# PHASE PORTRAITS OF SEPARABLE QUADRATIC SYSTEMS AND A BIBLIOGRAPHICAL SURVEY ON QUADRATIC SYSTEMS

TAO LI[1] AND JAUME LLIBRE[2]

ABSTRACT. Although planar quadratic differential systems and their applications have been studied in more than one thousand papers, we still have no complete understanding of these systems. In this paper we have two objectives.

First we provide a brief bibliographical survey on the main results about quadratic systems. Here we do not consider the applications of these systems to many areas as in Physics, Chemist, Economics, Biology, ...

Second we characterize the new class of planar separable quadratic polynomial differential systems. For such class of systems we provide the normal forms which contain one parameter, and using the Poincaré compactification and the blow up technique, we prove that there exist 10 non-equivalent topological phase portraits in the Poincaré disc for the separable quadratic polynomial differential systems.

## 1. Introduction and statement of the main result

Let $P(x,y)$ and $Q(x,y)$ be two real polynomials of degree 2. Then the differential system

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y), \tag{1}$$

is called a *planar quadratic polynomial differential system*, or in what follows simply a *quadratic system*. As usual the dot denotes derivative with respect to an independent variable $t$, called the time.

Quadratic systems began to be studied at the beginning of the twentieth century. According to Coppel [76] it seems that the first work on quadratic systems was written in 1904 by Büchel [41]. In 1966 Coppel [76] published a short survey on quadratic systems, another short survey on these systems appeared in 1982 by Chicone and Tian [63].

Quadratic systems have been intensively studied in the past several decades and a large number of valuable results were obtained, see the books [22, 189, 226] dedicated completely to quadratic systems and references therein, and

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also the references in [188]. But the investigations on quadratic systems are still very far from being completed.

It is well known that one of the main subjects in the dynamics of planar polynomial differential systems is to characterize their global phase portraits in the Poincaré disc, see subsection 2.2 for the definition of the Poincaré compactification and the Poincaré disc. For the quadratic systems a complete characterization of the global phase portraits is an extremely difficult task due to the fact that these systems depend on 12 parameters, see for instance [22]. Instead many particular subfamilies of quadratic systems have been analyzed, for instance:

- structurally stable quadratic systems modulo limit cycles [5, 192],
- structurally unstable quadratic systems of codimension one [12],
- all the configurations of singularities of quadratic systems [13, 16, 17, 18, 19, 20, 21, 22, 25, 27, 32, 33, 36, 68],
- quadratic systems with a center [35, 83, 92, 117, 118, 119, 128, 141, 175, 176, 177, 178, 179, 191, 193, 208, 214, 234],
- quadratic systems with an isochronous center [56, 75, 164],
- Hopf bifurcation in quadratic systems [35],
- quadratic systems with a weak focus of third order [9, 145],
- quadratic systems with a weak focus of second order [14],
- quadratic systems with a weak focus and an invariant straight line [15],
- quadratic systems with weak singularities [215],
- Lotka–Volterra quadratic systems [47, 203, 204, 222, 50, 45],
- Bernoulli quadratic systems [142],
- quadratic systems without finite singularities, also called quadratic foliations or chordal systems [95, 96, 113, 212],
- every quadratic system has finitely many limit cycles [34],
- Abel quadratic systems [155],
- weak Hilbert's 16th problem for quadratic systems: there are hundreds of papers see the references in the book of Christopher and Li [64] and [57, 58, 59, 74, 98, 99, 102, 103, 104, 105, 107, 108, 109, 121, 122, 123, 124, 125, 126, 129, 130, 206, 223, 225, 231],
- limit cycles in quadratic systems [11, 60, 61, 62, 69, 77, 85, 86, 87, 101, 106, 110, 116, 120, 131, 146, 163, 180, 181, 182, 183, 184, 190, 207, 209, 227, 228, 229, 233],
- quadratic-linear systems, i.e. one of the two equations of the system is defined by a polynomial of degree one [46, 162],
- integrability of quadratic–linear systems with [151, 152, 153],
- quadratic systems with a unique finite singularity [73],
- quadratic systems with a focus and one anti-saddle modulo limit cycles [8],
- quadratic systems with all points at infinity as singularities [3, 97, 174, 196, 198, 202],

- quadratic systems with a higher order singularity with two zero eigenvalues [112],
- quadratic systems with a finite and an infinite saddle-node [30, 31],
- quadratic systems with a semi-elemental triple node [29],
- singular points determine quadratic systems [24],
- restricted version of the Hilbert's 16th problem for quadratic [111],
- Darboux integrability for quadratic systems of [44, 47, 48, 55, 66, 78, 133, 140, 165, 171, 194, 195],
- Darboux invariants in quadratic systems [38, 138, 139],
- quadratic systems with a rational first integral [23, 49, 135, 136, 137],
- quadratic systems with a polynomial first integral [26, 51, 93],
- quadratic Hamiltonian systems [6, 7, 115, 175],
- quadratic systems of Darboux type [216],
- bounded quadratic systems [72, 80, 82, 127],
- phase portraits of quadratic systems in applications [186, 187],
- quadratic systems with a polynomial inverse integrating factor [70, 71],
- homogeneous quadratic systems [40, 79, 167, 169, 220, 221],
- quadratic systems with an integrable saddle [28, 39],
- quadratic systems with a weak saddle [43, 114],
- integrability of quadratic systems [159],
- quadratic systems with a symmetric center and simple infinite singular points [166, 217],
- singularly perturbed quadratic systems [147],
- algebraic limit cycles in quadratic systems [1, 52, 54, 67, 88, 89, 90, 91, 132, 143, 144, 148, 149, 154, 156, 157, 185, 224, 230],
- quadratic systems with a unique finite singular point of multiplicity two, possessing two zero eigenvalues [173],
- quadratic systems with a single finite singularity which in addition is simple [213],
- quadratic systems with a finite singular point of multiplicity four [211, 219],
- statistical measure of quadratic systems [10],
- quadratic systems with a singular point of multiplicity three [218],
- quadratic systems with invariant algebraic curves [94, 134, 150, 158, 160, 161],
- quadratic systems with invariant straight lines of total multiplicity greater than or equal to four [197, 199, 200, 201],
- quadratic systems with a semi-elemental triple node [4],
- quadratic systems with two parallel invariant straight lines [42],
- quadratic systems with complex conjugate invariant lines meeting at a finite point [205, 210],
- quadratic systems with invariant algebraic curves of arbitrarily high degree without rational first integrals [53, 65, 170].

