THE PLANAR DISCONTINUOUS PIECEWISE LINEAR REFRACTING SYSTEMS HAVE AT MOST ONE LIMIT CYCLE

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Abstract. In this paper we investigate the limit cycles of planar piecewise linear differential systems with two zones separated by a straight line. It is well known that when these systems are continuous they can exhibit at most one limit cycle, while when they are discontinuous the maximum number of limit cycles that they can exhibit is still open. For these last systems there are examples exhibiting three limit cycles.

The aim of this paper is to study the number of limit cycles for a special kind of planar discontinuous piecewise linear differential systems with two zones separated by a straight line which are known as refracting systems. First we obtain the existence and uniqueness of limit cycles for refracting systems of focus-node type. Second we prove that refracting systems of focus-focus type have at most one limit cycle, thus we give a positive answer to a conjecture on the uniqueness of limit cycle stated by Freire, Ponce and Torres in [10]. These two results complete the proof that any refracting system has at most one limit cycle.

1. Introduction

In the qualitative theory of the differential systems in the plane one of the most important problems is the determination and distribution of limit cycles, which is known as the famous Hilbert's 16-th problem [18, 28] and its weak form [4, 5, 12, 29].

Since many real world differential systems involve a discontinuity or a sudden change [2], in recent years there is a growing interest in the following planar piecewise smooth vector fields

$$\mathcal{X}(q) = \begin{cases} X^-(q) & \text{if } h(q) < 0, \\ X^+(q) & \text{if } h(q) > 0, \end{cases} \tag{1}$$

where the discontinuity boundary $\Sigma = \{q \in \mathbb{R}^2 : h(q) = 0\}$ divides the plane $\mathbb{R}^2$ into two regions $\Sigma^\pm = \{q \in \mathbb{R}^2 : \pm h(q) > 0\}$. The singularities $p^\pm$ of $X^\pm$ are called *visible* or *invisible* if $p^\pm \in \Sigma^\pm$ or $p^\pm \in \Sigma^\mp$, respectively.

Clearly the orbits are well defined in both zones $\Sigma^\pm$. While if an orbit arrives to the discontinuous boundary Σ, different things can occur.

Definition 1. *Let $X^\pm h(q) = \langle \nabla h(q), X^\pm(q)\rangle$. Then we can classify Σ into the following three open regions:*

(i) *crossing region $\Sigma^c = \{q \in \Sigma : X^+h(q)X^-h(q) > 0\}$, see Fig.1.1.*

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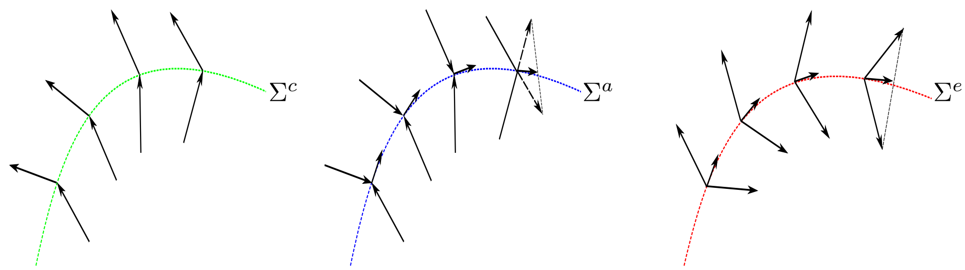


1.1 crossing region 1.2 attracting region 1.3 escaping region

FIGURE 1. Definition of the vector field on Σ.

(ii) *attracting region* $\Sigma^a = \{q \in \Sigma : X^+h(q) > 0, X^-h(q) < 0\}$, *see Fig.1.2;*
(iii) *escaping region* $\Sigma^e = \{q \in \Sigma : X^+h(q) < 0, X^-h(q) > 0\}$, *see Fig.1.3.*

The boundaries Σ^t of the above three regions are called $\Sigma-$*tangential point*, that is $\Sigma^t = \{q \in \Sigma : X^+h(q)X^-h(q) = 0\}$. If an isolated periodic orbit of systems (1) has sliding points, then it is called a *sliding limit cycle*, otherwise we call it a *crossing limit cycle*.

The most simplest piecewise smooth differential systems are the piecewise linear differential systems with a straight line of separation. Without loss of generality we can assume that the separating straight line is $x = 0$, then we have

$$
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} a_{1,1}^- & a_{1,2}^- \\ a_{2,1}^- & a_{2,2}^- \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1^- \\ b_2^- \end{pmatrix} & \text{if } x < 0, \\ \begin{pmatrix} a_{1,1}^+ & a_{1,2}^+ \\ a_{2,1}^+ & a_{2,2}^+ \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1^+ \\ b_2^+ \end{pmatrix} & \text{if } x > 0, \end{cases} \tag{2}
$$

where the dot denotes the derivative with respected to the tiem t. We call systems (2) with $x < 0$ (resp. $x > 0$) the left (resp. right) subsystems for convenience.

In 2012 Freire, Ponce and Torres [9] reduced the study of the planar piecewise linear differential systems (2) to the following Liénard canonical forms

$$
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} T^- & -1 \\ D^- & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ a^- \end{pmatrix} & \text{if } x < 0, \\ \begin{pmatrix} T^+ & -1 \\ D^+ & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} -b \\ a^+ \end{pmatrix} & \text{if } x > 0, \end{cases} \tag{3}
$$

where $T^\pm$ and $D^\pm$ denote the traces and determinants of the left and right subsystems, respectively.

If $b = 0, a^- = a^+$, then systems (3) become continuous differential systems. In 1990 Lum and Chua [25] did the following conjecture:

Conjecture 1. *Planar continuous piecewise linear differential systems* (3) *have at most one limit cycle.*

In 1998 Freire et al. [8] proved the conjecture 1 by qualitative analysis. Recently, Li and Llibre [19] provided the global phase portraits in the Poincaré disc of the planar continuous piecewise linear differential systems (3).

