CENTRAL CONFIGURATIONS OF THE CIRCULAR RESTRICTED 4-BODY PROBLEM WITH THREE EQUAL PRIMARIES IN THE COLLINEAR CENTRAL CONFIGURATION OF THE 3-BODY PROBLEM

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Abstract. In this paper we classify the central configurations of the circular restricted 4-body problem with three primaries with equal masses at the collinear configuration of the 3-body problem and an infinitisimal mass.

1. Introduction and results

The well-known Newtonian n-body problem concerns with the motion of n mass points with positive mass m_i moving under their mutual attraction in $\mathbb{R}^d$ in accordance with Newton's law of gravitation.

The equations of the motion of the n-body problem are :

$$\ddot{r}_i = -\sum_{j=1, j\neq i}^{n} \frac{m_j(r_i - r_j)}{r_{ij}^3}, \quad 1 \leq i \leq n,$$

where we have taken the unit of time in such a way that the Newtonian gravitational constant be one, and $r_i \in \mathbb{R}^d$ $(i = 1, \ldots, n)$ denotes the position vector of the i-body, $r_{ij} = |r_i - r_j|$ is the Euclidean distance between the i-body and the j-body.

The solutions of the 2-body problem (also called the Kepler problem) has been completely solved. Unfortunately the solutions for the n-body for $n > 2$ is still an open problem.

For the Newtonian n-body problem the simplest possible motions are such that the configuration formed by the n-bodies is constant up to rotations and scaling, such motions are called the *homographic solutions* of the n-body problem, and are the unique known explicit solutions of the n-body problem when $n > 2$. Only some special configurations of particles are allowed in the homographic solutions of the n-body problem, called by Wintner [64] *central configurations*. Also, central configurations are of utmost importance when studying bifurcations of the hypersurfaces of constant energy and angular momentum, for more details see Meyer [47] and Smale [60].

More precisely, let

$$M = m_1 + \cdots + m_n, \quad c = \frac{m_1 r_1 + \cdots + m_n r_n}{M},$$

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be the *total mass* and the *center of masses* of the n bodies, respectively.

A *configuration* $r = (r_1, \ldots, r_n)$ is called a *central configuration* if the acceleration vectors of the n bodies are proportional to their positions with respect to the center of masses with the same constant λ of proportionality, i.e.

$$\sum_{j=1, j\neq i}^{n} \frac{m_j(r_i - r_j)}{r_{ij}^3} = \lambda(r_j - c), \quad 1 \leq j \leq n, \tag{1}$$

where λ is the constant of proportionality.

Equations (1) are strongly nonlinear and to find the explicit central configurations $(r_1, \ldots, r_n)$ in function of the masses $m_1, \ldots, m_n$ when $n > 3$ is an unsolved problem.

Equations (1) are invariant under rotations, dilatations and translations on the plane. Two central configurations are related if we can pass from one to the other doing some of the mentioned transformations. This relation is of equivalence. When we talk about the number of central configurations we will talk about the number of classes of equivalence of central configurations.

There is an extensive literature on the study of central configurations, see for instance Euler [28], Lagrange [37], Hagihara [33], Llibre [40, 41], Meyer [47], Moeckel [49], Moulton [49], Saari [56], Smale [60], ..., and the papers quoted in these references.

In this paper we are interested in the planar central configurations of a circular restricted 4-body problem. Of course, for the central configurations of the 4-body problem there are many partial results, see for instance the papers [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 34, 35, 36, 38, 39, 42, 43, 44, 45, 46, 50, 51, 52, 53, 54, 55, 57, 58, 59, 63, 65, 66].

We note that the set of central configurations is invariant under translations, rotations, and homothecies with respect their center of mass. It is said that two central configurations are *equivalent* if after having the same center of mass (doing a translation if necessary) we can pass from one to the other through a rotation around its common center of mass and a homothecy. This defines a relation of equivalence in the set of central configurations. From now on when we talk about a central configuration, we are talking on a class of central configurations under this relation of equivalence.

The objective of the present article is to study the central configurations of the circular restricted 4-body problem with three equal primaries in the collinear central configuration of the 3-body problem. We recall that for the 3-body problem when the three masses are equal there is a unique collinear central configuration, where the mass in the middle equidistant from the other two, of course the equal masses can be permuted in the positions of this configuration.

Of course as in any circular restricted problem the objective is to describe the motion of the infinitesimal mass with respect to the primaries. Usually this problem is studied in a rotating system of coordinates where the positions of the primaries remain fixed, see for more details on the restricted problems the book of Szebehely [62].

