PLANAR BOUNDED PIECEWISE SMOOTH POLYNOMIAL VECTOR FIELD

SHIMIN LI[1,*], JAUME LLIBRE[2], XIANBO SUN[3]

Abstract. We prove that for any piecewise-smooth bounded polynomial vector field in $\mathbb{R}^2$ with finitely many finite $\mathcal{H}$-singular points (which include singular points, hyperbolic pseudo-equilibria and two fold singularities, the sum of the indices of all its finite $\mathcal{H}$-singular points is 1.

1. Introduction and statement of the main result

A planar polynomial differential system of the form

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = P(x,y), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = Q(x,y), \end{cases} \tag{1}$$

where $P(x,y)$ and $Q(x,y)$ are polynomials in the variables x and y is called a polynomial system of degree m if m is the maximum degree of the polynomials $P(x,y)$ and $Q(x,y)$. We denote $Z(x,y) = (P(x,y), Q(x,y))$ the associated vector field of system (1).

In the qualitative theory of planar polynomial differential systems [10, 23], one of the most important problems is the determination and distribution of limit cycles, which is known as the famous Hilbert's 16th problem. Since this problem is very difficult, mathematicians pay attention to the special forms of system (1), for Liénard systems see [4, 12], for Z_2-equivariant systems see [5, 15], for Hamiltonian systems see [21, 22].

Definition 1. *A vector field* (1) *is said bounded when all its orbits are bounded for* $t \geqslant 0$.

Let $\gamma(t) = (x(t,x_0), x(t,x_0))$ be the trajectory of system (1) with the initial value $\gamma(0) = (x_0, y_0) \in \mathbb{R}^2$. From [18] we know that if (1) is a bounded vector field, then the trajectory $\gamma(t)$ is defined for all $t \geqslant 0$ and the $\omega-$limit of any point (x_0, y_0) is not empty and compact. In other words, $Z(x,y)$ is a bounded vector field if for all $\gamma(0) \in \mathbb{R}^2$, there exists some compact set $K \subset \mathbb{R}^2$ such that $\gamma(t) \in K$ for each $t \in (0, +\infty)$. It is worth to note that most of the predator-prey systems are bounded because either the population of predator or population of prey cannot tend toward infinity in order to protect the ecological balance.

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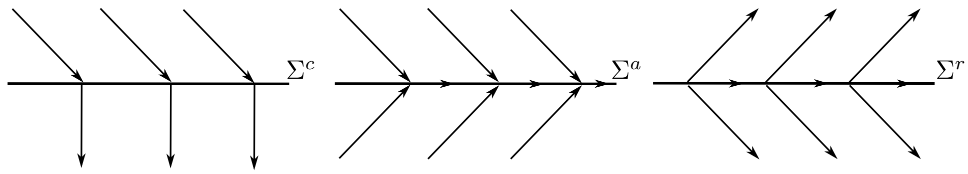


1.1 Crossing region 1.2 Attracting region 1.3 Repelling region

FIGURE 1. Definition of the vector field following Filippov's convention in the crossing, attracting and repelling regions.

Bounded quadratic vector fields (1) have been studied by [8, 9, 14]. Dickson and Perko [8] established necessary and sufficient conditions in order that a quadratic vector field be bounded. The authors of [14] studied the weak focus and bifurcations of limit cycles for the bounded quadratic vector fields. Dumortier and Hersens [9] present a survey of the known results on the bounded quadratic systems.

For a polynomial vector field we can define its infinite singular points using the Poincaré compactification, see for details Chapter 5 of [10]. In [6] the authors proved that:

Theorem 2. *Assume that a bounded polynomial vector field* (1) *has finitely many finite singular points and infinite singular points. Then the sum of the indices of all its finite singular points is* 1.

It is worth note that Theorem 2 has been generalized to the C^1 vector field in [7], and extended without the assumptions that there are finitely many infinite points in [1].