TABLE 1. The known results on the lower bounds for the maximum number of limit cycles of the discontinuous systems (3), where F, S, N denote focus/center, saddle and node respectively.

	F	S	N
F	3	3	3
S		2	2
N			2

For the discontinuous systems (3) most of the known results [11, 14, 15, 16, 22, 23, 30, 31] are concerned with the lower bounds of the number of limit cycles. According to the singularities of left and right subsystems (3), we can classify systems (3) into six types, see Table 1.

From Table 1 appeared the following conjecture:

Conjecture 2. *Planar discontinuous piecewise linear differential systems (3) have at most three crossing limit cycles.*

As far as we known conjecture 2 is still open and there were only several partial results for this conjecture. Llibre, Novaes and Teixeira [20, 21] proved that discontinuous systems (3) have at most two crossing limit cycles when $a^+a^- = 0$. Giannakopoulos and Pliete [13] showed that discontinuous systems (3) with a Z_2 symmetry have at most two crossing limit cycles. In [24] it is proved that if one of the subsystems (3) has a center then the maximum number of crossing limit cycles is two, and that this upper bound is sharped.

Definition 2. *If $X^+h(q) = X^-h(q)$ for all $q \in \Sigma$, then systems (1) are known as refracting systems.*

It is obvious that the whole discontinous line Σ of a refracting system is a crossing region. There are several papers classifying the generic singularities of the refracting systems, for dimension two see [7]; for dimension three see [3]; for dimension four see [17] and for arbitrary dimension see [1].

If $b = 0$ then systems (3) become planar discontinuous piecewise linear refracting systems and have been studied in several papers [15, 16, 27, 30, 31]. All the previous results shown that the planar discontinuous piecewise linear refracting systems (3) of types SS, NN, FS, SN have at most one limit cycle, see Table 2. More precisely we have

- SS see Theorems 3.4 and 3.5 of [15].
- NN see Theorem 3.1 of [16].
- FS see Theorem 1 of [27], or Theorem 3.1 of [31].
- SN see Theorem 3.1 of [30].

The dynamics of the planar discontinuous piecewise linear differential systems (3) are determined by

$$\Delta^{\pm} = (T^{\pm})^2 - 4D^{\pm}.$$

Table 2. The known results on the upper bounds for the maximum number of limit cycles of the refracting systems (3) before this paper.

	F	S	N
F	?	1	?
S		1	1
N			1

We define the modal parameters

$$m_{\{R,L\}} = \begin{cases} i & \text{if } \Delta^{\pm} < 0, \\ 0 & \text{if } \Delta^{\pm} = 0, \\ 1 & \text{if } \Delta^{\pm} > 0, \end{cases}$$

where $i^2 = -1$. Then the planar discontinuous piecewise linear refracting systems $(3)|_{b=0}$ into the following normal forms

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} 2\gamma_L & -1 \\ \gamma_L^2 - m_L^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ \alpha_L \end{pmatrix} & \text{if } x < 0, \\ \begin{pmatrix} 2\gamma_R & -1 \\ \gamma_R^2 - m_R^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ \alpha_R \end{pmatrix} & \text{if } x > 0, \end{cases} \tag{4}$$

where

$$\alpha_{\{R,L\}} = \begin{cases} \dfrac{2a^{\pm}}{\sqrt{|\Delta^{\pm}|}} & \text{if } \Delta^{\pm} \neq 0, \\ 2a^{\pm} & \text{if } \Delta^{\pm} = 0, \end{cases}$$

and

$$\gamma_{\{R,L\}} = \begin{cases} \dfrac{T^{\pm}}{\sqrt{|\Delta^{\pm}|}} & \text{if } \Delta^{\pm} \neq 0, \\ T^{\pm} & \text{if } \Delta^{\pm} = 0. \end{cases}$$

For a proof of these normal forms see [11].

Remark 1. *Systems (4) for $m = i$ have a focus; for $m = 1$ and $|\gamma| \geqslant 1$ have a node; for $m = 1$ and $|\gamma| < 1$ have a saddle; and For $m = 0$ have an improper node.*

2. Statements of the main results

It follows from Table 2 that the upper bounds for the maximum number of limit cycles of the planar discontinuous piecewise linear refracting systems (3) of type focus-node or focus-focus are still unknown. In the present paper we investigated the number of limit cycles for the above two remain unsolved types. We shall use the normal forms (4) instead of (3) because the former one have only four parameters. Without loss of generality we can assume that the left subsystem of (4) has a focus.

First we consider the planar discontinuous piecewise linear refracting systems (4) of type focus-node. Therefore, according wiht Remark 1, we have $m_L = i$, $m_R = 1$

and $|\gamma_R| > 1$, then systems (4) become

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} 2\gamma_L & -1 \\ \gamma_L^2+1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ \alpha_L \end{pmatrix} & \text{if } x < 0, \\ \begin{pmatrix} 2\gamma_R & -1 \\ \gamma_R^2-1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ \alpha_R \end{pmatrix} & \text{if } x > 0. \end{cases} \tag{5}$$

Theorem 1. *Planar discontinuous piecewise linear refracting systems of type focus-node* (5) *with* $|\gamma_R| > 1$ *have at most one limit cycle. Furthermore these systems have a unique limit cycle if and only if* $\gamma_R\gamma_L < 0$ *and* $\alpha_R < 0$, *which is stable if* $\gamma_L > 1$, *and unstable if* $\gamma_L < -1$.

