LIMIT CYCLES OF POLYNOMIAL DIFFERENTIAL SYSTEMS OF DEGREE 1 AND 2 ON THE CYLINDER

JOHANA JIMENEZ[1] AND JAUME LLIBRE[2]

Abstract. We consider planar polynomial differential systems of degree 1 and 2 on the cylinder and we study their limit cycles. We prove that such linear differential systems have at most one limit cycle and that such quadratic differential systems have at most two limit cycles. Moreover such upper bounds are reached.

1. Introduction and statement of the main results

The study of the limit cycles began with Poincaré, see [10]. Later on the existence of limit cycles was observed in nature, see for instance [1, 6, 11]. The limit cycles attracted the interest of many researchers and later on it became the main object to be studied in the statement of the second part of the 16th Hilbert problem, which wants to find an upper bound for the maximum number of limit cycles that a polynomial differential system of a fixed degree can have, see [4]. Hence in the last years, the study of the limit cycles of the planar differential systems has been one of the main problems of the qualitative theory of differential equations in the plane. See for instance [8, 9, 12] and the references quoted there.

We will work in the cylinder $[0,1]\times(0,1)/\sim$, with the equivalence relation $(0,y)\sim(1,y)$ for all $y\in(0,1)$.

In this paper we consider the planar polynomial differential systems

$$x = P(x,y), \qquad \dot{y} = Q(x,y), \tag{1}$$

where $P(x,y)$ and $Q(x,y)$ are polynomials of degree at most two on the cylinder, i.e. $P(0,y)=P(1,y)$ and $Q(0,y)=Q(1,y)$.

We say that a solution $(x(t,x_0,y_0),y(t,x_0,y_0))$ of system (1) such that $x(0,x_0,y_0)=x_0=0$ and $y(0,x_0,y_0)=y_0$, is a *periodic orbit of period* T on the cylinder if $x(T,x_0,y_0)=1$ and $y(0,x_0,y_0)=y(T,x_0,y_0)$.

It is known that a periodic orbits of a differential system can be isolated in the set of all periodic orbits of the system or belong to a continuum set of periodic orbits. When the periodic orbit is isolated then it is called a *limit cycle*. So the aim of this paper is to study the maximum number of limit cycles that the polynomial differential systems (1) of degree 1 and 2 on the cylinder can have. Moreover we will show that these maximum numbers are reached.

The main results of this paper are stated in the following subsections. More precisely, in subsection 1.1, we study the dynamics on the cylinder for the linear differential systems (1). And in subsection 1.2 we study the dynamics on the cylinder for the quadratic differential systems (1).

2020 *Mathematics Subject Classification.* Primary 34C05, 34C07, 34C25, 34C40.

Key words and phrases. Polynomial differential systems, Limit cycles, Normally hyperbolic.

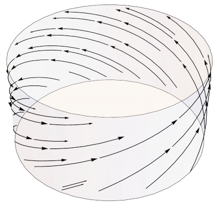

FIGURE 1. Orbits of the inear system (7) without periodic orbits.

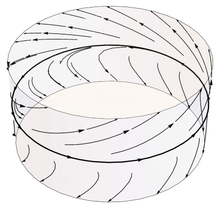

FIGURE 2. The unstable limit cycle of the linear differential system (8).

FIGURE 3. The circle filled with equilibrium points of the linear system (9).

FIGURE 4. A continuum of periodic orbits of the linear system (10).

1.1. **Dynamics of the linear differential systems on the cylinder.** In this subsection we consider differential systems of the form

$$P(x,y) = a_{00} + a_{10}x + a_{01}y, \qquad Q(x,y) = b_{00} + b_{10}x + b_{01}y, \tag{2}$$

where $a_{ij}, b_{ij} \in \mathbb{R}$ for $i, j \in \{0,1\}$ and such that $P(0,y) = P(1,y)$ and $Q(0,y) = Q(1,y)$. These linear differential systems on the cylinder are characterized in the following lemma.

Lemma 1.1. *The unique linear differential systems on the cylinder are*

$$\dot{x} = a_0 + a_1 y, \qquad \dot{y} = b_0 + b_1 y. \tag{3}$$

These systems are analytic on the cylinder.

For the linear differential systems on the cylinder we obtain the following results.

Theorem 1.1. *System* (3) *has at most one limit cycle on the cylinder.*

Proposition 1.1. *There are linear differential systems on the cylinder without periodic orbits, or with one limit cycle, or with a circle filled with equilibrium points, or with a continuum of periodic orbits. See Figures* 1–4*.*

In Proposition 1.1 we prove that the upper bound for the number of limit cycles found in Theorem 1.1 is reached.