A differential equation

$$\frac{dy}{dx} = F(x, y) \tag{2}$$

is said to be *separable* if $F(x, y)$ can be written as the form $F_1(x)F_2(y)$. In parallel we say that the planar differential system (1) is separable if written into the form

$$\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)} = F(x, y),$$

this differential equation is separable. The nonlinear separable differential systems are special because they can be solved by the separation method of variables. Although the solutions of a large number of separable differential systems may be complex and even can provide integrals which cannot be computed. Thus it is also necessary to study the separable differential systems using the qualitative theory of the differential systems.

In this paper we shall study the phase portraits in the Poincaré disc of the following family of separable quadratic polynomial differential systems

$$\dot{x} = a_1x^2 + a_2x + a_3, \qquad \dot{y} = b_1y^2 + b_2y + b_3, \tag{3}$$

where

$$a_1^2 + b_1^2 \neq 0, \qquad a_1^2 + a_2^2 + a_3^2 \neq 0, \qquad b_1^2 + b_2^2 + b_3^2 \neq 0. \tag{4}$$

The next theorem is our main result.

**Theorem 1.** *The global phase portraits of the separable quadratic polynomial differential systems* (3) *satisfying* (4) *are topologically equivalent to one of the* 10 *phase portraits in the Poincaré disc described in Figure* 1.

The paper is organized as follows. In section 2 we recall some basic notions and tools for studying the phase portraits of polynomial differential systems in the Poincaré disc. In particular, subsection 2.1 contains some definitions and results about finite and infinite singular points, and subsection 2.2 reviews the Poincaré compactification. In section 3 we provide the normal forms of systems (3) satisfying (4). In sections 4 and 5 the infinite and finite singular points are studied respectively. Finally, Theorem 1 is proved in section 6.

## 2. Preliminaries

In this section we shortly review some basic notions, results and tools which are involved in the investigation of the phase portraits of planar polynomial differential systems in the Poincaré disc, see [84, 232] for more details.

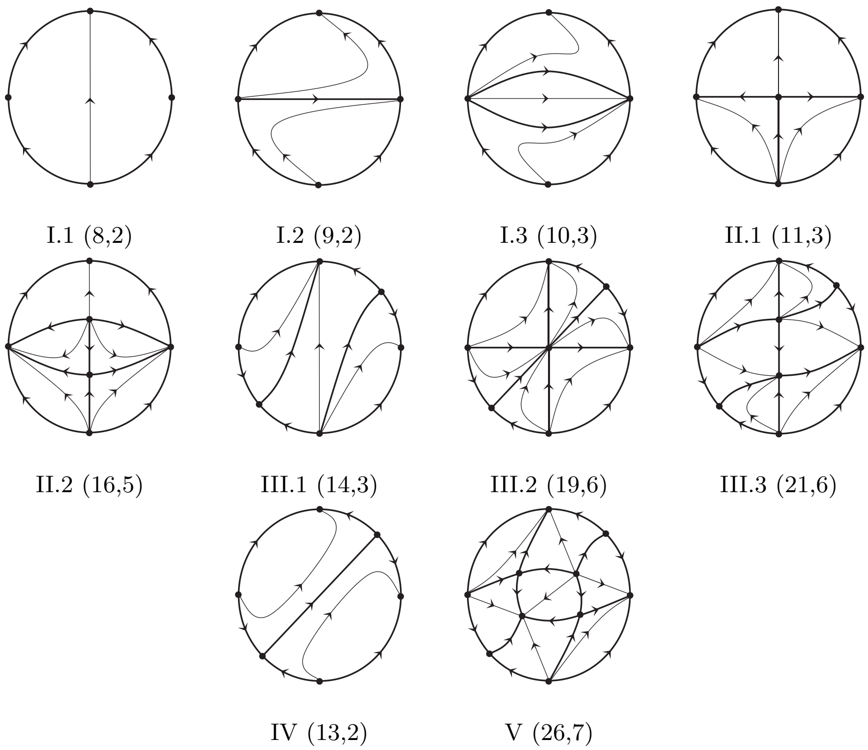



**Figure 1.** Phase portraits in the Poincaré disc of systems (3) satisfying (4). Here the pair (S,R) which appears in each phase portrait in the Poincaré disc denotes the number of separatrices S and the number of canonical regions R of the corresponding phase portrait.

2.1. **Singular points.** Let $Z(x, y) = (X(x, y), Y(x, y))$ be a planar smooth vector field. A point $p \in \mathbb{R}^2$ is said to be a *singular point* of $Z$ if $X(p) = Y(p) = 0$. Let $J$ be the Jacobian matrix of $Z$ at the singular point $p$, i.e.,

$$J = \begin{pmatrix} X_x(p) & X_y(p) \\ Y_x(p) & Y_y(p) \end{pmatrix},$$

where the subscripts $x$ and $y$ denote the partial derivative with respect to $x$ and $y$, respectively. Depending on the character of the matrix $J$, an isolated singular point $p$ is called

(i) a *hyperbolic* singular point if $J$ has two eigenvalues with nonzero real part,
(ii) a *semi-hyperbolic* singular point if $J$ has a unique zero eigenvalue,
(iii) a *nilpotent* singular point if $J$ has two zero eigenvalues and the matrix $J$ is not identically zero,
(iv) a *linearly zero* singular point if the matrix $J$ is identically zero.

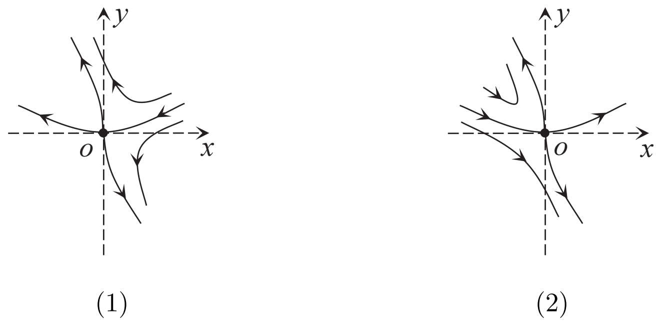



**Figure 2.** Phase portraits of saddle-nodes.

For a hyperbolic, semi-hyperbolic or nilpotent singular point $p$, the local phase portrait of $Z$ at $p$ has been thoroughly studied, see for instance Theorems 2.15, 2.19 and 3.5 of [84], respectively. In particular, a hyperbolic singular point $p$ is a *saddle* (resp. *node*) if $J$ at $p$ has two real eigenvalues with opposite (resp. same) sign, and a *strong focus* if the eigenvalues are complex with nonzero real part. If $J$ at $p$ has a pair purely imaginary eigenvalues, then it may be a *weak focus* or a *center*, see [84, 232]. Since Theorems 2.19 of [84] will be used repeatedly later on, we summarize it in what follows.