Second we investigate planar discontinuous piecewise linear refracting systems (4) of type focus-improper node, i.e. we assume that $m_L = i$ and $m_R = 0$, then systems (4) become

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} 2\gamma_L & -1 \\ \gamma_L^2+1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ \alpha_L \end{pmatrix} & \text{if } x < 0, \\ \begin{pmatrix} 2\gamma_R & -1 \\ \gamma_R^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ \alpha_R \end{pmatrix} & \text{if } x > 0. \end{cases} \tag{6}$$

Theorem 2. *Planar discontinuous piecewise linear refracting systems of type focus-improper node* (6) *have at most one limit cycle. Furthermore these systems have a unique limit cycle if and only if* $\gamma_R\gamma_L < 0$ *and* $\alpha_R < 0$, *which is stable if* $\gamma_L > 0$, *and unstable if* $\gamma_L < 0$.

Finally we study the limit cycles of the planar discontinuous piecewise linear refracting systems (4) of type focus-focus. Suppose that $m_R = i$ and $m_L = i$, then systems (4) become

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} 2\gamma_L & -1 \\ \gamma_L^2+1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ \alpha_L \end{pmatrix} & \text{if } x < 0, \\ \begin{pmatrix} 2\gamma_R & -1 \\ \gamma_R^2+1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ \alpha_R \end{pmatrix} & \text{if } x > 0. \end{cases} \tag{7}$$

We note that systems (7) have been studied in [9, 10]. The authors showed that systems (7) have at most one limit cycle when $\alpha_R \leqslant 0 \leqslant \alpha_L$ or $\alpha_L\alpha_R > 0$. While for the remain case $\alpha_L < 0 < \alpha_R$ they stated the following two conjectures based on extensive numerical simulations.

Conjecture 3. *Assuming* $\alpha_L < 0 < \alpha_R$ *and* $\gamma_L\gamma_R < 0$ *in systems* (7), *then the following statements hold.*

(a) *If* $\gamma_L < 0$ *and* $(\gamma_L + \gamma_R)(\hat{y} - z^*) < 0$, *then systems* (7) *have no crossing limit cycles, where* $\hat{y}$ *and* z^* *are defined in* (17).
(b) *If* $\gamma_L > 0$ *and* $(\gamma_L + \gamma_R)(\hat{y} - z^*) > 0$, *then systems* (7) *have no crossing limit cycles.*

Conjecture 4. *Assuming* $\alpha_L < 0 < \alpha_R$ *and* $\gamma_L\gamma_R < 0$ *in systems* (7), *then these systems have at most one crossing limit cycle.*

The third main result of this paper provides a positive answer to conjectures 3 and 4.

TABLE 3. The known results on the upper bounds for the maximum number of limit cycles of refracting systems (3) from this paper.

	F	S	N
F	1	1	1
S		1	1
N			1

Theorem 3. *Planar discontinuous piecewise linear refracting systems of type focus-focus* (7) *have at most one limit cycle. Furthermore these systems have a unique limit cycle if and only if* $\gamma_R\gamma_L < 0$ *and one of the following three conditions hold.*

(I) $\alpha_R \leqslant 0 \leqslant \alpha_L$ *and* $(\gamma_L+\gamma_R)(\alpha_L\gamma_R-\alpha_R\gamma_L) < 0$*, which is stable if* $\gamma_L+\gamma_R < 0$*, and unstable if* $\gamma_L+\gamma_R > 0$.
(II) $\alpha_L < 0, \alpha_R < 0$ *and* $\gamma_L(\gamma_L+\gamma_R) < 0$*, which is stable if* $\gamma_L > 0$*, and unstable if* $\gamma_L < 0$.
(III) $\alpha_L < 0 < \alpha_R$ *and* $\gamma_L(\gamma_R+\gamma_L)(\hat{y}-z^*) < 0$*, which is stable if* $\gamma_L > 0$*, and unstable if* $\gamma_L < 0$.

In summary from Table 1 and Theorems 1, 2 and 3 we have proved the following.

Corollary 4. *Planar discontinuous piecewise linear refracting systems* (3) *have at most one limit cycle, see Table 3.*

The rest of the paper is organized as follows. In section 3 we construct the Poincaré map of the refracting systems (4) which is crucial for analyzing the number of limit cycles. After we prove Theorems 1, 2 and 3 in sections 4, 5 and 6, respectively.

3. Preliminary results

Note that if a refracting system (4) has a limit cycle, then it must intersect the discontinuity straight line $x = 0$, because both subsystems of (4) are linear differential systems. According with Proposition 3.7 of [9], and recalling that $b = 0$ and $\gamma_L \neq 0$, we obtain the following necessary conditions for the existence of limit cycles:

$$\gamma_L\gamma_R < 0.$$

Without loss of generality we can assume that $\gamma_L > 0, \gamma_R < 0$, otherwise doing the change of variables $X = x, Y = -y, T = -t$, we change $\gamma_L < 0, \gamma_R > 0$ into the former one.

In order to study the crossing limit cycles of the planar discontinuous piecewise linear refracting systems (4), we need to analyze their Poincaré maps as follows.

First we define the left Poincaré map of systems (4). From the left subsystems of (4) we have $\dot{x}|_{x=0} = -y$, then the orbit of systems (4) starting at the point $(0, y)$ with $y > 0$ will go into the left zone under the flow of the left subsystem, and after this orbit reaches $x = 0$ again at some point $(0, P_L(y))$ with $P_L(y) < 0$, see Fig. 2.

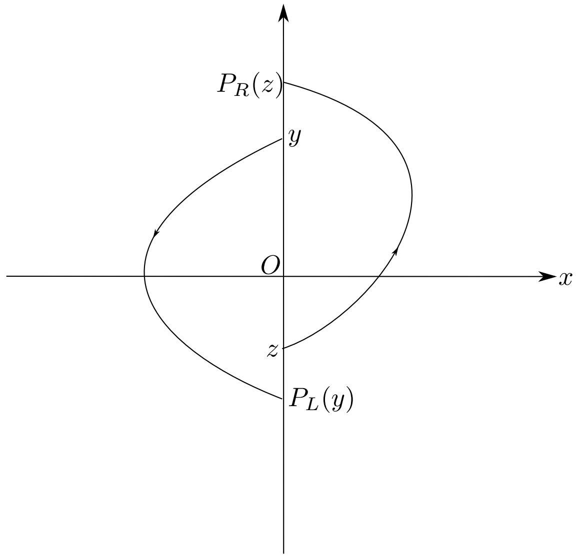


FIGURE 2. The left and right Poincaré map of a refracting system (4).