More precisely, taking the unit of mass equal to the masses of the three primaries and since a central configuration is invarinat under rotations and homothecies through its center of mass without loss of generality we can assume that the position vector r_j of the three primaries with masses $m_1 = m_2 = m_3 = 1$ are

$$r_1 = (x_1, y_1) = (-1, 0),\ r_2 = (x_2, y_2) = (0, 0),\ r_3 = (x_3, y_3) = (1, 0). \tag{2}$$

We denote the position of the infinitesimal mass $m_4 = 0$ by $r_4 = (x_4, y_4) = (x, y)$. Then our main result is the following one.

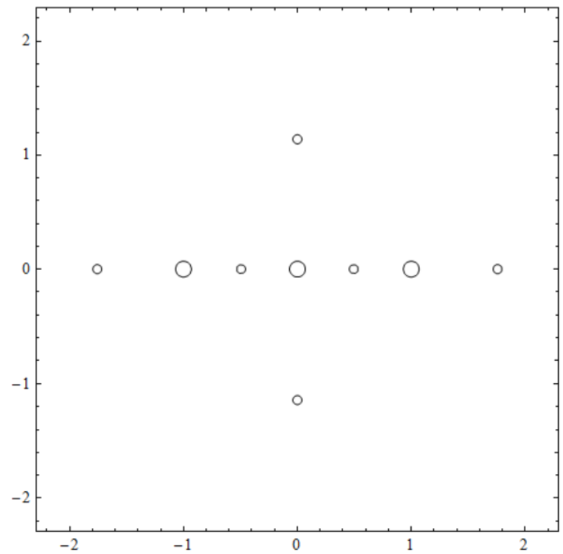


FIGURE 1. The six central configurations of the circular restricted 4-body problem with three equal primaries in the collinear central configuration of the 3-body problem. The three primaries are indicated with the big circles, and the position of the infinitesimal mass in the corresponding six central configurations is indicated with a small circle.

Theorem 1. *The circular restricted* 4*-body problem with three primaries of equal masses* $m_1 = m_2 = m_3 = 1$ *with position vectors given in* (2)*, and one infinitessimal mass* $m_4 = 0$ *with position vector* $r_4 = (x_4, y_4) = (x, y)$ *have the following six central configurations with* $r_4 = p_j$ *for* $j = 1, \ldots, 6$ *being:*

- (i) $p_1 = (x, y) = (0, 1.1394282249562009..)$, *where the value of the coordinate* y *is a root of the polynomial* $-16 - 48y^2 + 40y^3 - 48y^4 + 120y^5 + 23y^6 + 120y^7 - 75y^8 + 40y^9 - 75y^{10} - 25y^{12}$;
- (ii) $p_2 = (x, y) = (0, -1.1394282249562009..)$;
- (iii) $p_3 = (x, y) = (1.7576799791694022.., 0)$, *where the value of the coordinate* y *is a root of the polynomial* $-4 + 5x^3 - 12x^4 - 10x^5 + 5x^7$;
- (iv) $p_4 = (x, y) = (0.49466649101736443.., 0)$, *where the value of the coordinate* y *is a root of the polynomial* $-4 + 8x^2 + 21x^3 - 4x^4 - 10x^5 + 5x^7$;
- (v) $p_5 = (x, y) = (-0.49466649101736443.., 0)$;

(vi) $p_6 = (x, y) = (-1.7576799791694022.., 0)$.

See Figure 1.

The proof of Theorem 1 in given in the next section.

2. Proof of Theorem 1

From (1) we obtain the following eight equations for the central configurations of the 4-body problem in the plane

$$
\begin{aligned}
e_j = & \sum_{j=1, j\neq i}^{4} \frac{m_j(x_i - x_j)}{r_{ij}^3} = \lambda(x_j - c_1), \quad 1 \le j \le 4, \\
e_{j+5} = & \sum_{j=1, j\neq i}^{4} \frac{m_j(y_i - y_j)}{r_{ij}^3} = \lambda(y_j - c_2), \quad 1 \le j \le 4,
\end{aligned} \tag{3}
$$