In recent years piecewise smooth differential systems [3, 11] attract more and more attention mainly due to their applications. A planar piecewise smooth (PWS) differential system is defined by

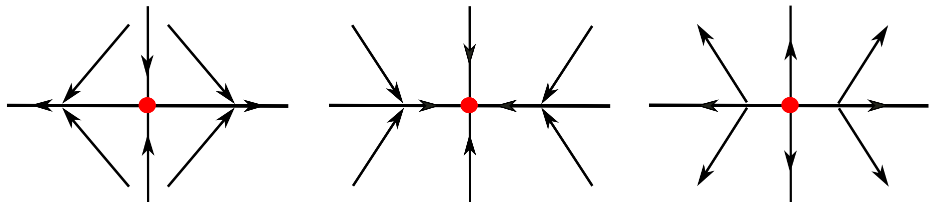
$$\left\{\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= P^{\pm}(x,y),\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= Q^{\pm}(x,y), \end{aligned}\right. \tag{2}$$

where the vector field $(P^+(x,y), Q^+(x,y))$ is defined in $y \geq 0$, and the vector field $(P^-(x,y), Q^-(x,y))$ is defined in $y \leq 0$. Note that the whole plane $\mathbb{R}^2$ is partitioned into two open zones $\Sigma^{\pm} : \{(x,y)|\pm y > 0\}$ by the discontinuous boundary $\Sigma = \{(x,y)|y = 0\}$. Similarly, let $Z^{\pm}(x,y) = (P^{\pm}(x,y), Q^{\pm})(x,y)$ be the piecewise smooth vector filed associated to the piecewise differential system (2).

Definition 3. *The discontinuous boundary* Σ *can be divided into three open regions:*

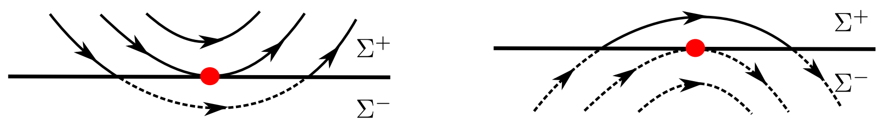
(i) *Crossing region* $\Sigma^c = \{p \in \Sigma | Q^+(p)Q^-(p) > 0\}$*, see Figure 1.1;*
(ii) *Attracting region* $\Sigma^a = \{p \in \Sigma | Q^+(p) < 0, Q^-(p) > 0\}$*, see Figure 1.2;*
(iii) *Repelling region* $\Sigma^r = \{p \in \Sigma | Q^+(p) > 0, Q^-(p) < 0\}$*, see Figure 1.3.*

In the crossing region Σ^c the trajectories of $Z^+(x,y)$ and $Z^-(x,y)$ can be concatenated naturally. However in the attracting region Σ^a (resp. repelling region Σ^r), the trajectories must be continued through Σ^a (resp. Σ^r) and they slid in Σ^a



2.1 Pseudo-saddle 2.2 Stable pseudo-node 2.3 Unstable pseudo-node

FIGURE 2. Definition of the hyperbolic pseudo-equilibrium of a sliding vector field (3).



3.1 visible fold of Z^+ 3.2 invisible fold of Z^+

FIGURE 3. Definition of the fold singularities.

(resp. Σ^r) in forward (resp. backward) time. Thus both the attracting region and the repelling region are called sliding regions, that is $\Sigma^s = \Sigma^a \cup \Sigma^r$. Following the Filippov's convex method we construct the sliding vector filed in the form

$$Z_s(x,y) = \lambda Z^+(x,y) + (1-\lambda)Z^-(x,y), \tag{3}$$

where $\lambda \in (0,1)$ is such that $Z_s(x,y)$ is tangent to Σ^s. Then

$$\lambda = \frac{Q^-(x,y)}{Q^-(x,y) - Q^+(x,y)}. \tag{4}$$

Definition 4. *The point $p := (x,y) \in \Sigma^s$ is a hyperbolic pseudo-equilibrium if $Z_s(p) = 0$ and $Z_s^{'}(p) \neq 0$, where the derivative is taken along the tangent direction of the Σ. Moreover, we will call stable pseudo-node to any point $p \in \Sigma^a$ such that $Z_s(p) = 0$ and $Z_s^{'}(p) < 0$, unstable pseudo-node to any point $p \in \Sigma^r$ such that $Z_s(p) = 0$ and $Z_s^{'}(p) > 0$, pseudo-saddle to any point $p \in \Sigma^a$ such that $Z_s(p) = 0$ and $Z_s^{'}(p) > 0$ or $p \in \Sigma^r$ such that $Z_s(p) = 0$ and $Z_s^{'}(p) < 0$, see Figure 2.*

The boundaries of the above three regions Σ^c, Σ^a and Σ^r are singularities called tangencies, that is $\Sigma^t = \{p \in \Sigma | Q^+(p)Q^-(p) = 0\}$.