Lemma 1.1, Proposition 1.1, and Theorem 1.1 are proved in section 3. The kind of stability of the limit cycles and of the circle filled with equilibrium points in Proposition 1.1 and Theorem 1.1 are given inside their proofs.

1.2. **Dynamics of the quadratic differential systems on the cylinder.** Now we consider polynomial differential systems of the form

$$P(x,y) = \sum_{i+j=0}^{2} a_{ij} x^i y^j, \qquad Q(x,y) = \sum_{i+j=0}^{2} b_{ij} x^i y^j, \tag{4}$$

where $a_{ij}, b_{ij} \in \mathbb{R}$ for $i, j \in \{0,1,2\}$ and such that $P(0,y) = P(1,y)$ and $Q(0,y) = Q(1,y)$. These quadratic differential systems on the cylinder are characterized in the following lemma.

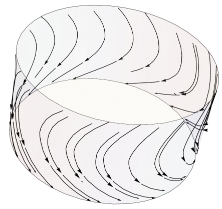

FIGURE 5. The quadratic system (11) without periodic orbits.

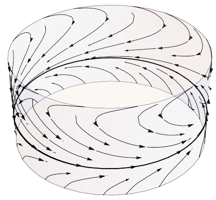

FIGURE 6. The stable limit cycle of the quadratic system (12).

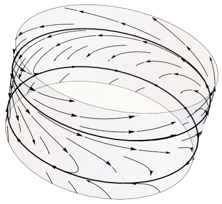

FIGURE 7. One stable and one unstable limit cycle of the quadratic system (13).

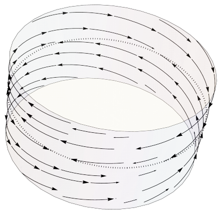

FIGURE 8. One circle filled with equilibrium points of the quadratic system (15).

Lemma 1.2. *The unique quadratic differential systems on the cylinder are*

$$\dot{x} = a_0 + a_1 y + a_2 y^2, \quad \dot{y} = b_0 + b_1 y + b_2 y^2. \tag{5}$$

These systems are analytic on the cylinder.

For the quadratic differential systems on the cylinder we get the following results.

Theorem 1.2. *System* (5) *has at most two limit cycles on the cylinder.*

Proposition 1.2. *There are quadratic differential systems on the cylinder without periodic orbits, or with* 1 *or* 2 *limit cycles, or with* 1 *or* 2 *circles filled with equilibrium points, or with one limit cycle and one circle filled with equilibrium points, or with a continuum of periodic orbits. See Figures* 5–11.

In Proposition 1.2 we prove that the upper bound for the number of limit cycles found in Theorem 1.2 is reached.

Lemma 1.2, Proposition 1.2 and Theorem 1.2 are proved in section 3. The kind of stability of the limit cycles and of the circle filled with equilibrium points in Proposition **??** and Theorem 1.2 are given inside their proofs.

2. PRELIMINARIES

In the proofs of Propositions 1.1 and 1.2 we used the definition of submanifold normally hyperbolic.

Let φ_t be a smooth flow on a manifold M and consider that N is a submanifold of M filled with equilibrium points of φ_t. N is called *normally hyperbolic* if there exists a splitting of the tangent bundle of M over N into subbundles such that $TM = E^s \oplus E^u \oplus TN$. And they are invariant under $D\varphi_t$ for all $t \in \mathbb{R}$. Moreover, $D\varphi_t$ contracts E^s exponentially, $D\varphi_t$ expands E^s exponentially, and TN is the tangent bundle of N.

For a normally hyperbolic submanifold always exists stable and unstable manifolds, more precisely.

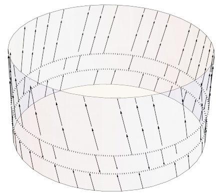

FIGURE 9. The two circles filled with equilibrium points of the quadratic system (16).

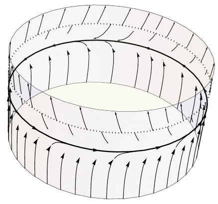

FIGURE 10. One stable limit cycle and one circle filled with equilibrium points of the quadratic system (17).

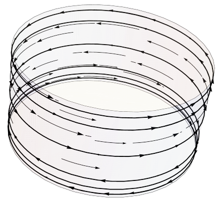

FIGURE 11. A continuum of periodic orbits of the quadratic system (19).