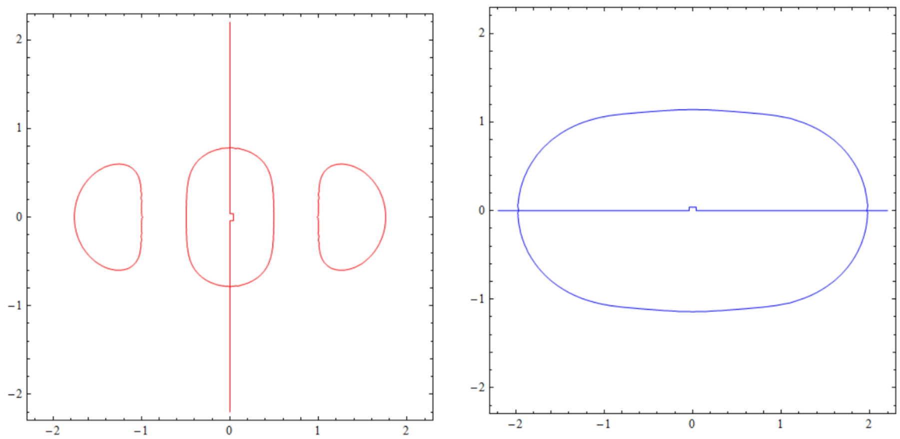
where $c = (c_1, c_2)$. Substituting in (3) the expressions (2), $m_1 = m_2 = m_3 = 1$, $m_4 = 0$ and $r_4 = (x_4, y_4) = (x, y)$, corresponding to our circular restricted 4-body problem these eight equations reduce to

$$
\begin{aligned}
& e_1 = -e_3 = \frac{5}{4} + \lambda = 0, \\
& e_2 = e_5 = e_6 = e_7 = 0, \\
& e_4 = -\lambda x - \frac{x}{(x^2+y^2)^{3/2}} - \frac{1+x}{((x+1)^2+y^2)^{3/2}} + \frac{1-x}{((x-1)^2+y^2)^{3/2}} = 0, \\
& e_8 = y\left(-\lambda - \frac{1}{(x^2+y^2)^{3/2}} - \frac{1}{((x-1)^2+y^2)^{3/2}} - \frac{1}{((x+1)^2+y^2)^{3/2}}\right) = 0.
\end{aligned}
$$

Therefore $\lambda = -5/4$, and the position vector of $r_4 = (x, y)$ in order to have a central configuration of the circular restricted 4-body problem must be a real solution of the system

$$
\begin{aligned}
e_4 &= \frac{5}{4}x - \frac{x}{(x^2+y^2)^{3/2}} - \frac{1+x}{((x+1)^2+y^2)^{3/2}} + \frac{1-x}{((x-1)^2+y^2)^{3/2}} = 0, \\
e_8 &= y\left(\frac{5}{4} - \frac{1}{(x^2+y^2)^{3/2}} - \frac{1}{((x-1)^2+y^2)^{3/2}} - \frac{1}{((x+1)^2+y^2)^{3/2}}\right) = 0.
\end{aligned}
$$

In Figure 2 we have shwon the curves $e_4(x, y) = 0$ and $e_8(x, y) = 0$, and in Figure 3 the intersection of these two curves. We see that these two curves intersect in six points inside the rectangle $R = \{(x, y) \in \mathbb{R}^2 : -2.2 \le x \le 2.2, -2.2 \le y \le 2.2\}$. Computing the coordenates of these six points numerically using the Newton method (see for instance [61]), we get the six points p_j which appear in the statement of Theorem 1. Of course we have omitted the three points where are located the the three primaries in the intersections of the two curves $e_4(x, y) = 0$ and $e_8(x, y) = 0$, because there really these two curves are not defined. Now we shall prove that these six points obtained numerically really are solutions of the system $e_4(x, y) = e_8(x, y) = 0$.



(a) The curve $e_4(x, y) = 0$ (b) The curve $e_8(x, y) = 0$

FIGURE 2. The curves $e_4(x, y) = 0$ and $e_8(x, y) = 0$ in the rectangle $(x, y) \in [-2.2, 2.2] \times [-2.2, 2.2]$.

We note that equations $e_4(x, y) = e_8(x, y) = 0$ are invariant if we change x by $-x$, and y by $-y$, so if (x, y) is a solution of the system $e_4(x, y) = e_8(x, y) = 0$ also $(-x, y)$, $(x, -y)$ and $(-x, -y)$ are solutions. So in order to prove Theorem 1 we only need to study the solutions of system $e_4(x, y) = e_8(x, y) = 0$ satisfying $x \geq 0$ and $y \geq 0$. Moreover, from Figure 3 we see that all the are of the form $(x, 0)$ or $(0, y)$, and since in the the origin $(0, 0)$ there is one primary, we must look only for the solutions $(x, 0)$ or $(0, y)$ with $x > 0$ and $y > 0$.