Definition 5. *A point $p \in \Sigma^t$ is a fold singularity of $Z^+(x,y)$ if $Q^+(p) = 0$, $Q_x^+(p)P^+(p) + Q_y^+(p)Q^+(p) \neq 0$. Moreover, the fold singularity $p \in Z^+(x,y)$ is visible (resp. invisible) if $Q_x^+(p)P^+(p) + Q_y^+(p)Q^+(p) > 0$ (resp. < 0). Similarly, a point $p \in \Sigma^t$ is a fold singularity of $Z^-(x,y)$ if $Q^-(p) = 0, Q_x^-(p)P^-(p) + Q_y^-(p)Q^-(p) \neq 0$. Moreover, the fold singularity $p \in Z^-(x,y)$ is visible (resp. invisible) if $Q_x^+(p)P^+(p) + Q_y^+(p)Q^+(p) < 0$ (resp. > 0). See Figure 3.*

Definition 6. *A point $p \in \Sigma$ is a two fold singularity when it is a fold singularity in both $Z^+(x,y)$ and $Z^-(x,y)$. The six different types of two fold singularity are shown in Figure 4. A point $p \in \Sigma$ is a fold-regular singularity when p is a fold singularity of $Z^+(x,y)$ (resp. $Z^-(x,y)$) and a regular point of $Z^-(x,y)$ (resp. $Z^+(x,y)$).*

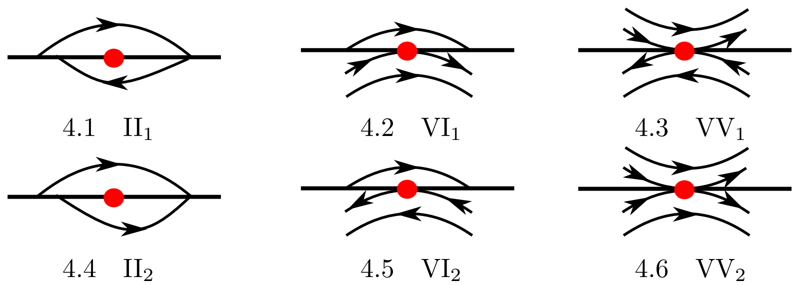


FIGURE 4. Two fold singularities.

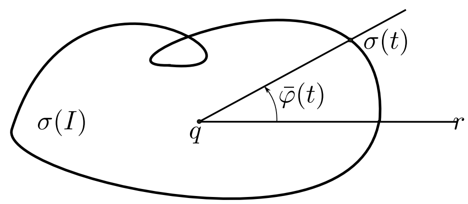


FIGURE 5. Definition of $\varphi(t)$.

Definition 7. *A point p of the vector field $Z^{\pm}(x,y)$ defined in* (2) *is called $\mathcal{H}$-singular point if one of the following conditions hold:*

- *$p \notin \Sigma$ is either a singular point of $Z^{+}(x,y)$ with $y>0$, or a singular point of $Z^{-}(x,y)$ with $y<0$.*
- *$p \in \Sigma^s$ is a hyperbolic pseudo-equilibrium of $Z_s(x,y)$, see Definition 4.*
- *$p \in \Sigma^t$ is a two fold singularity of $Z^{\pm}(x,y)$, see Definition 6.*

Definition 8. *A piecewise-smooth vector field* (2) *is said bounded when all its orbits are bounded for $t \geqslant 0$.*

In [16] the authors proved that the fact that there exist bounded vector fields $Z^{+}(x,y)$ and $Z^{-}(x,y)$, such that $Z^{\pm}(x,y)$ is not bounded, and then they obtained the sufficient and necessary conditions for the piecewise-smooth quadratic system (2) to be bounded.