**Theorem 2.** *Let the origin $O$ be an isolated singular point of the vector field $Z = (X(x,y), Y(x,y)) = (\widetilde{X}(x,y), \lambda y + \widetilde{Y}(x,y))$, where $\widetilde{X}$ and $\widetilde{Y}$ are analytic in a neighborhood of the origin with $\widetilde{X}(0,0) = \widetilde{Y}(0,0) = D\widetilde{X}(0,0) = D\widetilde{Y}(0,0) = 0$ and $\lambda > 0$. Let $y = f(x)$ be the solution of the equation $\lambda y + \widetilde{Y}(x,y) = 0$ in a neighborhood of $O$, and suppose that the function $g(x) = \widetilde{X}(x, f(x))$ has the expression $g(x) = a_m x^m + o(x^m)$, where $m \geq 2$ and $a_m \neq 0$.*

- (a) *If $m$ is odd and $a_m < 0$, then $O$ is a topological saddle. Moreover, the two stable separatrices are tangent to the $x$-axis at $O$.*
- (b) *If $m$ is odd and $a_m > 0$, then $O$ is an unstable topological node.*
- (c) *If $m$ is even, then $O$ is a saddle-node, i.e., a point whose neighborhood is separated into two parts by one unstable separatrix that is tangent to the positive $y$-axis at $O$ and one unstable separatrix that is tangent to the negative $y$-axis at $O$. When $a_m < 0$ (resp. $a_m > 0$), the left (resp. right) part is a parabolic sector with unstable invariant manifolds of $O$ and the right (resp. left) part is two hyperbolic sectors separated by one stable separatrix that is tangent to the positive (resp. negative) $x$-axis at $O$, see Figure 2(1) (resp. (2)).*

**Remark 3.** The case $\lambda < 0$ can be reduced to $\lambda > 0$ reversing the time. Moreover the case $Z = (\lambda x + \widehat{X}, \widehat{Y})$ can be reduced to $Z = (\widetilde{X}, \lambda y + \widetilde{Y})$ by the change of variables $(x,y) \to (y,x)$. As we will see the two changes will be repeatedly used to study the local phase portrait at a semi-hyperbolic singular point.

The local phase portrait of a linearly zero singular point $p$ can be determined by using the *blow up technique*, see [2, 81, 84] for more details.

2.2. **Poincaré compactification.** It is well known that an important tool for studying the phase portraits of a quadratic polynomial vector field $Z = (X, Y)$ is the *Poincaré compactification.* For completeness we shortly review it in this subsection, for more details see Chapter 5 of [84], or [232].

Consider the 2-dimensional sphere $\mathbb{S}^2 := \{(s_1, s_2, s_3) \in \mathbb{R}^3 : s_1^2 + s_2^2 + s_3^2 = 1\}$, called the *Poincaré sphere*, and its tangent plane at the point $(0, 0, 1)$, this plane is identified with the plane $\mathbb{R}^2$ where the vector field $Z$ is defined. According with [84] the vector field $Z$ in $\mathbb{R}^2$ can be extended analytically to a vector field $p(Z)$ on $\mathbb{S}^2$ by using the central projection $f$ which maps each point $Q$ in $\mathbb{R}^2$ onto two points on $\mathbb{S}^2$ using the straight line through $Q$ and the origin $(0, 0, 0)$. Notice that we obtain two copies of $Z$ on $\mathbb{S}^2$ by the central projection $f$, one in the northern hemisphere and the other in the southern hemisphere. So we have a vector field $Z^*$ defined on $\mathbb{S}^2 \setminus \mathbb{S}^1$, where $\mathbb{S}^1 := \{(s_1, s_2, s_3) \in \mathbb{S}^2 : s_3 = 0\}$ is the equator of $\mathbb{S}^2$. Doing the rescaling $s_3^2 Z^*$ we extend the quadratic polynomial vector field $Z^*$ to the whole sphere, and as usually we denote the vector field $y_3^3 Z^*$ by $p(Z)$, called the *Poincaré compactification* of the vector field $Z$. The dynamics of $p(Z)$ near the equator $\mathbb{S}^1$ corresponds to the dynamics of $Z$ in a neighborhood of the infinity of $\mathbb{R}^2$.

It is sufficient to consider the Poincaré compactification restricted to the northern hemisphere $\{(s_1, s_2, s_3) \in \mathbb{S}^2 : s_3 > 0\}$ union the equator $\mathbb{S}^1$ in order to study the phase portrait of the vector field $Z$. Moreover the phase portrait is drawn on the so-called *Poincaré disc* $\mathbb{D}^2 := \{(s_1, s_2) \in \mathbb{R}^2 : s_1^2 + s_2^2 \leq 1\}$, obtained projecting the northern hemisphere union the equator onto $\mathbb{D}^2$ using the projection $\pi(s_1, s_2, s_3) = (s_1, s_2)$.