First we look for the solutions $(0, y)$ with $y > 0$, then system $e_4(x, y) = e_8(x, y) = 0$ reduce to

$$\frac{5y}{4} - \frac{2y}{(y^2+1)^{3/2}} - \frac{1}{y^2} = 0, \tag{4}$$

or equivalently to

$$8y^3 = (1 + y^2)^{3/2}(-4 + 5y^3).$$

Squaring the both sides of the this equation we get the equation

$$-16 - 48y^2 + 40y^3 - 48y^4 + 120y^5 + 23y^6 + 120y^7 - 75y^8 + 40y^9 - 75y^{10} - 25y^{12} = 0.$$

This polynomial equation has only two real roots

$$0.7625005146027564.. \quad \text{and} \quad 1.1394282249562009..,$$

but only the second root satisfies equation (4). This provides the solution p_1 of Theorem 1, and consequently also the solution p_2.

Now we look for the solutions $(x, 0)$ with $x > 0$ of the system $e_4(x, y) = e_8(x, y) = 0$. For these solutions the system reduce to

$$\frac{5x}{4} - \frac{1}{x^2} + \frac{1-x}{|1-x|^{3/2}} = \frac{1+x}{|1+x|^{3/2}}, \tag{5}$$

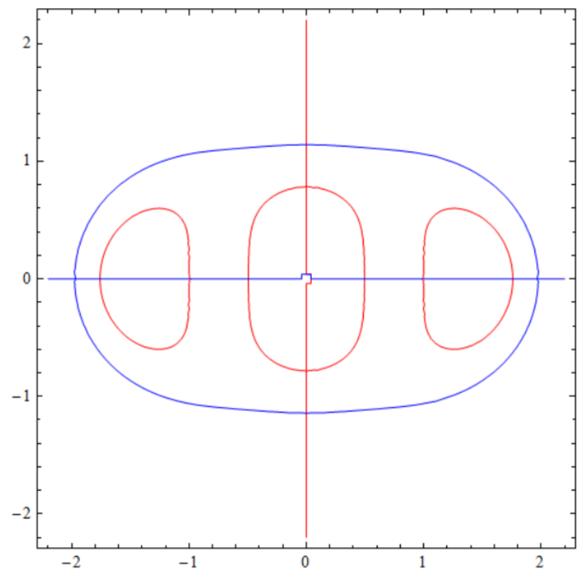


FIGURE 3. In this picture we can see the six intersection points between the two curves $e_4(x,y)=0$ and $e_8(x,y)=0$ different from the positions of the primaries, which provide the six central configurations of the circular restricted 4-body problem with three primaries of equal masses at the collinear configuration of the 3-body problem and an infinitisimal mass

squaring the both sides of the previous equality we obtain

$$\frac{1}{x^4}-\frac{5}{2x}+\frac{25x^2}{16}+\frac{1}{(x-1)^4}-\frac{1}{(x+1)^4}-\frac{(x-1)(5x^3-4)}{2x^2|x-1|^3}=0.$$

Writting this equation with a common denominator, which only vanishes at the positions of the primeries, its numerator equal zero can be written as

$$\begin{aligned}&8(x-1)^5x^2(x+1)^4(5x^3-4)=|x-1|^3(16-64x^2-40x^3+96x^4+288x^5\\&-39x^6-112x^7-84x^8+160x^9+150x^{10}-40x^{11}-100x^{12}+25x^{14}).\end{aligned}$$

Squaring again the both sides of the this equality we get

$$\begin{aligned}&(x-1)^6(-4+5x^3-12x^4-10x^5+5x^7)(-4+8x^2-11x^3-4x^4-10x^5+5x^7)\\&(-4+8x^2+21x^3-4x^4-10x^5+5x^7)(-4+16x^2+5x^3+4x^4-10x^5+5x^7)=0.\end{aligned}$$

The real zero $x=1$ is not good because it correspond to the position of a primary. The unique real root of the polynomial $-4+5x^3-12x^4-10x^5+5x^7$ is 1.7576799791694022.. which also is a zero of equation (5), and consequently provides the central configuration p_3, and by the symmetries of the equations of the central configurations also provides the central configuration p_6.

The unique real root of the polynomial $-4+8x^2+21x^3-4x^4-10x^5+5x^7$ is 0.4946664910173645.. which also is a zero of equation (5), and consequently provides the central configuration p_4, and due to the symmetries of the equations of the central configurations also provides the central configuration p_5.

The real roots of the polynomials $-4+8x^2-11x^3-4x^4-10x^5+5x^7$ and $4+16x^2+5x^3+4x^4-10x^5+5x^7$ are not zeros of the equation (5). This completes the proof of Theorem 1.

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