BOUNDED POLYNOMIAL VECTOR FIELDS II

JAUME LLIBRE

Abstract. Let X be a polynomial vector field in $\mathbb{R}^n$ of degree m, and let X_m be its homogeneous part of degree m. The main purpose of this paper is to give necessary conditions for the boundedness of X in terms of X_m and of the parity of m. Thus, for instance we prove that if X is bounded and m is even then X_m has a straight line of critical points. For $m = 2$ this result was conjectured by Kaplan and Yorke in [**KY**].

1. INTRODUCTION

Let $X : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field and let $\gamma(t, x)$ be the integral curve of X which passes through $x \in \mathbb{R}^n$ when $t = 0$, defined on its maximal interval I_x. We say that X is *bounded* if for each $x \in \mathbb{R}^n$ there exists a compact subset K of $\mathbb{R}^n$ such that $\gamma(t, x) \in K$ for all $t \in I_x \cap (0, +\infty)$.

If all the components of a vector field X in $\mathbb{R}^n$ are polynomial functions then we say that X is *polynomial.* Let $X = (P^1, \dots, P^n)$ be a polynomial vector field in $\mathbb{R}^n$. We say that the *degree* of X is k if $k = \max\{\text{degree}(P^1), \dots, \text{degree}(P^n)\}$, and we say that X is *homogeneous of degree* m if its degree is m and each P^i is a homogeneous polynomial of degree m or is identically zero.

The condition that all the solutions of a polynomial vector field are bounded is a great restriction, but one that is necessary in many physically motivated systems, as for instance the Lorenz system which is an example of bounded polynomial vector field in $\mathbb{R}^3$ of degree 2 (for more details on the Lorenz system, see for instance [**GH**]).

The bounded homogeneous polynomial vector fields in $\mathbb{R}^n$ of degree 2 were studied by Markus, Kaplan and Yorke (see [**KY**] for arbitrary n and [**Ma**] for odd n). They proved that if X is a bounded homogeneous polynomial vector field in $\mathbb{R}^n$ of degree 2, then X has a straight line of critical points.

The main goal of this paper is to prove the following two theorems.

THEOREM 1. *Let X be a bounded polynomial vector field in $\mathbb{R}^n$ of even degree m, and let X_m be its homogeneous part of degree m. Then X_m has a straight line of critical points.*

Notice that Theorem 1 gives a necessary condition for a polynomial vector field in $\mathbb{R}^n$ of even degree to be bounded.