For working with $p(Z)$ on $\mathbb{S}^2$ we need the local charts $(U_i, \phi_i)$ and $(V_i, \psi_i)$ for $i = 1, 2, 3$, where

$$U_i := \{(s_1, s_2, s_3) \in \mathbb{S}^2 : s_i > 0\}, \qquad V_i := \{(s_1, s_2, s_3) \in \mathbb{S}^2 : s_i < 0\},$$

$\phi_i : U_i \to \mathbb{R}^2$ and $\psi_i : V_i \to \mathbb{R}^2$ are defined by

$$\phi_i(s_1, s_2, s_3) = -\psi_i(s_1, s_2, s_3) = (s_m/s_i, s_n/s_i) = (u, v),$$

where $m < n$ and $m, n \neq i$. Then the expression of $p(Z)$ in the local chart $U_1$ is given by

$$\dot{u} = z^2 \left( -uX\left(\frac{1}{z}, \frac{u}{z}\right) + Y\left(\frac{1}{z}, \frac{u}{z}\right) \right), \qquad \dot{z} = -z^3 X\left(\frac{1}{z}, \frac{u}{z}\right). \tag{5}$$

In the local chart $U_2$ the vector field $p(Z)$ is given by

$$\dot{v} = z^2\left(X\left(\frac{v}{z}, \frac{1}{z}\right) - vY\left(\frac{v}{z}, \frac{1}{z}\right)\right), \qquad \dot{z} = -z^3 Y\left(\frac{v}{z}, \frac{1}{z}\right). \tag{6}$$

The compactified vector field $p(Z)$ in the local chart $U_3$ is just $Z$. The expression of $p(Z)$ in $V_i$ is the expression in $U_i$ multiplied by $-1$ for $i = 1, 2, 3$.

The singular points of $p(Z)$ in $\mathbb{S}^2 \setminus \mathbb{S}^1$ correspond to the finite singular points of $Z$, and they can be obtained by using the local chart $U_3$. The infinite singular points of $Z$ are the singular points of systems (5) and (6) lying in $\mathbb{S}^1$, we remark that their coordinate $z = 0$. It is worth mentioning that we only need to look at $U_1|_{z=0}$ and $U_2|_{(0,0)}$ to study the infinite singular points of $Z$.

Two Poincaré compactified vector fields on $\mathbb{D}^2$ are *topologically equivalent* if there is a homeomorphism from one vector field to the other sending orbits to orbits, and preserving or reversing the direction of all the orbits.

According to [168, 172] a *separatrix* of $\pi(p(Z))$ on $\mathbb{D}^2$ is one of following orbits: finite singular points, limit cycles, all orbits at the infinity, and the two boundaries of a hyperbolic sector at a finite or infinite singular point. We denote the set formed by all separatrices by $\Sigma$. Then $\Sigma$ is a closed set as it is proved in [172]. An open connected component of $\mathbb{D}^2 \setminus \Sigma$ is called a *canonical region* of $\pi(p(Z))$. Then the *separatrix configuration* $\Sigma^*$ of $\pi(p(Z))$ is the union of the set $\Sigma$ with an orbit for each canonical region. Two separatrix configurations $\Sigma_1^*$ and $\Sigma_2^*$ are *topologically equivalent* if there exists a homeomorphism from $\Sigma_1^*$ to $\Sigma_2^*$ sending the orbits of one separatrix configuration to the orbits of the other, and preserving or reversing the direction of all the orbits.

Finally we recall the Neumann's Theorem [172], which characterizes the topological equivalence of two global phase portraits in the Poincaré disc $\mathbb{D}^2$.

**Theorem 4.** *Two Poincaré compactified polynomial vector fields $\pi(p(Z_1))$ and $\pi(p(Z_2))$ with finitely many separatrices are topologically equivalent if and only if their separatrix configurations are topological equivalent.*

We shall use this theorem for proving Theorem 1.

## 3. Normal forms

In this section we study the normal forms of system (3) satisfying (4). The next lemma is the main result of this section.

**Lemma 5.** *System* (3) *satisfying* (4)*, after a linear change of variables and a scaling of its independent variable $t$, can be written in one of the following normal forms:*

(I) $\dot{x} = 1, \quad \dot{y} = y^2 + \xi;$
(II) $\dot{x} = x, \quad \dot{y} = y^2 + \xi;$
(III) $\dot{x} = x^2, \quad \dot{y} = y^2 + \xi;$
(IV) $\dot{x} = x^2 + 1, \quad \dot{y} = y^2 + \xi;$
(V) $\dot{x} = x^2 - 1, \quad \dot{y} = y^2 + \xi.$

*Proof.* Since $a_1^2 + b_1^2 \neq 0$ by condition (4), we only need to consider the case of $b_1 \neq 0$ by the change of variables $(x, y) \to (y, x)$.

If $a_1 = a_2 = 0$, we have $a_3 \neq 0$ from (4), and then the change of variables

$$(x, y) \to (a_3 x, y/b_1 - b_2/(2b_1))$$

transforms system (3) into system (I) with $\xi = b_1 b_3 - b_2^2/4$.

If $a_1 = 0$ and $a_2 \neq 0$, then the change

$$(x, y, t) \to (x - a_3/a_2, a_2 y/b_1 - b_2/(2b_1), t/a_2)$$

transforms system (3) into system (II) with $\xi = b_1 b_3/a_2^2 - b_2^2/(4a_2^2)$.

If $a_1 \neq 0$ doing the change

$$(x, y, t) \to ((\Delta x - a_2)/(2a_1), \Delta y/(2b_1) - b_2/(2b_1), 2t/\Delta),$$

where

$$\Delta := \begin{cases} 1 & \text{if } a_2^2 - 4a_1 a_3 = 0, \\ \sqrt{|a_2^2 - 4a_1 a_3|} & \text{if } a_2^2 - 4a_1 a_3 \neq 0, \end{cases}$$

system (3) becomes system (III) (resp. (IV), (V)) when $a_2^2 - 4a_1 a_3 = 0$ (resp. $< 0$, $> 0$). Here $\xi = 4b_1 b_3/\Delta^2 - b_2^2/\Delta^2$. □

From Lemma 5 we only need to study the phase portraits of systems (I)–(V) in order to obtain all phase portraits of system (3) satisfying (4) in the Poincaré disc.

## 4. Infinite singular points

In this section we study the infinite singular points of systems (I)–(V).