A path in the plane $\mathbb{R}^2$ is a continuous map from the interval $I=[0,1]$ to $\mathbb{R}^2$ ($\sigma: I \to \mathbb{R}^2$); that is, we assign to every $t \in [0,1]$ the point $\sigma(t)=(\sigma_1(t),\sigma_2(t))$ in the plane, such that $\sigma_i: I \to \mathbb{R}$ are continuous maps. We say that the path σ is closed if $\sigma(0)=\sigma(1)$.

Let q be a point of $\mathbb{R}^2$ which does not belong to $\sigma(I)$ and let r be a ray with origin at q. For every point $\sigma(t)$ we denote by $\bar{\varphi}(t)$ the angle formed by the rays r and $\overline{q\sigma(t)}$. The angle $\bar{\varphi}(t)$ is an element of the circle $\mathbb{R}/2\pi\mathbb{Z}$. The function $\bar{\varphi}: I \to \mathbb{R}/2\pi\mathbb{Z}$ is continuous with respect to the parameter t; see Figure 5.

Let q be a point of $\mathbb{R}^2$ which does not belong to $\sigma(I)$. Then the difference $\varphi(1)-\varphi(0)$ is a multiple of 2π, independent of the chosen ray r. We define the quotient

$$i(q,\sigma) = \frac{\varphi(1) - \varphi(0)}{2\pi},$$

which is an integer called the (topological) index of the closed path σ around the point q.

Given an isolated singularity p of a vector field $Z(x,y)$ in $\mathbb{R}^2$, there is a neighborhood V of p on which there is no other singularity of $Z(x,y)$. Consider now a closed path $\sigma : I \to V \setminus \{p\}$ such that $\sigma(I)$ is a small circle surrounding p. We define the (topological) index of p equal to $i(Z \circ \sigma, p)$; it is equal to the number of turns of the closed path $Z \circ \sigma$ around the origin of coordinates of $\mathbb{R}^2$ in a counter–clockwise sense. The index of p is independent of the chosen closed path σ, see for more details Chapter 6 of [10].

It is easy to check from the definition of index of a singular point that the index of a hyperbolic pseudo-saddle and a hyperbolic pseudo-node are -1 and 1, respectively. While the indices of the two fold singularities II_1, II_2, VI_1, VV_1, VI_2 and VV_2 are $1, 0, 0, -1, 0$ and 0, respectively. Note that the index of two fold singularities II_2, VI_1, VV_2 are 0 because the regularisation of $Z^{\pm}(x,y)$ has no singular points, for the proof see statement (ii) of Proposition 17.

Motivated by the works [1, 6, 7], the purpose of this paper is to extend Theorem 2 to the piecewise smooth vector field (2). The main result of this manuscript can be stated as follows:

Theorem 9. *For any bounded piecewise-smooth vector field* (2) *with finitely many finite $\mathcal{H}$-singular points such that*

$$F(x,y) := P^+(x,y)Q^-(x,y) - P^-(x,y)Q^+(x,y) \neq 0 \tag{5}$$

if $|y|$ is sufficiently small, the sum of the indices for all finite $\mathcal{H}$-singular points is 1.

2. Proof of Theorem 9

The proof of Theorem 9 will be completed doing the following four steps.

Step 1. We approximate the piecewise smooth vector field (2) by an one-parameter family of continuous vector field, which is known as regularization. We refer readers to [19] for more detail on regularization.

Definition 10. *A φ_ε-regularization of a piecewise smooth vector field* (2) *is an one parameter family of vector field $Z_\varepsilon(x,y)$ defined by*

$$Z_\varepsilon(x,y) := (F_1(x,y), F_2(x,y)) = (1 - \varphi_\varepsilon(y))\, Z^-(x,y) + \varphi_\varepsilon(y) Z^+(x,y), \tag{6}$$

where $\varphi_\varepsilon(y) = \varphi\left(\frac{y}{\varepsilon}\right)$, and $\varphi : \mathbb{R} \to \mathbb{R}$ is an analytic transition function satisfying $\varphi(t) = 0$ if $y \leqslant -\varepsilon$, $\varphi(t) = 1$ if $y \geqslant \varepsilon$ and $\varphi'(t) > 0$ if $y \in (-\varepsilon, \varepsilon)$.

