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The limit dynamics for the vacuum Einstein equations in a homogeneous universe

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ABSTRACT

We study the dynamics of the Bianchi IX universe when one of the structure constants tends to zero, i.e. we study the dynamics of the Bianchi VII universe. We prove that there is a surface filled of periodic orbits surrounding an equilibrium point, and that except for another surface filled of equilibria the remainder orbits comes and go to infinity.

1. Introduction and statement of the main results

Bianchi universes are homogeneous cosmological models non-necessarily isotropic in 3-dimensional spaces created by Luigi Bianchi [1]. Here I study the qualitative dynamics of the model Bianchi VII, who can be obtained as the limit of Bianchi IX when one of the structure constants tends to zero.

The vacuum Einstein equations for a homogeneous universe are

$$\begin{aligned}(\ln a^2)'' &= (\mu b^2 - \nu c^2)^2 - \lambda^2 a^4,\\ (\ln b^2)'' &= (\lambda a^2 - \nu c^2)^2 - \mu^2 b^4,\\ (\ln c^2)'' &= (\lambda a^2 - \mu b^2)^2 - \nu^2 c^4,\end{aligned} \tag{1}$$

satisfying that

$$XY + XZ + YZ - \left(\lambda^2 a^4 + \mu^2 b^4 + \nu^2 c^4 - 2\left(\lambda\mu a^2 b^2 + \mu\nu b^2 c^2 + \lambda\nu a^2 c^2\right)\right) = 0, \tag{2}$$

where $X = (\ln a^2)'$, $Y = (\ln b^2)'$ and $Z = (\ln c^2)'$, see equations (22)–(24) of [2], and for more details [2]. As usual the prime denotes derivative with respect to the time t.

In order to study the dynamics of the differential system (1) of three second order differential equations we write this system as the following differential system of six first order differential equations

$$\begin{aligned}x' &= X,\\ X' &= (\mu e^y - \nu e^z)^2 - \lambda^2 e^{2x},\\ y' &= Y,\\ Y' &= (\lambda e^x - \nu e^z)^2 - \mu^2 e^{2y},\\ z' &= Z,\\ Z' &= (\lambda e^x - \mu e^y)^2 - \nu^2 e^{2z},\end{aligned} \tag{3}$$

where $x = \ln a^2$, $y = \ln b^2$ and $z = \ln c^2$. This differential system has the first integral

$$H = XY + XZ + YZ - \left(\lambda^2 e^{2x} + \mu^2 e^{2y} + \nu^2 e^{2z} - 2\left(\lambda\mu e^{x+y} + \mu\nu e^{y+z} + \lambda\nu e^{x+z}\right)\right),$$

because the function $H = H(x, X, y, Y, z, Z)$ is constant on the solutions of the differential system (3) because

$$\frac{dH}{dt} = \frac{\partial H}{dx}x' + \frac{\partial H}{dX}X' + \frac{\partial H}{dy}y' + \frac{\partial H}{dY}Y' + \frac{\partial H}{dz}z' + \frac{\partial H}{dZ}Z' = 0.$$

In the paper [2] the authors obtained numerically information about the dynamics of the differential system (1) when $\lambda \to 0$. Our objective is to study analytically the differential system (1) when $\lambda \to 0$ and to describe its dynamics.

The differential system (3) when $\lambda \to 0$ becomes

$$\begin{aligned}x' &= X,\\ X' &= (\mu e^y - \nu e^z)^2,\\ y' &= Y,\\ Y' &= \nu^2 e^{2z} - \mu^2 e^{2y},\\ z' &= Z,\\ Z' &= \mu^2 e^{2y} - \nu^2 e^{2z}.\end{aligned} \tag{4}$$

Our main result is the following one.

Theorem 1. *The orbits of the differential system (4) are one of the following three types.*

(a) Equilibrium points: system (4) has a surface filled of equilibria.

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