**Lemma 6.** *In the local chart $U_1$ system* (I) *has a unique infinite singular point, namely the origin, which is linearly zero having the local phase portrait presented in*

(a) *Figure* 3(2) *if $\xi > 0$,*
(b) *Figure* 3(3) *if $\xi = 0$,*
(c) *Figure* 4(2) *if $\xi < 0$.*

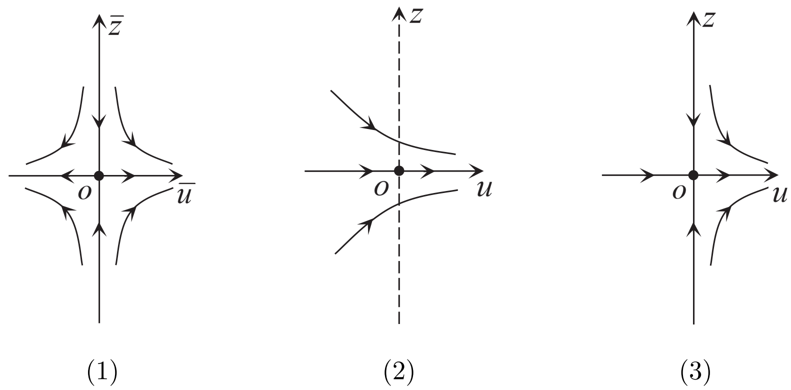



**Figure 3.** Blow up of system (7) with $\xi \geq 0$ at $O$.

*Proof.* In the local chart $U_1$ system (I) writes

$$\dot{u} = u^2 + \xi z^2 - uz^2, \qquad \dot{z} = -z^3. \tag{7}$$

When $z = 0$ the origin $O$ is the unique singular point of system (7) and it is linearly zero.

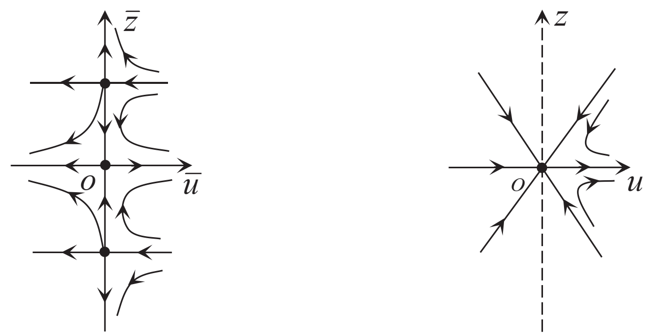
To determine the local phase portrait of system (7) at $O$ we use the blow up technique as follows. Since the characteristic polynomial of the linear part of system (7) at $O$ is $\mathcal{F}(u, z) = z(u^2 + \xi z^2)$, $z = 0$ is a characteristic direction. Thus, using the the $u$-directional blow up $(u, z) = (\bar{u}, \bar{u}\bar{z})$ we change system (7) into

$$\dot{\bar{u}} = \bar{u}(1 + \xi\bar{z}^2 - \bar{u}\bar{z}^2), \qquad \dot{\bar{z}} = -\bar{z}(1 + \xi\bar{z}^2), \tag{8}$$

after cancelling the common factor $\bar{u}$ doing the change of time $t \to t/\bar{u}$. For $\bar{u} = 0$ the origin $O$ is always a singular point of system (8) and it is a hyperbolic saddle. Since $\bar{u} = 0$ and $\bar{z} = 0$ are two invariant straight lines of (8), the two stable (resp. unstable) invariant manifolds of the saddle are contained in the $\bar{z}$-axis (resp. $\bar{u}$-axis), see Figure 3(1).

If $\xi \geq 0$, then $O$ is the unique singular point of system (8) when $\bar{u} = 0$. Moreover $z = 0$ is the unique characteristic direction of system (7) at $O$ if $\xi > 0$, while if $\xi = 0$, system (7) has exactly two characteristic directions at $O$, $z = 0$ and $u = 0$. Note that these characteristic directions are invariant straight lines of system (7). Hence going back through the used blow up in the last paragraph and using the direction of vector field on these invariant straight lines, we get the local phase portrait of system (7) at $O$ as shown in Figure 3(2) if $\xi > 0$, and in Figure 3(3) if $\xi = 0$, i.e. statements (a) and (b) hold.

If $\xi < 0$ then system (8) for $\bar{z} = 0$ has two additional singular points $(0, p^{\pm})$ with $p^{\pm} = \pm 1/\sqrt{-\xi}$. They are semi-hyperbolic because the two eigenvalues at $(0, p^{\pm})$ are 0 and 2. We next apply Theorem 2 to determine the local phase portraits at these two points. Translating $(0, p^{+})$ to the




(1) (2)

**Figure 4.** Blow up of system (7) with $\xi < 0$ at $O$.

origin we see that the quantities $\lambda, m, a_m$ in Theorem 2 associated to system (8) are $\lambda = 2, m = 2, a_m = 1/\xi < 0$, respectively. Hence a straightway application of Theorem 2(c) yields that $(0, p^+)$ is a saddle-node with two unstable separatrices in the $\bar{z}$-axis and one stable separatrix in $\{(\bar{u}, \bar{z}) : \bar{u} > 0, \bar{z} = p^+\}$, combining that $\bar{u} = 0$ and $\bar{z} = p^+$ are two invariant straight lines of system (8). Similarly, translating $(0, p^-)$ to the origin we get $\lambda = 2, m = 2, a_m = 1/\xi < 0$, and thus $(0, p^-)$ is a saddle-node with two unstable separatrices in the $\bar{z}$-axis and one stable separatrix in $\{(\bar{u}, \bar{z}) : \bar{u} > 0, \bar{z} = p^-\}$ by Theorem 2(c) again. This concludes the local phase portraits of system (8) at $(0, p^{\pm})$, see Figure 4(1) where the local phase portrait at $O$ is also drawn. Consequently, the local phase portrait of system (7) at $O$ is as shown in Figure 4(2), going back to system (7) through the blow up $(u, z) = (\bar{u}, \bar{u}\bar{z})$. This completes the proof of statement (c). □

**Lemma 7.** *In the local chart* $U_1$ *system* (II) *has a unique infinite singular point, namely the origin, which is linearly zero having the local phase portrait presented in*

(a) *Figure* 5(2) *when* $\xi > 0$,
(b) *Figure* 5(3) *when* $\xi = 0$,
(c) *Figure* 6(2) *when* $\xi < 0$.

*Proof.* In the local chart $U_1$ system (II) becomes

$$\dot{u} = u^2 - uz + \xi z^2, \qquad \dot{z} = -z^2. \tag{9}$$

For $z = 0$ system (9) has a unique singular point, namely the origin $O$. Clearly, it is linearly zero.