It is obvious that

$$Z_\varepsilon(x,y) = Z^{\pm}(x,y), \quad |y| \geq \varepsilon. \tag{7}$$

The strip with $y \in (-\varepsilon, \varepsilon)$ is the region where the piecewise smooth vector field is regularized. Let $y = \varepsilon u$ in (6), then we obtain the so called slow-fast system

$$(8) \qquad \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} & = (1-\varphi(u))P^-(x, \varepsilon u) + \varphi(u)P^+(x, \varepsilon u) := F_1(x, \varepsilon u), \\ \varepsilon\dfrac{\mathrm{d}u}{\mathrm{d}t} & = (1-\varphi(u))Q^-(x, \varepsilon u) + \varphi(u)Q^+(x, \varepsilon u) := F_2(x, \varepsilon u). \end{cases}$$

Using the geometry singular perturbation theory and the blow up technique to (8), we can obtain the dynamics of piecewise smooth vector field (2) near the discontinuous boundary Σ, see for instance [13, 17, 20].

Lemma 11. *If the piecewise smooth vector field* (2) *is bounded, then the regularized continuous vector field* (6) *is bounded.*

Proof. From (6) it follows that when $\varepsilon \to 0$ the vector field (5) tends to the vector field (2) in $\mathbb{R}^2 \setminus \{y = 0\}$. So the vector field (2) is bounded in $\mathbb{R}^2 \setminus \{y = 0\}$.

On $y = 0$ cannot be a sliding orbit escaping to infinity, otherwise from the definition of a sliding orbit in its neighborhood would be orbits in $\mathbb{R}^2 \setminus \{y = 0\}$ escaping to infinity. Hence we obtain that also on the straight line $y = 0$ the orbits are bounded, and the lemma is proved.

□

Step 2. We shall prove that the vector field $Z_\varepsilon(x, y)$ has finitely many finite singular points.

Lemma 12. *If the piecewise smooth vector field $Z^\pm(x, y)$ defined in* (2) *has finitely many finite $\mathcal{H}-$singular points, then the regularized continuous vector field $Z_\varepsilon(x, y)$ defined in* (6) *has finitely many finite singular points.*

In fact if a piecewise smooth vector field (2) has an isolated finite singular point $p_0 = (x_0, y_0) \notin \Sigma$, then we can choose $\varepsilon_0 > 0$ sufficiently small such that $|y_0| \geqslant \varepsilon_0$. Thus p_0 is a singular point of $Z_\varepsilon(x, y)$ for $0 < \varepsilon \leqslant \varepsilon_0$. Now we just need to consider the case $p \in \Sigma$. Without loss of generality we assume that $p = (0, 0) \in \Sigma$. Lemma 12 can be proved from the following six propositions.

Proposition 13. *Let $p \in \Sigma^c$ be a crossing point of $Z^\pm(x, y)$. Then there exist a neighborhood V of p in Σ and ε_0 such that for every $0 < \varepsilon \leqslant \varepsilon_0$, $Z_\varepsilon(x, y)$ has no singular points in V.*

Proof. For the proof see Proposition 6 of [19]. □

Proposition 14. *Let $p \in \Sigma^s := \Sigma^a \cup \Sigma^r$ with $Z_s(p) \neq 0$. Then there exist a neighborhood V of p in Σ and ε_0 such that for every $0 < \varepsilon \leqslant \varepsilon_0$, $Z_\varepsilon(x, y)$ has no singular points in V.*

Proof. For the proof see Proposition 6 of [19]. □

Proposition 15. *Let $p \in \Sigma^s = \Sigma^a \cup \Sigma^r$ be hyperbolic pseudo-equilibrium of $Z_s(p)$. Then there exist a neighborhood V of p in Σ and ε_0 such that for every $0 < \varepsilon \leqslant \varepsilon_0$, $Z_\varepsilon(x, y)$ has a unique singular point in V.*