Now we define the right Poincaré map of systems (4). From the right subsystems of (4) we know that $\dot{x}|_{x=0} = -y$. The orbit of systems (4) starting from points $(0, z)$ with $z < 0$ goes into the right zone under the action of the flow of the right linear subsystems of (4), and after this orbit reaches $x = 0$ again at some point $(0, P_R(z))$ with $P_R(z) > 0$, see again Fig. 2.

Clearly the crossing limit cycles of planar discontinuous piecewise linear refracting systems (4) are in correspondence with the zeros of the Poincaré map

$$P_L(y) - P_R^{-1}(y) \quad \text{with} \quad y \in (0, +\infty), \tag{8}$$

or equivalently

$$P_L^{-1}(z) - P_R(z) \quad \text{with} \quad z \in (-\infty, 0). \tag{9}$$

We recall the following results on the existence and uniqueness of limit cycles for planar discontinuous piecewise linear differential systems without sliding regions proved in [26].

Consider the following piecewise linear differential systems

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{cases} \begin{pmatrix} \mu_1^- & \mu_2^- \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} \mu_0^- \\ 0 \end{pmatrix} & \text{if } Y < 0, \\ \begin{pmatrix} \mu_1^+ & \mu_2^+ \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} \mu_0^+ \\ 0 \end{pmatrix} & \text{if } Y > 0. \end{cases} \tag{10}$$

Definition 3. *We say that a point $p \in \Sigma = \{Y = 0\}$ is a $\Sigma-$monodromic singularity of systems* (10) *if either p is a tangential point, or a singularity of one of the subsystems of* (10)*, and there exists a neighborhood of p such that the orbits of systems* (10) *turn around p either in forward or in backward time.*

Theorem 5. [26] *Assume that systems* (10) *have a* $\Sigma-$*monodromic singularity. Then systems* (10) *have no limit cycles when* $\mu_1^+\mu_1^- \geqslant 0$, *and have at most one limit cycle when* $\mu_1^+\mu_1^- < 0$. *Moreover there is a choice of the parameters for which the limit cycle exists.*

For studying planar discontinuous piecewise linear refracting systems having a focus we shall consider the auxiliary function

$$\varphi_\gamma(t) = 1 - \mathrm{e}^{\gamma t}(\cos t - \gamma \sin t),$$

introduced in [9].

Proposition 6. *The function* $\varphi_\gamma(t)$ *has the following properties.*

(I) $\varphi_\gamma^{'}(t) < 0$ *if* $t \in (\pi, 2\pi)$.
(II) *If* $\gamma < 0$, *then* $\varphi_\gamma(t) > 0$.
(III) *If* $\gamma > 0$, *then there is a unique* $\hat{t} \in (\pi, 2\pi)$ *such that* $\varphi_\gamma(\hat{t}) = 0$, $\varphi_\gamma(t) > 0$ *for* $t \in (\pi, \hat{t})$ *and* $\varphi_\gamma(t) < 0$ *for* $t \in (\hat{t}, 2\pi)$.

Proof. Since $\varphi_\gamma'(t) = (1+\gamma^2)\mathrm{e}^{\gamma t}\sin t$, the function $\varphi_\gamma(t)$ is decreasing for $t \in (\pi, 2\pi)$.

Notice that $\varphi_\gamma(\pi) = 1 + \mathrm{e}^{\gamma\pi}$ and $\varphi_\gamma(2\pi) = 1 - \mathrm{e}^{2\gamma\pi}$. Then, if $\gamma < 0$ we have $\varphi_\gamma(t) > 0$ on $(\pi, 2\pi)$, and if $\gamma > 0$ there exists a $\hat{t} \in (\pi, 2\pi)$ so that $\varphi_\gamma(t) > 0$ on $(\pi, \hat{t})$, and $\varphi_\gamma(t) < 0$ on $(\hat{t}, 2\pi)$. □

4. Proof of Theorem 1

It is obvious that if the right subsystems of (5) have a visible node, then refracting systems (5) cannot have limit cycles. Thus a necessary condition for the existence of limit cycles of systems (5) is that $\alpha_R < 0$.

We divide the proof of Theorem 1 into two cases.

*Case*1: $\alpha_L \geqslant 0$. Then the left subsystems of (5) have an invisible focus when $\alpha_L > 0$, and an equilibrium on Σ when $\alpha_L = 0$. Doing the change of variables

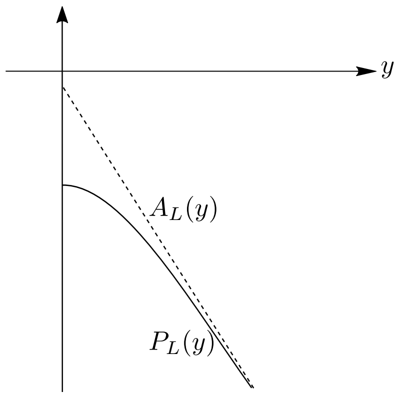
$$X = 2\gamma_L x - y, \quad Y = x, \quad \text{if } x < 0, \text{ or}$$

$$X = 2\gamma_R x - y, \quad Y = x, \quad \text{if } x > 0,$$

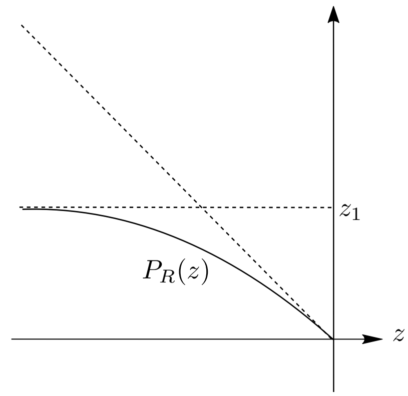
then the refracting systems (5) become

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{cases} \begin{pmatrix} 2\gamma_L & -(\gamma_L^2+1) \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} \alpha_L \\ 0 \end{pmatrix} & \text{if } Y < 0, \\ \begin{pmatrix} 2\gamma_R & -(\gamma_R^2-1) \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} \alpha_R \\ 0 \end{pmatrix} & \text{if } Y > 0. \end{cases} \tag{11}$$