Next we use the blow up technique to study the local phase portrait of system (9) at $O$. Since the characteristic polynomial of the linear part of

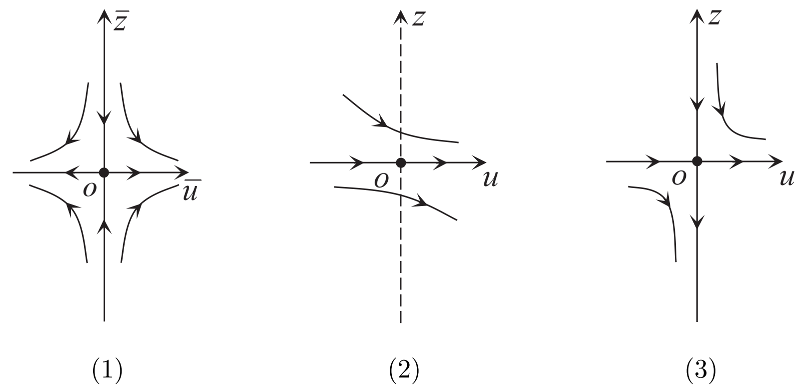



**Figure 5.** Blow up of system (9) with $\xi \geq 0$ at $O$.

system (9) at the origin $O$ is $\mathcal{F}(u, z) = z(u^2 + \xi z^2)$, $z = 0$ is a characteristic direction. Thus using the blow up technique with the change $(u, z) = (\bar{u}, \bar{u}\bar{z})$ we obtain

$$\dot{\bar{u}} = \bar{u}(1 - \bar{z} + \xi\bar{z}^2), \qquad \dot{\bar{z}} = -\bar{z}(1 + \xi\bar{z}^2), \tag{10}$$

after cancelling the common factor $\bar{u}$ doing the change $t \to t/\bar{u}$. For $\bar{u} = 0$ the origin $O$ is always a singular point of system (10), particularly it is a hyperbolic saddle with two unstable (resp. stable) manifolds contained in the $\bar{u}$-axis (resp. $\bar{z}$-axis), see Figure 5(1).

If $\xi \geq 0$, then $O$ is the unique singular point of system (10) for $\bar{u} = 0$. Moreover $z = 0$ is the unique characteristic direction of system (9) at $O$ if $\xi > 0$, while if $\xi = 0$, system (9) has exactly two characteristic directions at $O$, $z = 0$ and $u = 0$. Since these characteristic directions are invariant straight lines of system (9), going back to system (9) through the last blow up and using the direction of vector fields on these lines, we get the local phase portrait of system (9) at $O$ as it is shown in Figure 5(2) if $\xi > 0$, and in Figure 5(3) if $\xi = 0$, i.e. statements (a) and (b) hold.

If $\xi < 0$, system (10) for $\bar{u} = 0$ has two additional singular points, $(0, p^{\pm})$ with $p^{\pm} = \pm 1/\sqrt{-\xi}$ where the eigenvalues are $-p^{\pm}$ and 2. Thus $(0, p^{-})$ is an unstable hyperbolic node if $p^{-} < 0$, and $(0, p^{+})$ is a hyperbolic saddle if $p^{+} > 0$. This concludes the local phase portraits of system (10) at $O$ and $(0, p^{\pm})$ as shown in Figure 6(1). Going back to system (9) we get the local phase portrait of system (9) at $O$ as it is shown in Figure 6(2), consequently statement (c) holds. $\square$

**Lemma 8.** *In the local chart $U_1$ each one of the systems* (III), (IV) *and* (V) *has two infinite singular points: the origin and* $(1, 0)$. *Moreover the origin is a stable hyperbolic node, and* $(1, 0)$ *is a hyperbolic saddle with two unstable separatrices contained in the $u$-axis.*

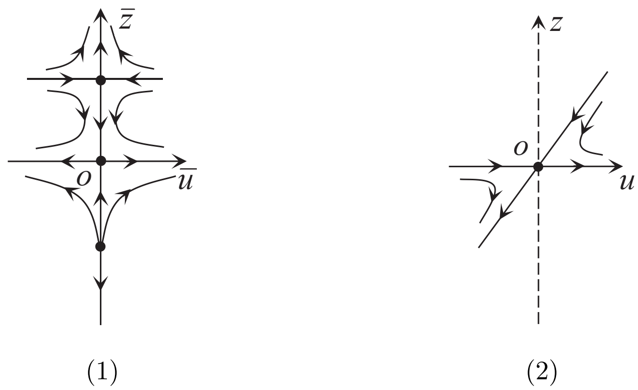



**Figure 6.** Blow up of system (9) with $\xi < 0$ at the origin.

*Proof.* In the local chart $U_1$ systems (III), (IV) and (V) are

$$\begin{aligned} \dot{u} &= -u + u^2 + \xi z^2, & \dot{z} &= -z, \\ \dot{u} &= -u + u^2 + \xi z^2 - uz^2, & \dot{z} &= -z - z^3, \\ \dot{u} &= -u + u^2 + \xi z^2 + uz^2, & \dot{z} &= -z + z^3, \end{aligned} \tag{11}$$

respectively. Thus each system in (11) has two singular points $O$ and $(1,0)$ for $z = 0$. For each system the eigenvalues at $O$ (resp. $(1,0)$) are always $-1$ (resp. 1) and $-1$. Thus both $O$ and $(1,0)$ are hyperbolic, particularly $O$ is a stable node and $(1,0)$ is a saddle. Since $z = 0$ is an invariant straight line of each system in (11), the two unstable manifolds of $(1,0)$ are contained in the $u$-axis. This ends the proof of Lemma 8. □

The last four lemmas studied the infinite singular points of systems (I)-(V) in the local chart $U_1$. The next lemma verifies whether the origin is an infinite singular point of these systems in the local chart $U_2$.

**Lemma 9.** *In the local chart $U_2$ the origin is an infinite singular point for each one of systems* (I)-(V). *Moreover it is a stable hyperbolic node.*

*Proof.* We observe that systems (I)-(V) can be written in a unified form as

$$\dot{x} = \kappa_1 x^2 + \kappa_2 x + \kappa_3, \qquad \dot{y} = y^2 + \xi, \tag{12}$$

where $\kappa_i \in \{-1, 0, 1\}$, $i = 1, 2, 3$. In the local chart $U_2$ system (12) writes

$$\begin{aligned} \dot{v} &= -v + \kappa_1 v^2 + \kappa_2 vz + \kappa_3 z^2 - \xi v z^2, \\ \dot{z} &= -z - \xi z^3, \end{aligned}$$

which obviously has the origin as a singular point. Since all the two eigenvalues at the origin are $-1$, it is a stable hyperbolic node. □

## 5. Finite singular points

This section is devoted to study the finite singular points of systems (I)-(V).

