mu} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & -\mu^2 & 0 \end{pmatrix},$$

and

$$Y(t) = \begin{pmatrix} y_1(t) & y_2(t) & y_3(t) & y_4(t) & y_5(t) \\ y_6(t) & y_7(t) & y_8(t) & y_9(t) & y_{10}(t) \\ y_{11}(t) & y_{12}(t) & y_{13}(t) & y_{14}(t) & y_{15}(t) \\ y_{16}(t) & y_{17}(t) & y_{18}(t) & y_{19}(t) & y_{20}(t) \\ y_{21}(t) & y_{22}(t) & y_{23}(t) & y_{24}(t) & y_{25}(t) \end{pmatrix}.$$

The solution $Y(t)$ at order $O(\varepsilon)$ of the variational equation (18) such that $Y(0)$ is the 5×5 identity matrix is

$$\begin{pmatrix} \dfrac{\mu^2+C}{\mu^2+1} & -\dfrac{t\mu^2}{2\mu^2+2} & \dfrac{(C-1)\sqrt{-2h}}{\mu\left(\mu^2+1\right)^2} & \dfrac{\mu^2-C\mu^2}{\alpha\mu^2+\alpha} & \dfrac{\sqrt{2}\mu^2\left(t\sqrt{\mu^2+1}+S\right)}{\alpha\left(\mu^2+1\right)^{3/2}} \\ \dfrac{S}{\sqrt{\mu^2+1}} & \dfrac{\mu^2+C}{\mu^2+1} & \dfrac{\sqrt{-h}\left(t\sqrt{\mu^2+1}\mu^2+2S\right)}{\sqrt{2}\mu\left(\mu^2+1\right)^{3/2}} & -\dfrac{S\mu^2}{\alpha\sqrt{\mu^2+1}} & \dfrac{(C-1)\mu^2}{\alpha\left(\mu^2+1\right)} \\ 0 & 0 & 1 & 0 & \dfrac{\sqrt{-2h}\mu\left(t\sqrt{\mu^2+1}-S\right)}{\alpha\left(\mu^2+1\right)^{3/2}} \\ \dfrac{\alpha-C\alpha}{\mu^2+1} & -\dfrac{t\alpha}{2\mu^2+2} & -\dfrac{(C-1)\sqrt{-h}\alpha}{\mu\left(\mu^2+1\right)^2} & \dfrac{C\mu^2+1}{\mu^2+1} & \dfrac{S\left(S\mu^2+t\sqrt{\mu^2+1}\right)}{\left(1+\mu^2\right)^{3/2}} \\ \dfrac{S\alpha}{\sqrt{\mu^2+1}} & \dfrac{(C-1)\alpha}{\mu^2+1} & -\dfrac{\sqrt{-h}\alpha\left(t\sqrt{\mu^2+1}-2S\right)}{\sqrt{2}\mu\left(\mu^2+1\right)^{3/2}} & -\dfrac{S\mu^2}{\sqrt{\mu^2+1}} & \dfrac{C\mu^2+1}{\mu^2+1} \end{pmatrix},$$

where $C=\cos(\sqrt{\mu^2+1}\,t)$ and $S=\sin(\sqrt{\mu^2+1}\,t)$. Then the monodromy matrix $Y(2\pi/\sqrt{\mu^2+1})$ is

$$\begin{pmatrix} 1 & -\dfrac{\pi\mu^2}{(\mu^2+1)^{3/2}} & 0 & 0 & \dfrac{2\sqrt{2}\pi\mu^2}{\alpha\,(\mu^2+1)^{3/2}} \\ 0 & 1 & \dfrac{i\sqrt{2}\sqrt{h}\pi\mu}{(\mu^2+1)^{3/2}} & 0 & 0 \\ 0 & 0 & 1 & 0 & \dfrac{2i\sqrt{2}\sqrt{h}\pi\mu}{\alpha\,(\mu^2+1)^{3/2}} \\ 0 & -\dfrac{\pi\alpha}{(\mu^2+1)^{3/2}} & 0 & 1 & \dfrac{2\pi S}{(\mu^2+1)^{3/2}} \\ 0 & 0 & -\dfrac{i\sqrt{2}\sqrt{h}\pi\alpha}{\mu\,(\mu^2+1)^{3/2}} & 0 & 1 \end{pmatrix},$$

and their eigenvalues

$$1,\quad 1,\quad 1,\quad 1-\frac{2\pi\sqrt{h}}{(\mu^2+1)^{3/2}},\quad 1+\frac{2\pi\sqrt{h}}{(\mu^2+1)^{3/2}},$$

are the multipliers of the periodic orbit γ. That is, three multipliers equal to 1 and two complex multipliers.

In order to apply Theorem 1 and to obtain that the unique C^1 first integrals of the differential system (3) are H, F and any C^1 function in the variables H and F we need to verify that the three vectors $\nabla H(x,y,z,u,v)$, $\nabla F(x,y,z,u,v)$ and $(\dot{x},\dot{y},\dot{z},\dot{u},\dot{v})$ of system (3) are linearly independent on the points (x,y,z,u,v) of the periodic orbit (17). Indeed, an easy computation shows that the matrix 3×5 having the three files formed by the components of the vectors $\nabla H(x,y,z,u,v)$, $\nabla F(x,y,z,u,v)$ and $(\dot{x},\dot{y},\dot{z},\dot{u},\dot{v})$ has rank three. This completes the proof of Theorem 2.

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