It is easy to check that $(0,0)$ is the unique $\Sigma-$monodromic singularity of systems (11). According with Theorem 5, systems (11) have no limit cycles when $\gamma_R\gamma_L \geqslant 0$, and systems (11) have a unique limit cycle when $\gamma_R\gamma_L < 0$. The stability of the limit cycle follows using the Poincaré-Bendixson Theorem, see for instance Corollary 1.20 of [6].



3.1 left Poincaré map



3.2 right Poincaré map

FIGURE 3. Graphics of the left and right Poincaré map of a refracting system (5).

Case 2: $\alpha_L < 0$. Now we cannot use Theorem 5 to prove the uniqueness of limit cycles, because $(0,0)$ is not a $\Sigma-$monodromic singularity of systems (5).

The left Poincaré maps of a system (5) can be stated as follows, for a proof see Proposition 6 of [11].

Lemma 7. *The parameter representation of the left Poincaré map $P_L(y)$ of a refracting system* (5) *is*

$$y = \frac{\alpha_L}{(1+\gamma_L^2)} \frac{\varphi_{\gamma_L}(t)}{\mathrm{e}^{\gamma_L t} \sin t},$$
$$P_L(y) = -\frac{\alpha_L}{(1+\gamma_L^2)} \frac{\varphi_{-\gamma_L}(t)}{\mathrm{e}^{-\gamma_L t} \sin t},$$

where $\pi < t < \hat{t}$, see Fig. 3.1. *Moreover we have*

(i) $\lim_{y\to 0^+} P_L^{'}(y) = 0$; $\lim_{y\to+\infty} P_L^{'}(y) = -\mathrm{e}^{\pi\gamma_L}$.
(ii) $P_L^{'}(y) < 0$; $P_L^{''}(y) < 0$.
(iii) $P_L(y)$ *has* $A_L(y) = -\mathrm{e}^{\pi\gamma_L} y + \dfrac{2\alpha_L\gamma_L}{1+\gamma_L^2}(1+\mathrm{e}^{\pi\gamma_L})$ *as an asymptote.*

The right Poincaré maps of a system (5) can be stated as follows, for a proof see Proposition 7 of [11].

Lemma 8. *The parameter representation of the right Poincaré map $P_R(z)$ of a refracting system (5) is*

$$z = \alpha_R \frac{\mathrm{e}^{-\gamma_R t} - \cosh t + \gamma_R \sinh t}{(\gamma_R^2 - 1)\sinh t},$$
$$P_R(z) = -\alpha_R \frac{\mathrm{e}^{\gamma_R t} - \cosh t - \gamma_R \sinh t}{(\gamma_R^2 - 1)\sinh t},$$

where $t \geqslant 0$, see Fig. 3.2. Moreover we have:

(i) $\lim_{z\to-\infty} P_R(z) = z_1 = \dfrac{\alpha_R}{\gamma_R - 1}$; $\lim_{z\to 0^-} P_R^{'}(z) = -1$; $\lim_{z\to-\infty} P_R^{'}(z) = 0$.

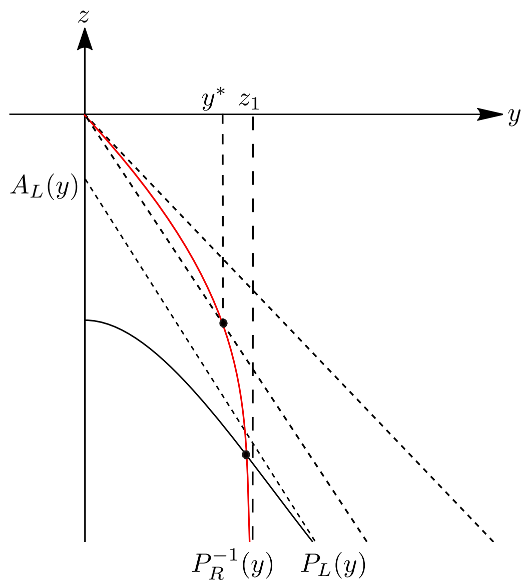


FIGURE 4. The intersection points of the graphs $P_L(y)$ and $P_R^{-1}(y)$.

(ii) $P_R^{'}(z) < 0$, $P_R^{''}(z) < 0$.
(iii) $P_R(z)$ *has* $z = z_1$ *as an asymptote.*

From (8) we know that the number of limit cycles of a refracting system (5) is in correspondence with the number of positive zeros of $P_L(y) - P_R^{-1}(y)$.

Let $(y^*, -\mathrm{e}^{\pi\gamma_L} y^*)$ be the intersection point of the graphs of $-\mathrm{e}^{\pi\gamma_L} y$ and $P_R^{-1}(y)$, see Figure 4. It is obvious that at such a point we have

$$(P_R^{-1})'(y^*) < -\mathrm{e}^{\pi\gamma_L}. \tag{12}$$

We assume that the graphs of $P_L(y)$ and $P_R^{-1}(y)$ have two intersection points, which are $y = y_1$ and $y = y_2$, where

$$y^* < y_1 < y_2 < z_1.$$

Then the following conditions hold:

$$P_L(y_1) = P_R^{-1}(y_1), \quad P_L(y_2) = P_R^{-1}(y_2).$$

By the Rolle's theorem there exists an intermediate point $\bar{y}$, such that

$$P_L'(\bar{y}) = (P_R^{-1})'(\bar{y}).$$

We claim that the above equality is impossible. On one hand, from Lemma 7 we have

$$-\mathrm{e}^{\pi\gamma_L} < P_L^{'}(y) < 0, \quad y > 0.$$

On the other hand, from Lemma 8 and (12) we know

$$(P_R^{-1})^{'}(y) < (P_R^{-1})^{'}(y^*) < -\mathrm{e}^{\pi\gamma_L}, y > y^*.$$

Thus we get the required contradiction, and Theorem 1 is proved.