**Lemma 10.** *Systems* (I) *and* (IV) *have no finite singular points.*

*Proof.* Lemma 10 follows from $\dot{x} > 0$ systems (I) and (IV). □

**Lemma 11.** *On the finite singular points of system* (II) *we have the following statements.*

- (a) *If $\xi > 0$, there is no finite singular points.*
- (b) *If $\xi = 0$, there is a unique finite singular point, the origin. Moreover it is a semi-hyperbolic saddle-node with two unstable separatrices contained in the $x$-axis and one stable separatrix contained in the negative $y$-axis.*
- (c) *If $\xi < 0$, there are two finite singular points, $(0, \pm\sqrt{-\xi})$. Moreover $(0, \sqrt{-\xi})$ is an unstable hyperbolic node and $(0, -\sqrt{-\xi})$ is a hyperbolic saddle with two stable separatrices contained in the $y$-axis.*

*Proof.* The number of finite singular points of system (II) equals the number of roots of the equation $y^2 + \xi = 0$. Thus statement (a) holds directly if $\xi > 0$.

If $\xi = 0$, then $O$ is the unique finite singular point, where the eigenvalues are 1 and 0, so that it is semi-hyperbolic. To determine the local phase portrait we use the change $(x, y) \to (y, x)$ to transform system (II) into the system of Theorem 2, i.e. the system

$$\dot{x} = x^2, \qquad \dot{y} = y. \tag{13}$$

A direct application of Theorem 2(c) concludes that $O$ is a saddle-node of system (13) with two unstable separatrices contained in the $y$-axis and one stable separatrix contained in the negative $x$-axis. Here the $x$-axis and $y$-axis are invariant straight lines of (13). Therefore statement (b) holds going back to system (II).

If $\xi < 0$, system (II) has two finite singular points $(0, \pm\sqrt{-\xi})$, where the eigenvalues are 1 and $\pm 2\sqrt{-\xi}$. This yields statement (c) joining the fact that $x = 0$ is an invariable straight line. □

**Lemma 12.** *On the finite singular points of system* (III) *we have the following statements.*

- (a) *If $\xi > 0$, there is no finite singular points.*
- (b) *If $\xi = 0$, there is a unique finite singular point, the origin. Moreover, it is linearly zero with the local phase portrait presented in Figure* 7(2).

(c) *If $\xi < 0$, there are two finite singular points, $(0, \pm\sqrt{-\xi})$. Moreover $(0, \sqrt{-\xi})$ is a semi-hyperbolic saddle-node with two unstable separatrices contained in the $y$-axis and one stable separatrix contained in $\{(x, y) : x < 0, y = \sqrt{-\xi}\}$, $(0, -\sqrt{-\xi})$ is a semi-hyperbolic saddle-node with two stable separatrices contained in the $y$-axis and one unstable separatrix contained in $\{(x, y) : x > 0, y = -\sqrt{-\xi}\}$.*

*Proof.* As in system (II) the number of finite singular points of system (III) is equal to the number of roots of the equation $y^2 + \xi = 0$. Thus statement (a) holds directly if $\xi > 0$.

If $\xi = 0$, then $O$ is the unique finite singular point of system (III) and it is linearly zero. Next we use the blow up technique to study the local phase portrait at $O$. Since the characteristic polynomial of system (III) is $\mathcal{F}(x, y) = xy(x - y)$, $y = 0$ is a characteristic direction. Thus doing the $x$-directional blow up $(x, y) = (\bar{x}, \bar{x}\bar{y})$ we get the system

$$\dot{\bar{x}} = \bar{x}, \qquad \dot{\bar{y}} = -\bar{y} + \bar{y}^2, \tag{14}$$

after cancelling the common factor $\bar{x}$ using the change $t \to t/\bar{x}$. For $\bar{x} = 0$ system (14) has two singular points, $(0, 1)$ and $O$. Since the eigenvalues at $(0, 1)$ (resp. $O$) are 1 and 1 (resp. $-1$), $(0, 1)$ is an unstable hyperbolic node and $O$ is a hyperbolic saddle with two stable (resp. unstable) separatrices contained in the $\bar{y}$-axis (resp. $\bar{x}$-axis), see Figure 7(1). Finally we get the local phase portrait of system (III) at $O$ as shown in Figure 7(2), going back to system (III) through the used blow up. This yields statement (b).

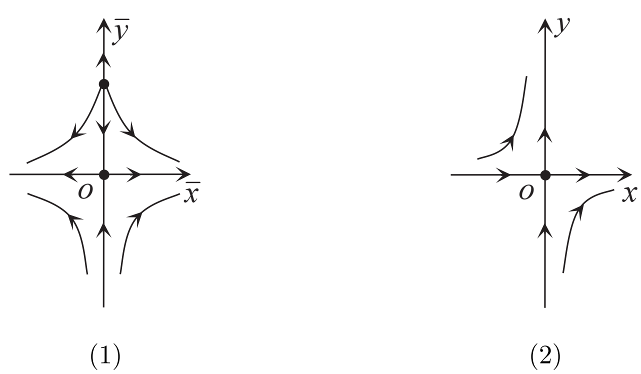



**Figure 7.** Blow up of system (III) with $\xi = 0$ at $O$.

If $\xi < 0$, system (III) has two exactly singular points, $(0, \pm\sqrt{-\xi})$, where the eigenvalues are 0 and $\pm 2\sqrt{-\xi}$. So they are semi-hyperbolic. To determine the local phase portrait at $(0, \sqrt{-\xi})$ we use the change $(x, y) \to (x, y + \sqrt{-\xi})$ to transform system (III) into the normal form of system of Theorem 2, i.e.