Theorem 2.1. *Let N be a normally hyperbolic submanifold filled with equilibrium points of φ_t. Then there exist smooth stable and unstable manifolds tangent to $E^s \oplus TN$ and $E^u \oplus TN$, along N, respectively. Moreover both N and the stable and unstable manifolds are permanent under small perturbations of the flow φ_t.*

For more details on the normally hyperbolic theory see [5].

3. Proofs of the main results

Proof of Lemma 1.1. Considering differential system (1), with the linear polynomials $P(x,y)$ and $Q(x,y)$ given in (2). We have that the linear differential system (1) is well defined on the cylinder if $P(0,y) = P(1,y)$ and $Q(0,y) = Q(1,y)$. And these conditions occurs if and only if $a_{01} = b_{01} = 0$. Since now all the higher derivates of the polynomials P and Q coincide the system is analytic, and the lemma is proved. $\square$

Proof of Theorem 1.1. We observe that the periodic orbits $(x(t,x_0,y_0), y(t,x_0,y_0))$ of period T of system (3) on the cylinder are generated by the zeros of the function $\Pi(y_0) = y(T,x_0,y_0) - y(0,x_0,y_0)$ for $y_0 \in (0,1)$ when $x(T,x_0,y_0) = 1$ and $y(t,x_0,y_0) < 1$ for all $t \in (0,T)$. In order to analyse such zeros we consider two different cases, namely either $b_1 = 0$ or $b_1 \neq 0$.

Case 1: $b_1 = 0$. We obtain that the solution of system (3) satisfying $x_0 = 0$ and $y_0 \in (0,1)$ is given by

$$x(t,0,y_0) = \frac{1}{2}t(2a_0 + a_1 b_0 t + 2a_1 y_0), \qquad y(t,0,y_0) = b_0 t + y_0.$$

Then $\Pi(y_0) = b_0 T$. Hence, if $b_0 \neq 0$ we obtain that system (3) does not have periodic orbits on the cylinder. And when $b_0 = 0$ system (3) has a continuum of periodic orbits on the cylinder, because for every $y_0 \in (0,1)$ the circle $y = y_0$ generates a periodic orbit on the cylinder.

Case 2. $b_1 \neq 0$. Then the equation $\dot{y} = 0$ has the zero $y_1 = -b_0/b_1$. We shall prove that $y = y_1$ is the unique solution that generates a limit cycle on the cylinder. Here we study two subcases, namely $a_1 = 0$ or $a_1 \neq 0$.

Subcase 2.1: $a_1 = 0$. If $a_0 \neq 0$, we obtain

$$x(t,0,y_0) = a_0 t, \qquad y(t,0,y_0) = \frac{-b_0 + e^{b_1 t}(b_0 + b_1 y_0)}{b_1}.$$

In this case $x(t,0,y_0) = 1$ for $T = 1/a_0$. Hence $\Pi(y_0) = (-1 + e^{b_1/a_0})(b_0 + b_1 y_0)/b_1$. Thus, if $y_0 \neq -b_0/b_1$, system (3) does not have perodic orbits on the cylinder. So the unique periodic orbit of system (3) on the cylinder is $y_0 = y_1 = -b_0/b_1$ if $y_1 \in (0,1)$, and consequently it is a limit cycle. Moreover it is stable (resp. unstable) if $b_1 < 0$ (resp. $b_1 > 0$). And it travels clockwise (resp. counterclockwise) sense if $a_0 < 0$ (resp. $a_0 > 0$).

If $a_0 = 0$, we obtain that $\dot{x} \equiv 0$, thus when $y_1 \in (0,1)$ the circle $y = y_1$ of the cylinder is filled with equilibrium points of the linear differential system (3). Moreover the Jacobian matrix of the system on such equilibrium points has eigenvalues $\lambda_1 = 0$ and $\lambda_1 = b_1$. Hence from section 2 the circle c_1 is normally hyperbolic. For Theorem 2.1 it is a stable (resp. an unstable) circle if $b_1 < 0$ (resp. $b_1 > 0$).

Subcase 2.2: $a_1 \neq 0$. Then the polynomial $a_0 + a_1 x$ can have one real solution, namely $x_1 = -a_0/a_1$. Here we consider two additional subcases.

Subcase 2.2.1: If $a_0 b_0 \neq 0$, then

$$x(t,0,y_0) = \frac{a_1(e^{b_1 t} - 1)(b_0 + b_1 y_0) + b_1(a_0 b_1 - a_1 b_0)t}{b_1^2}, \quad y(t,0,x_0) = \frac{-b_0 + e^{b_1 t}(b_0 + b_1 y_0)}{b_1}.$$

If $a_0 b_1 - a_1 b_0 = 0$, then $y_1 = x_1 = -a_0/a_1$ and solving $x(T,0,y_0) = 1$ for

$$T = \frac{a_0 \log\left(\dfrac{a_0^2 + a_1 b_0 + a_0 a_1 y_0}{a_0(a_0 + a_1 y_0)}\right)}{a_1 b_0}, \quad \text{and } \Pi(y_0) = b_0/a_0 \neq 0.$$

Hence in this subcase system (3) does not have periodic orbits.