5. Proof of Theorem 2

For a refracting system (6) the left Poincaré map is also given by Lemma 7, and the right Poincaré map can be stated as follows, see Proposition 7 of [11].

Lemma 9. *The parameter representation of the right Poincaré map $P_R(z)$ of a refracting system (6) is*

$$\begin{aligned} z &= \alpha_R \frac{e^{-\gamma_R t} - 1 + \gamma_R t}{\gamma_R^2 t}, \\ P_R(z) &= -\alpha_R \frac{e^{\gamma_R t} - 1 - \gamma_R t}{\gamma_R^2 t}, \end{aligned} \tag{13}$$

where $t \geqslant 0$. Moreover,

(i) $\lim\limits_{z\to-\infty} P_R(z) = z_2 = \dfrac{\alpha_R}{\gamma_R}$; $\lim\limits_{z\to 0^-} P_R'(z) = -1$; $\lim\limits_{z\to-\infty} P_R'(z) = 0$.
(ii) $P_R'(z) < 0$; $P_R''(z) < 0$.
(iii) *$P_R(z)$ has $z = z_2$ as an asymptote.*

Proof. (i) From (13) we have

$$\lim_{z\to-\infty} P_R(z) = \lim_{t\to+\infty} -\alpha_R \frac{e^{\gamma_R t} - 1 - \gamma_R t}{\gamma_R^2 t} = \frac{\alpha_R}{\gamma_R}.$$

A direct computation shows that

$$P_R'(z) = -e^{2\gamma_R t} \frac{e^{-\gamma_R t} - 1 + \gamma_R t}{e^{\gamma_R t} - 1 - \gamma_R t}. \tag{14}$$

Thus

$$\begin{aligned} \lim_{z\to 0^-} P_R'(z) &= \lim_{t\to 0^+} -e^{2\gamma_R t} \frac{e^{-\gamma_R t} - 1 + \gamma_R t}{e^{\gamma_R t} - 1 - \gamma_R t} = -1, \\ \lim_{z\to-\infty} P_R'(z) &= \lim_{t\to+\infty} -e^{2\gamma_R t} \frac{e^{-\gamma_R t} - 1 + \gamma_R t}{e^{\gamma_R t} - 1 - \gamma_R t} = 0. \end{aligned}$$

(ii) Substituting (13) into (14) we obtain

$$\begin{aligned} P_R'(z) &= e^{2\gamma_R t} \frac{z}{P_R(z)}, \\ P_R''(z) &= -e^{3\gamma_R t} \gamma_R^2 t^2 \frac{\gamma_R t - \sinh(\gamma_R t)}{P_R(z)}. \end{aligned}$$

Recall that $z \leqslant 0, P_R(z) > 0, \gamma_R < 0$ and $t \geqslant 0$, so we get (ii). □

The proof of Theorem 2 is similar to the proof of Theorem 1 and we omit it here.

6. Proof of Theorem 3

From (9) we know that the number of crossing limit cycles of refracting systems (7) are in correspondence with the negative zeros of the function $P_L^{-1}(z) - P_R(z)$.

We define the function

$$f(\gamma, t) = e^{-\gamma t}\varphi_\gamma(t) = e^{-\gamma t} - \cos t + \gamma \sin t. \tag{15}$$

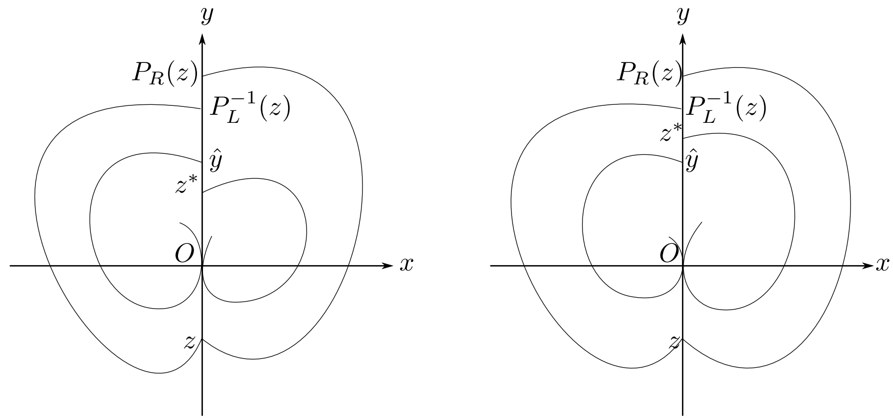


Fig.5.1 $\hat{y} > z^*$ Fig.5.2 $\hat{y} < z^*$

FIGURE 5. Poincaré maps of refracting systems (7) with $\alpha_L < 0 < \alpha_R$.

According with Proposition 6, If $\gamma > 0$ there exists a unique $\hat{t} \in (\pi, 2\pi)$ such that $f(\gamma, \hat{t}) = 0$. It is easy to check that for $t \in (\pi, \hat{t})$,

$$f(\gamma, t) > 0, \quad f(-\gamma, t) > 0,$$

or equivalently

$$e^{-\gamma t} > \cos t - \gamma \sin t, \quad e^{\gamma t} > \cos t + \gamma \sin t. \tag{16}$$

Lemma 10. [11] *The parametric representation of the inverse of left Poincaré map $P_L^{-1}(z)$ of systems (7) is*

$$z = -\frac{\alpha_L}{1+\gamma_L^2} \frac{f(-\gamma_L, t_L)}{\sin t_L},$$
$$P_L^{-1}(z) = \frac{\alpha_L}{1+\gamma_L^2} \frac{f(\gamma_L, t_L)}{\sin t_L},$$

where $t_L \in (\pi, \hat{t}_L)$ such that $z(\hat{t}_L) = 0$.