Proof. For the proof see Proposition 8 of [19]. □

Proposition 16. *Let $p \in \Sigma$ be a fold-regular point of $Z^{\pm}(x,y)$. Then there exist a neighborhood V of p in Σ and an $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \leqslant \varepsilon_0$, $Z_\varepsilon(x,y)$ has no singular points in V.*

Proof. For the proof see Proposition 9 of [19]. □

Proposition 17. *Let $p \in \Sigma$ be a two fold singularity of $Z^{\pm}(x,y)$, the following statements hold.*

(i) If $p \in II_1, VI_2, VV_1$, then there exist a neighborhood V of p in Σ and an $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \leqslant \varepsilon_0$, $Z_\varepsilon(x,y)$ has a unique singular point in V.

(ii) If $p \in II_2, VV_2, VI_1$, there exist a neighborhood V of p in Σ and an $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \leqslant \varepsilon_0$, $Z_\varepsilon(x,y)$ has no singular points in V.

Proof. Without loss of generality we assume that $p = (0,0)$ is a two fold singularity of $Z^{\pm}(x,y)$, we have $Q^+(0,0) = Q^-(0,0) = 0$, $P^+(0,0)Q^+_x(0,0) \neq 0$ and $P^-(0,0)Q^-_x(0,0) \neq 0$.

In the case II_1 we have $P^+(0,0)Q^+_x(0,0) < 0$ and $P^-(0,0)Q^-_x(0,0) > 0$. Without loss of generality we assume that $P^+(0,0) > 0$ and $P^-(0,0) < 0$, see Figure 4.1. Recall that $F(x,y) = P^+(x,y)Q^-(x,y) - P^-(x,y)Q^+(x,y)$, it is obvious that $F(0,0) = 0$, and $F_x(0,0) = P^+(0,0)Q^+_x(0,0) - P^+(0,0)Q^-_x(0,0) > 0$. Then there exists a neighborhood V of $(0,0)$, in which $F(x,y) = 0$ has a unique solution $x = \alpha(y)$, $y \in J_V$, a neighborhood of $y = 0$, i.e. $F(\alpha(y), y) \equiv 0$ in J_V according to the implicit function theorem. Similar treatment as that in the proof of Proposition 15, we can conclude that $Z_\varepsilon(x,y)$ has a unique singular point in V.

In the case VI_2 we have $P^+(0,0)Q^+_x(0,0) < 0$ and $P^-(0,0)Q^-_x(0,0) < 0$. Without loss of generality we assume that $P^+(0,0) > 0$ and $P^-(0,0) < 0$, see Figure 4.5. Since $F_x(0,0) \neq 0$, following the proof of II_1, we ensure that $Z_\varepsilon(x,y)$ has a unique singular point in V.

In the case VV_1 we have $P^+(0,0)Q^+_x(0,0) > 0$ and $P^-(0,0)Q^-_x(0,0) < 0$. We assume that $P^+(0,0) > 0$ and $P^-(0,0) < 0$, see Figure 4.3. Similar to the case II_1, we can prove that $Z_\varepsilon(x,y)$ has a unique singular point in V.

In the case VI_1 we have $P^+(0,0)Q^+_x(0,0) < 0$ and $P^-(0,0)Q^-_x(0,0) < 0$. We assume without loss of generality that $P^+(0,0) > 0$ and $P^-(0,0) > 0$, see Figure 4.5. We claim that $Z_\varepsilon(x,y)$ cannot have singular points in any small neighborhood V of $(0,0)$. Indeed, if $P^-(0,0) > P^+(0,0)$, then from (6) we have that $\phi_\varepsilon(-\varepsilon) = \dfrac{P^-(0,0)}{P^-(0,0) - P^+(0,0)} > 1$, and hence $\dfrac{P^-(x,y)}{P^-(x,y) - P^+(x,y)} > 1$ in a small neighborhood of $(0,0)$ by continuity, which is in contradiction with $\varphi_\varepsilon(y) < 1$. If $P^-(0,0) < P^+(0,0)$, then $\dfrac{P^-(0,0)}{P^-(0,0) - P^+(0,0)} < 0$, and $\dfrac{P^-(x,y)}{P^-(x,y) - P^+(x,y)} < 0$ in a small neighborhood of $(0,0)$ by continuity. Thus $\varphi_\varepsilon(y)$ does not satisfy the requirement that $\varphi_\varepsilon(y) \in [0,1]$. If $P^-(0,0) = P^+(0,0)$, then we can choose a suitable neighborhood V of $(0,0)$, such that $\left|\dfrac{P^-(x,y)}{P^-(x,y) - P^+(x,y)}\right| > 1$, and so

