PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 3, November 1981

MINIMAL PERIODIC ORBITS OF CONTINUOUS MAPPINGS OF THE CIRCLE

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ABSTRACT. Let f be a continuous map of the circle into itself and suppose that $n > 1$ is the least integer which occurs as a period of a periodic orbit of f. Then we say that a periodic orbit $\{p_1, \ldots, p_n\}$ is minimal if its period is n. We classify the minimal periodic orbits, that is, we describe how the map f must act on the minimal periodic orbits. We show that there are $\varphi(n)$ types of minimal periodic orbits of period n, where φ is the Euler phi-function.

1. Introduction and statement of results. Let $C(X, X)$ denote the set of continuous maps of a space X into itself. A point $p \in X$ is a *periodic point* of a map $f \in C(X, X)$ if $f^n(p) = p$ for some positive integer n. The *period* of p is the least such integer n, and the *orbit* of p is the set $\{f^k(p): k = 1, \ldots, n\}$. We refer to such an orbit as a *periodic orbit of period n*.

Let $P(f)$ denote the set of positive integers n such that f has a periodic point of period n. The following theorem for periodic orbits of maps of the closed interval I is proved in [5] (see also [3]).

THEOREM (ŠTEFAN). *Let $f \in C(I, I)$. Suppose $n \in P(f)$ where n is odd and $n > 1$, but $j \notin P(f)$ for all $j \in \{3, 5, \ldots, n-2\}$. Let $\{p_1, \ldots, p_n\}$ be a periodic orbit of f of period n with $p_1 < p_2 < \cdots < p_n$. Let $t = (n+1)/2$. Then either* (A) *or* (B) *holds*:

(A)
$$\begin{aligned} f(p_{t-k}) &= p_{t+k} && \text{for } k = 1, \ldots, t-1, \\ f(p_{t+k}) &= p_{t-k-1} && \text{for } k = 0, \ldots, t-2, \quad \text{and} \\ f(p_n) &= p_t. \end{aligned}$$

(B)
$$\begin{aligned} f(p_{t-k}) &= p_{t+k+1} && \text{for } k = 0, \ldots, t-2, \\ f(p_{t+k}) &= p_{t-k} && \text{for } k = 1, \ldots, t-1, \quad \text{and} \\ f(p_1) &= p_t. \end{aligned}$$

In this paper we obtain a similar result for periodic orbits of maps of the circle S^1. For distinct points $a, b \in S^1$, let (a, b) and $[a, b]$ denote the open and closed intervals, respectively, from a counterclockwise to b.

THEOREM A. *Let $f \in C(S^1, S^1)$. Suppose $n \in P(f)$ where $n > 1$, and $j \notin P(f)$ for all $j \in \{1, 2, \ldots, n-1\}$. Let $P = \{p_1, \ldots, p_n\}$ be a periodic orbit of f of period n*

Received by the editors November 4, 1980.
1980 *Mathematics Subject Classification*. Primary 54H20.