Lemma 11. [11] *The parametric representation of the right Poincaré map $P_R(z)$ of systems (7) is*

$$z = \frac{\alpha_R}{1+\gamma_R^2} \frac{f(-\gamma_R, t_R)}{\sin t_R},$$
$$P_R(z) = -\frac{\alpha_R}{1+\gamma_R^2} \frac{f(\gamma_R, t_R)}{\sin t_R},$$

where $t_R \in (\pi, \hat{t}_R)$ such that $z(\hat{t}_R) = 0$.

From Fig. 5 we have

$$\begin{aligned} \hat{y} &= P_L^{-1}(0) = \alpha_L e^{-\gamma_L \hat{t}_L} \sin \hat{t}_L, \\ z^* &= P_R(0) = -\alpha_R e^{\gamma_R \hat{t}_R} \sin \hat{t}_R. \end{aligned} \tag{17}$$

Lemma 12. *For $\gamma > 0$ we consider the function*

$$F(\gamma, t) = \frac{f(\gamma, t)}{f(-\gamma, t)},$$

where $f(\gamma,t)$ is given in (15)*. Then $F_\gamma^{'}(\gamma,t)<0$ and $F_t^{'}(\gamma,t)>0$ on $(\pi,\hat{t})$.*

Proof. Since $t\in(\pi,2\pi)$ we have that

$$f_\gamma^{'}(\gamma,t)=-t\mathrm{e}^{-\gamma t}+\sin t<0,\quad f_\gamma^{'}(-\gamma,t)=t\mathrm{e}^{\gamma t}-\sin t>0.$$

Consequently we get

$$F_\gamma^{'}(\gamma,t)=\frac{f_\gamma^{'}(\gamma,t)f(-\gamma,t)-f(\gamma,t)f_\gamma^{'}(-\gamma,t)}{f^2(-\gamma,t)}<0,$$

because $f(\gamma,t)>0$ and $f(-\gamma,t)>0$.

From (16) we can deduce that

$$\begin{aligned}F_t^{'}(\gamma,t) &= \frac{\mathrm{e}^{\gamma t}\left(-(\gamma^2-1)\sin t+2\gamma(\cos t-\mathrm{e}^{-\gamma t})\right)}{f^2(-\gamma,t)}\\ &\quad+\frac{\mathrm{e}^{-\gamma t}\left((\gamma^2-1)\sin t+2\gamma(\cos t-\mathrm{e}^{\gamma t})\right)}{f^2(-\gamma,t)}\\ &> \frac{\mathrm{e}^{\gamma t}\left(-(\gamma^2-1)\sin t-2\gamma^2\sin t\right)+\mathrm{e}^{-\gamma t}\left((\gamma^2-1)\sin t-2\gamma^2\sin t\right)}{f^2(-\gamma,t)}\\ &= \frac{(1+\gamma^2)(\mathrm{e}^{\gamma t}-\mathrm{e}^{-\gamma t})}{f^2(-\gamma,t)}>0.\end{aligned}$$

□

From the Implicit Function Theorem we have the following corollary.

Corollary 13. *Assume that $c>0$ is an arbitrary constant. Then from $F(\gamma,t)=c$ we obtain the function $t=g(\gamma,c)$, which is increasing with respect to the variable γ.*

Lemma 14. *Assume that $\gamma<0$ and $t\in(\pi,\hat{t})$. Tthen*

$$(\gamma+t)\sin t-\gamma t\cos t<0,$$

where $f(-\gamma,\hat{t})=0$.

Proof. Note that

$$\left(\frac{\gamma+t}{t}-\gamma\frac{\cos t}{\sin t}\right)^{'}=\frac{-\gamma(\sin^2 t-t^2)}{t^2\sin^2 t}\leqslant 0,$$

we only need to show that $(\gamma+\hat{t})\sin\hat{t}-\gamma\hat{t}\cos\hat{t}<0$.

Since $f(-\gamma,\hat{t})=0$ and $f_t^{'}(-\gamma,\hat{t})\leqslant 0$, we have

$$\mathrm{e}^{\gamma\hat{t}}-\cos\hat{t}+\gamma\sin\hat{t}=0,$$

$$\gamma\mathrm{e}^{\gamma\hat{t}}+\sin\hat{t}-\gamma\cos\hat{t}\leqslant 0.$$

Thus we obtain

$$-2\gamma\cos\hat{t}+(1-\gamma^2)\sin\hat{t}\leqslant 0,$$

and then

$$(\gamma+\hat{t})\sin\hat{t}-\gamma\hat{t}\cos\hat{t}\leqslant\left((\gamma+\hat{t})+\frac{\hat{t}(\gamma^2-1)}{2}\right)\sin\hat{t}<0,$$

which finishes the proof. $\square$

Corollary 15. *For $\gamma > 0$ we define the function*

$$C(t) = \mathrm{e}^{-\gamma t}\left((\gamma + t)\sin t - \gamma t\cos t\right) + \mathrm{e}^{\gamma t}\left((\gamma - t)\sin t - \gamma t\cos t\right) + 2\gamma(t - \cos t\sin t), \tag{18}$$

then $C(t) < 0$ if $t \in (\pi, \hat{t})$.

Proof. If $(\gamma - t)\sin t - \gamma t\cos t \leqslant 0$, then from Lemma 14 we have $C(t) < 0$ because $\gamma < 0$ and $t - \cos t\sin t > 0$.