$\varphi_\varepsilon(y) = \dfrac{P^-(x,y)}{P^-(x,y) - P^+(x,y)}$ also does not satisfy the requirement in any small neighborhood of $(0,0)$.

In the case II_2 we have $P^+(0,0)Q_x^+(0,0) < 0$ and $P^-(0,0)Q_x^-(0,0) > 0$. Without loss of generality we assume that $P^+(0,0) > 0$ and $P^-(0,0) > 0$, see Figure 4.4. Similar to the case VI_1, we can conclude that $Z_\varepsilon(x,y)$ has no unique singular points in V.

In the case VV_2 we have $P^+(0,0)Q_x^+(0,0) > 0$ and $P^-(0,0)Q_x^-(0,0) < 0$. Without loss of generality we assume that $P^+(0,0) > 0$ and $P^-(0,0) > 0$, see Figure 4.6. Similar to the case VI_1, we can conclude that $Z_\varepsilon(x,y)$ has no singular points in V.

The proposition follows. □

Step 3. We introduce the stereographic projection, which project the continuous vector field $Z_\varepsilon(x,y) := (F_1(x,y), F_2(x,y))$ defined in (6) in the plane $\mathbb{R}^2$ to the tangent vector field $V_\varepsilon(u,v,w) := (V_1, V_2, V_3)$ on the sphere $\mathbb{S}^2 = \{(u,v,w)|u^2 + v^2 + (w-1/2)^2 = 1/4\}$. More precisely, the stereographic projection $\rho : \mathbb{R}^2 \mapsto \mathbb{S}^2$ is the intersection of the straight line passing through the point $(x,y,0)$ of the plane $\mathbb{R}^2$ and the north pole $(0,0,1)$ with the sphere $\mathbb{S}^2$. Thus the sphere $\mathbb{S}^2$ is tangent to $\mathbb{R}^2$ at the south pole point $(0,0,0)$, see Figure 6. Now we consider the projection of the vector field $Z_\varepsilon(x,y)$ from $\mathbb{R}^2$ to $\mathbb{S}^2$ via stereographic projection which assigns to each point $p := (x,y) \in \mathbb{R}^2$ the point $\bar{p} := (u,v,w) \in \mathbb{S}^2$ through the relations $x = \dfrac{u}{1-w}, y = \dfrac{v}{1-w}$, then we have

$$
\begin{aligned}
&V_1(u,v,w) = (1-w-u^2)F_1\left(\frac{u}{1-w}, \frac{v}{1-w}\right) - uvF_2\left(\frac{u}{1-w}, \frac{v}{1-w}\right),\\
(9)\quad &V_2(u,v,w) = -uvF_1\left(\frac{u}{1-w}, \frac{v}{1-w}\right) + (1-w-v^2)F_2\left(\frac{u}{1-w}, \frac{v}{1-w}\right),\\
&V_3(u,v,w) = u(1-w)F_1\left(\frac{u}{1-w}, \frac{v}{1-w}\right) + v(1-w)F_2\left(\frac{u}{1-w}, \frac{v}{1-w}\right).
\end{aligned}
$$

Note that system (9) is not defined at the north pole point $p = (0,0,1)$, thus we extend it to $\mathbb{S}^2$ by a change of time scale $t = (1-w)^n\tau$, where n is the maximum of the degrees of the polynomials F_1 and F_2, and then we obtain the $\tilde{V}_\varepsilon = (1-w)^m V_\varepsilon$. We shall call the stereographic compactification of $Z_\varepsilon(x,y)$, $S(\tilde{V}_\varepsilon)$ the induced vector field in the sphere $\mathbb{S}^2$. It is obvious that the dynamics of the orbits of $Z_\varepsilon(x,y)$ near infinity are determined by the dynamics of the orbits of $S(\tilde{V}_\varepsilon)$ near $p = (0,0,1)$.