$$\dot{x} = x^2, \qquad \dot{y} = 2\sqrt{-\xi}y + y^2. \tag{15}$$

Then the quantities $\lambda, m, a_m$ in Theorem 2 associated to system (15) are $\lambda = 2\sqrt{-\xi} > 0, m = 2, a_m = 1$. Thus together with the invariant lines $x = 0$ and $y = 0$, $O$ is a saddle-node of system (15) with two unstable separatrices contained in the $y$-axis and one stable separatrix contained in the negative $x$-axis. Coming back to system (III) we finally get that $(0, \sqrt{-\xi})$ is a saddle-node with two unstable separatrices contained in the $y$-axis and one stable separatrix contained in $\{(x, y) : x < 0, y = \sqrt{-\xi}\}$. Regarding the local phase portrait at $(0, -\sqrt{-\xi})$, using the change $(x, y, t) \to (x, y - \sqrt{-\xi}, -t)$ we can carry out a similar analysis, and thus $(0, -\sqrt{-\xi})$ is a saddle-node with two stable separatrices contained in the $y$-axis and one unstable separatrix contained in $\{(x, y) : x > 0, y = -\sqrt{-\xi}\}$. That is, statement (c) holds. $\square$

**Lemma 13.** *For system* (V) *all finite singular points must lie in the invariant straight lines $x = 1$ and $x = -1$.*

- (a) *In the line $x = 1$ system* (V) *has*
  - (a1) *no singular points if $\xi > 0$;*
  - (a2) *a unique singular point $(1, 0)$ if $\xi = 0$, and it is a semi-hyperbolic saddle-node with two unstable separatrices contained in the $x$-axis and one stable separatrix contained in $\{(x, y) : x = 1, y < 0\}$;*
  - (a3) *two singular points $(1, \pm\sqrt{-\xi})$ if $\xi < 0$, and $(1, \sqrt{-\xi})$ is an unstable hyperbolic node and $(1, -\sqrt{-\xi})$ is a hyperbolic saddle with two stable separatrices contained in the line $x = 1$.*
- (b) *In the line $x = -1$ system* (V) *has*
  - (b1) *no singular points if $\xi > 0$;*
  - (b2) *a unique singular point $(-1, 0)$ if $\xi = 0$, and it is a semi-hyperbolic saddle-node with two stable separatrices contained in the $x$-axis and one unstable separatrix contained in $\{(x, y) : x = -1, y > 0\}$;*
  - (b3) *two singular points $(-1, \pm\sqrt{-\xi})$ if $\xi < 0$, and $(-1, -\sqrt{-\xi})$ is a stable hyperbolic node and $(-1, \sqrt{-\xi})$ is a hyperbolic saddle with two unstable separatrices contained in the line $x = -1$.*

*Proof.* We neglect the proof of this lemma because it is completely similar to the proof of Lemma 11 by using Theorem 2 and the fact that $x = \pm 1$ are invariant straight lines. $\square$

## 6. Proof of Theorem 1

To prove Theorem 1 we can equivalently study all global phase portraits for each one of the systems (I)-(V) by Lemma 5. We see that all finite singular points of each one of systems (I)-(VI) must lie on invariant straight lines. This means that there are no periodic orbits, and consequently no limit cycles. Combining the investigation of infinite singular points in section 4 with the one of finite singular points in section 5, we conclude that all separatrices can be determined in a natural and unique way except for the ones in system (IV) with $\xi > 0$, and particularly the following statements hold.

(i) The global phase portrait of system **(I)** is topologically equivalent to the one of
  (i.a) Figure 1(I.1) if $\xi > 0$ from Lemmas 6(a), 9 and 10;
  (i.b) Figure 1(I.2) if $\xi = 0$ from Lemmas 6(b), 9 and 10;
  (i.c) Figure 1(I.3) if $\xi < 0$ from Lemmas 6(c), 9 and 10.

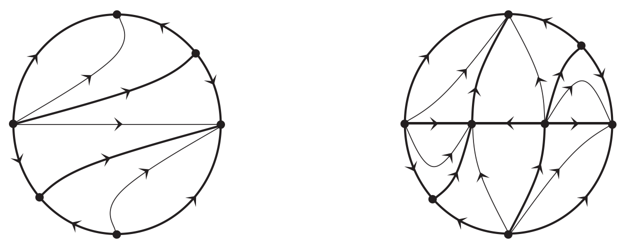
(ii) The global phase portrait of system **(II)** is topologically equivalent to the one of
  (ii.a) Figure 1(I.1) if $\xi > 0$ from Lemmas 7(a), 9 and 11(a);
  (ii.b) Figure 1(II.1) if $\xi = 0$ from Lemmas 7(b), 9 and 11(b);
  (ii.c) Figure 1(II.2) if $\xi < 0$ from Lemmas 7(c), 9 and 11(c).

(iii) The global phase portrait of system **(III)** is topologically equivalent to the one of
  (iii.a) Figure 1(III.1) if $\xi > 0$ from Lemmas 8, 9 and 12(a);
  (iii.b) Figure 1(III.2) if $\xi = 0$ from Lemmas 8, 9 and 12(b);
  (iii.c) Figure 1(III.3) if $\xi < 0$ from Lemmas 8, 9 and 12(c).

(iv) Since $y = x$ is a solution of system (IV) with $\xi = 1$, it follows from Lemmas 8, 9 and 10 that the global phase portrait of system **(IV)** is topologically equivalent to the one of
  (iv.a) Figure 1(III.1) if $\xi > 1$;
  (iv.b) Figure 1(IV) if $\xi = 1$;
  (iv.c) Figure 8(a) if $\xi < 1$.

(v) The global phase portrait of system **(V)** is topologically equivalent to the one of
  (v.a) Figure 1(III.1) if $\xi > 0$ from Lemmas 8, 9 and 13(a1)(b1);
  (v.b) Figure 8(b) if $\xi = 0$ from Lemmas 8, 9 and 13(a2)(b2);
  (v.c) Figure 1(V) if $\xi < 0$ from Lemmas 8, 9 and 13(a3)(b3).

From Theorem 4 we easily observe that the global phase portrait of Figure 8(a) is topologically equivalent to the one of Figure 1(III.1), and the global phase portraits of Figure 8(b) is topologically equivalent to the one of Figure 1(III.3). Consequently, according to the items (i), $\cdots$, (v), and Theorem 4, we finally obtain the 10 non-topologically equivalent global phase portraits in Figure 1. This ends the proof of Theorem 1.



(a) (S,R)=(14,3) (b) (S,R)=(21,6)

**Figure 8.** Two global phase portraits.

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[1] Department of Mathematics, Sichuan University, 610064 Chengdu, Sichuan, P.R. China

*Email address*: t.l.scumath@gmail.com

[2] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

*Email address*: jllibre@mat.uab.cat