If $(\gamma - t)\sin t - \gamma t\cos t > 0$, since $\mathrm{e}^{\gamma t} < 1 < \mathrm{e}^{-\gamma t}$, then

$$\begin{aligned} C(t) &\leqslant (\gamma + t)\sin t - \gamma t\cos t + (\gamma - t)\sin t - \gamma t\cos t + 2\gamma(t - \cos t\sin t) \\ &= 2\gamma(\sin t - t\cos t + t - \cos t\sin t) \\ &= 2\gamma(t + \sin t)(1 - \cos t) < 0. \end{aligned}$$

$\square$

Proposition 16. *Suppose that $c_i > 0$ for $i = 1, 2$. Then the graphs of $F(\gamma, t) = c_1$ and $\gamma t = -c_2$ have at most one intersection point.*

Proof. We give a proof by contradiction. Assume that the graphs of $F(\gamma, t) = c_1$ and $\gamma t = -c_2$ have two intersection points. The difference of two slopes

$$-\frac{F_\gamma^{'}(\gamma, t)}{F_t^{'}(\gamma, t)} + \frac{t}{\gamma} = \frac{C(t)}{\gamma F_t^{'}(\gamma, t)},$$

where $C(t)$ is given in (18).

Since the difference of the two slopes in two intersection points have different signs, we get the required contradiction because $C(t) < 0$ and $F_t^{'}(\gamma, t) > 0$. $\square$

Proposition 17. *Assume that $-\gamma_R < \gamma_L < 0$, $f(\pm\gamma_i, t_i) > 0$ for $i = R, L$, and*

$$F(-\gamma_L, t_L) = F(\gamma_R, t_R),$$

if $t_i \in (\pi, 2\pi)$. Then $\gamma_L t_L + \gamma_R t_R > 0$.

Proof. We have that $\gamma_L t_L + \gamma_R t_R \neq 0$, because if $\gamma_L t_L + \gamma_R t_R = 0$ then the curves $F(\gamma, t) = c_1$ and $\gamma t = -c_2$ have two intersection points $(-\gamma_L, t_L)$ and (γ_R, t_R), in contradiction with Proposition 16.

We claim that $\gamma_L t_L + \gamma_R t_R > 0$. If γ_R big enough, obviously the statement holds. If there exist $\gamma_i, t_i, i = L, R$ such that $\gamma_L t_L + \gamma_R t_R < 0$, then increasing γ_R tending it to $+\infty$, we get for some value of γ_R that $\gamma_L t_L + \gamma_R t_R > 0$. So by the continuity with respect to the variable γ_R, we obtain that $\gamma_L t_L + \gamma_R t_R = 0$ for some suitable γ_R, in contradiction with the fact that $\gamma_L t_L + \gamma_R t_R \neq 0$. Hence the claim is proved. $\square$

6.1. **Proof of Conjecture 3.** (a) Since $-\gamma_R < \gamma_L < 0$ and $\hat{y} < z^*$, it follows that $P_L^{-1}(0) - P_R(0) = \hat{y} - z^* < 0$ and $\lim\limits_{z\to-\infty}(P_L^{-1}(z) - P_R(z)) < 0$. Note that if $\bar{z} \in (-\infty, 0)$ is the biggest zero of $P_L^{-1}(z) - P_R(z)$, then $(P_L^{-1}(z) - P_R(z))'|_{z=\bar{z}} < 0$.

From direct computations and by Proposition 17 we have

$$(P_L^{-1}(z) - P_R(z))'|_{z=\bar{z}} = \frac{\bar{z}}{P_R(\bar{z})}(\mathrm{e}^{-\gamma_L t_L} - \mathrm{e}^{\gamma_R t_R}) > 0.$$

Thus we have a contradiction.

(b) The proof of statement (b) is similar and we omit it.

6.2. **Proof of Conjecture 4.** From Conjecture 3 we just need to prove that the refracting systems (7) have at most one limit cycle when $-\gamma_R < \gamma_L < 0$ and $\hat{y} \geqslant z^*$.

As in the proof of Conjecture 3 we can deduce that $P_L^{-1}(z) - P_R(z)$ has no zeros in $(-\infty, 0)$ when $\hat{y} = z^*$. In the following we consider the case $\hat{y} > z^*$. Then we have $P_L^{-1}(0) - P_R(0) = \hat{y} - z^* > 0$ and $\lim\limits_{z\to-\infty}(P_L^{-1}(z) - P_R(z)) < 0$. Clearly $P_L^{-1}(z) - P_R(z)$ has at least one zero in $(-\infty, 0)$. Again as in the proof of Conjecture 3 we can obtain that $P_L^{-1}(z) - P_R(z)$ has at most one zero in $(-\infty, 0)$.

6.3. **Proof of Theorem 3.** We divide the proof of Theorem 3 into the following three cases.

Case (I): $\alpha_R \leqslant 0 \leqslant \alpha_L$. From Theorem 1 of [10] we know that the refracting systems (5) have at most one crossing limit cycle. If $\gamma_R\gamma_L < 0$ and $(\gamma_R+\gamma_L)(\alpha_L\gamma_R-\alpha_R\gamma_L) < 0$ there is a unique limit cycle, which is stable for $\gamma_L + \gamma_R < 0$ and unstable for $\gamma_L + \gamma_R > 0$.

Case (II): $\alpha_L < 0$ and $\alpha_R < 0$. The uniqueness of the limit cycles of the refracting systems (5) can be obtained from Theorem 2 of [10]. If $\gamma_L\gamma_R < 0$ and $\gamma_L(\gamma_L+\gamma_R) < 0$ there is a unique limit cycle, which is stable for $\gamma_L > 0$ and unstable for $\gamma_L < 0$.

Case (III): $\alpha_L < 0 < \alpha_L$. Refracting systems (5) have at most one limit cycle due to Conjecture 4. From the proof of Conjecture 4 we know that the refracting systems (5) have a unique limit cycle when $-\gamma_R < \gamma_L < 0$ and $\hat{y} > z^*$, which is stable when $\gamma_L > 0$.

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