Lemma 18. *If $Z_\varepsilon(x,y)$ has finitely many finite singular points, then $S(\tilde{V}_\varepsilon)$ is a continuous tangent vector field in the sphere $\mathbb{S}^2$ has finitely many singular points.*

Proof. $S(\tilde{V}_\varepsilon)$ is a continuous tangent vector field because $Z_\varepsilon(x,y)$ is continuous. Suppose that (x^*,y^*) is a singular point of $Z_\varepsilon(x,y) = (F_1(x,y), F_1(x,y))$, that is $F_1(x^*,y^*) = F_2(x^*,y^*) = 0$. From (9), we have

$$(u^*, v^*, w^*) = \left(\frac{x^*}{1+(x^*)^2+(y^*)^2}, \frac{y^*}{1+(x^*)^2+(y^*)^2}, \frac{(x^*)^2+(y^*)^2}{1+(x^*)^2+(y^*)^2}\right)$$

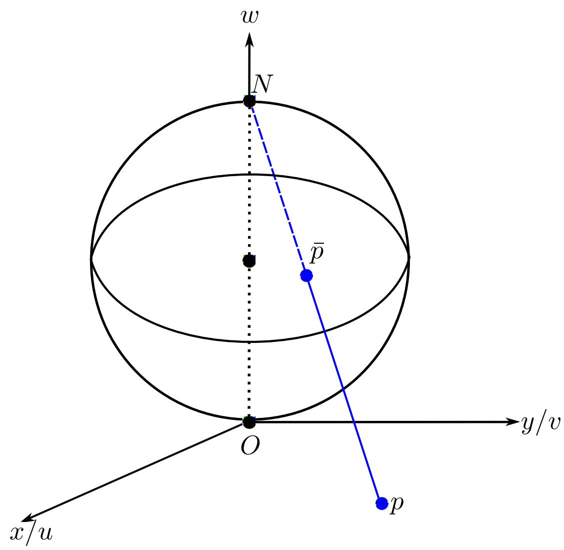


FIGURE 6. Stereographic projection. $O = (0,0,0)$, $N = (0,0,1)$, $p = (x,y,0)$, $\bar{p} = (u,v,w)$,

is a singular point of $S(\tilde{V}_\varepsilon)$. Since $Z_\varepsilon(x,y)$ have finitely many finite singular points, we can conclude that $S(\tilde{V}_\varepsilon)$ have finitely many singular points. $\square$

Step 4. According to Lemma 18, $S(\tilde{V}_\varepsilon)$ is continuous on $\mathbb{S}^2$ with finitely many singular points, the sum of the indices $I_{S(\tilde{V}_\varepsilon)} = 2$ by Poincaré-Hopf Theorem (see for instance Theorem 6.30 of [10]). From Lemma 11, we know that $Z_\varepsilon(x,y)$ is a bounded vector field, and then there are no orbits of $S(\tilde{V}_\varepsilon)$ whose ω-limit is $N = (0,0,1)$. Thus we have both the number of elliptic sectors $e(N) = 0$ and the number of hyperbolic sector $h(N) = 0$.

According to the Poincaré index formula (see for more details Proposition 6.32 of [10]), the index of the north pole point $i_N = 1 + (e(N) - h(N))/2 = 1$. From the above analysis, we can conclude that the sum of the indices at all the finite singular points of $Z_\varepsilon(x,y)$ is 1 because the north pole locally is an unstable node due to the fact that the vector field is bounded.

From (6) the phase portraits of the vector field $Z_\varepsilon(x,y)$ tends in a continuous way to the phase portrait of the vector field $Z^{\pm}(x,y)$. Since there are finitely many finite singular points of $Z_\varepsilon(x,y)$ and of $Z^{\pm}(x,y)$ the sum of their indices are preserved by continuous deformations (see for instance [2]), hence the sum of the indices of the $\mathcal{H}$-singular points of $Z^{\pm}(x,y)$ is also 1.

This completes the proof of Theorem 9.

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