If $a_0 b_1 - a_1 b_0 \neq 0$ and $b_0 + b_1 y_0 = 0$, solving $x(T,0,y_0) = 1$ we obtain that $T = b_1/(a_0 b_1 - a_1 b_0)$, and $\Pi(y_0) = (-1 + e^{b_1^2/(a_0 b_1 - a_1 b_0)})(b_0 + b_1 y_0)/b_1$, which only have one real root, in $y_0 = y_1 = -b_0/b_1$. Therefore system (3) has one limit cycle on the cylinder if $y_0 = -b_0/b_1 \in (0,1)$, it is stable (resp. unstable) when $b_1 < 0$ (resp. $b_1 > 0$). Moreover it travels clockwise (resp. counterclockwise) if $b_1(a_0 b_1 - a_1 b_0) < 0$ (resp. $b_1(a_0 b_1 - a_1 b_0) > 0$).

If $(a_0 b_1 - a_1 b_0)(b_0 + b_1 y_0) \neq 0$, solving $x(T,0,y_0) = 1$ we get

$$T = \frac{a_1(b_0 + b_1 y_0) - b_1^2}{a_0 b_1 - a_1 b_0} - \mathcal{W}\left(\frac{a_1(b_0 + b_1 y_0)}{a_0 b_1 - a_1 b_0} e^{(a_1(b_0 + b_1 y_0) - b_1^2)/(a_0 b_1 - a_1 b_0)}\right), \tag{6}$$

where $\mathcal{W}$ is the Lambert Function, for more details see [2]. Moreover $\Pi(y_0) = (-1 + e^{b_1 T})(b_0 + b_1 y_0)/b_1 \neq 0$, because $T \neq 0$ and $b_0 + b_1 y_0 \neq 0$. Therefore system (3) does not have periodic orbits on the cylinder.

Subcase 2.2.2: $a_0 b_0 = 0$. We have three options, either $a_0 = 0, b_0 \neq 0$, or $a_0 \neq 0, b_0 = 0$, or $a_0 = b_0 = 0$. When $a_0 = 0$ and $b_0 \neq 0$, similar to above case, it is possible to prove that system (3) admits at most one limit cycle such that it is generated by the circle $y_0 = -b_0/b_1$ when $y_0 \in (0,1)$. When $a_0 \neq 0$ and $b_0 = 0$, we obtain T as in (6) and $\Pi(y_0) = (-1 + e^{b_1 T})y_0$, which is different to zero for $y_0 \neq 0$, therefore system (3) does not have periodic orbits on the cylinder. Finally, if $a_0 = b_0 = 0$, we obtain $T = \log((b_1 + a_1 y_0)/a_1 y_0)/b_1$ and $\Pi(y_0) = b_1/a_1 \neq 0$, this is, system (3) does not have periodic orbits on the cylinder.

This completes the proof of the theorem. $\square$

Proof of Proposition 1.1. In what follows we provide linear differential systems without periodic orbits, or having one limit cycle, or a circle filled with equilibrium points, or a continuum of periodic orbits on the cylinder.

First we consider the linear differential system on the cylinder

$$\dot{x} = 4 + 2y, \qquad \dot{y} = 3 - \frac{3}{2}y. \tag{7}$$

Its solution with $x_0 = 0$ and $y_0 \in (0, 1)$ is

$$x(t, 0, y_0) = \frac{4}{3}\big(-2 + 6t + e^{-3t/2}(-2 + y_0) + y_0\big), \; y(t, 0, y_0) = 2 + e^{-3t/2}(-2 + y_0).$$

If $-2 + y_0 = 0$, then $T = 1/8$ and $\Pi(y_0) = (-1 + e^{3/16})(-2 + y_0) = 0$, but this $y_0 \notin (0, 1)$, so system (7) does not have periodic orbits on the cylinder.

If $-2 + y_0 \neq 0$, then $T = 11/24 - y_0/6 + 2/3\mathcal{W}\big(4e^{-11/16+y_0/4}(-2 + y_0)\big)$ and $\Pi(y_0) = (-1 + e^{-3T/2})(-2 + y_0) \neq 0$. Therefore system (7) does not have periodic orbits on the cylinder. See the phase portrait of system (7) in Figure 1.

