LIMIT CYCLES OF PIECEWISE POLYNOMIAL DIFFERENTIAL SYSTEMS WITH THE DISCONTINUITY LINE $xy = 0$

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Abstract. In this paper we study the maximum number of limit cycles bifurcating from the periodic orbits of the center $\dot{x} = -y((x^2 + y^2)/2)^m, \dot{y} = x((x^2 + y^2)/2)^m$ with $m \geq 0$ under discontinuous piecewise polynomial (resp. polynomial Hamiltonian) perturbations of degree n with four zones separated by the discontinuity set $\{(x, y) \in \mathbb{R}^2 : xy = 0\}$. For such perturbations of degree n, using the averaging theory up to any order we provide upper bounds for the maximum number of the bifurcated limit cycles and show their reachability for orders one and two. Besides, we explore the effect of 4-star-symmetry on the maximum number of limit cycles bifurcating from the unperturbed periodic orbits. In particular, our result implies that 4-star-symmetry almost halves the maximum number. It is worth mentioning that our work gives some new lower bounds for the maximum number of limit cycles of some piecewise polynomial systems.

1. Introduction and statement of main result

Hilbert's 16th problem, an important subject in the qualitative theory of differential systems, asks for the maximum number of limit cycles that planar polynomial differential systems with a fixed degree can have. Since David Hilbert [18] proposed it in 1900, a large number of works were devoted to the study of this problem, see the survey paper [22]. But it is still an open problem up to now, even for quadratic differential systems. As Hilbert's 16th problem turns out to be extremely difficult, some researchers have particularized it to identify the maximum number of limit cycles bifurcating from a periodic annulus, when we perturb it inside the class of all planar polynomial differential systems with a fixed degree $n \geq 1$. In essence, this is the weak Hilbert's 16th problem, see [2, 19, 22]. If this periodic annulus is formed by the linear center $\dot{x} = -y, \dot{y} = x$, Iliev [19] proved that $[3(n - 1)/2]$ is a lower bound for the maximum number, where $[\cdot]$ denotes the integer part function. They also gave that $[N(n - 1)/2]$ is an upper bound for the maximum number using the Melnikov method of order N. However the exact maximum number is unclear so far except for some special perturbations, e.g. Liénard family [16, 26]. In 2010, Buică, Giné and Llibre [6] extended the work of Iliev [19] considering the polynomial perturbations of the center

$$\dot{x} = -y\left(\frac{x^2 + y^2}{2}\right)^m, \qquad \dot{y} = x\left(\frac{x^2 + y^2}{2}\right)^m \tag{1}$$

with $m \geq 0$. Note that $H(x, y) = (x^2 + y^2)/2$ is a first integral of system (1).

Discontinuous events are widespread in the real world, such as stick-slip motion in oscillators with dry friction [14, 23], switching in electronic circuits [4, 5], and impact in mechanical

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