Now we consider the linear differential system on the cylinder

$$\dot{x} = -1 + 3y, \qquad \dot{y} = -1 + 2y. \tag{8}$$

The solution with $x_0 = 0$ and $y_0 \in (0, 1)$ is

$$x(t, 0, y_0) = \frac{1}{4}\big(3 + 2t - 6y_0 + e^{2t}(-3 + 6y_0)\big), \quad y(t, 0, y_0) = \frac{1}{2} + e^{2t}\bigg(-\frac{1}{2} + y_0\bigg).$$

If $-1 + 2y_0 = 0$, then $T = 2$. Hence $\Pi(y_0) = 1/2(-1 + e^2)(-1 + 2y_0) = 0$. Therefore system (7) has one limit cycle on the cylinder. Moreover, if $b_1 = 2 > 0$ and $b_1(a_0b_1 - a_1b_0) = 2 > 0$, then the limit cycle is unstable and travels counterclockwise sense. See Figure 2.

If $-1 + 2y_0 \neq 0$, then $T = 1/2 + 3y_0 - 1/2\mathcal{W}\big(e^{1+6y_0}(-3 + 6y_0)\big)$, and $\Pi(y_0) = 1/2(-1 + e^{2T})(-1 + 2y_0) \neq 0$. Hence system (8) does not have periodic orbits on the cylinder.

The following linear differential system on the cylinder has a circle filled with equilibrium points.

$$\dot{x} = 1 - 2y, \qquad \dot{y} = \frac{3}{2} - 3y. \tag{9}$$

Indeed, we observed that $x_1 = -a_0/a_1 = 1/2 = -b_0/b_1 = y_1$, then the circle $y = y_1$ is filled with equilibrium points of (9). Since the Jacobian matrix of the system at these equilibrium points has the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -3$, then the circle c_1 is normally hyperbolic and it is stable because $b_1 = -3$. See Figure 3.

Consider the linear differential system on the cylinder

$$\dot{x} = \frac{1}{2} - y, \qquad \dot{y} = 0. \tag{10}$$

Since $\dot{y} = 0$ we have that $\Pi(y_0) = 0$ for $y_0 \in (0, 1)$. Hence for every $y_0 \in (0, 1)$ the circle $y = y_0$ is a periodic orbit on the cylinder. Hence system (10) has a continuum of periodic orbits on the cylinder. As $a_1 = -1 < 0$, we have that the periodic orbit $y = y_0$ travels in clockwise sense if $y_0 > -a_0/a_1 = 1/2$, otherwise, it travels counterclockwise. See Figure 4.

This completes the proof of Proposition 1.1. □

Proof of Lemma 1.2. We consider system (1), with the polynomials $P(x, y)$ and $Q(x, y)$ in (4). We have that this quadratic differential system is well defined on the cylinder if

$P(0,y) = P(1,y)$ and $Q(0,y) = Q(1,y)$, i.e. $a_{10} = -a_{20}$, $b_{10} = -b_{20}$ and $a_{11} = b_{11} = 0$. Moreover these quadratic differential systems are analytic on cylinder if

$$\frac{\partial P}{\partial x}(0,y) = \frac{\partial P}{\partial x}(1,y),\ \frac{\partial P}{\partial y}(0,y) = \frac{\partial P}{\partial y}(1,y),\ \frac{\partial Q}{\partial x}(0,y) = \frac{\partial Q}{\partial x}(1,y),\ \frac{\partial Q}{\partial y}(0,y) = \frac{\partial Q}{\partial y}(1,y),$$

because then all the derivates of higher order of the polynomials P and Q coincide. These last conditions are satisfied when $a_{10} = b_{10} = 0$. This completes the proof of the lemma. □

Proof of Theorem 1.2. Consider the quadratic differential systems (5). If $a_2 b_2 \neq 0$ the roots of the polynomial $a_0 + a_1 y + a_2 y^2$ are $x_{2,3} = (-a_1 \pm \sqrt{\widetilde{\Delta}})/(2a_2)$, where $\widetilde{\Delta} = a_1^2 - 4a_0 a_2$. While the roots of the polynomial $b_0 + b_1 y + b_2 y^2$ are $y_{2,3} = (-b_1 \pm \sqrt{\Delta})/(2b_2)$, where $\Delta = b_1^2 - 4b_0 b_2$, respectively. When $a_2 = 0$ the polynomial $a_0 + a_1 y + a_2 y^2$ has one real root, namely $x_1 = -a_0/a_1$, for $a_1 \neq 0$. And for $a_1 = 0$, we obtain $\dot{x} = a_0$. Analogously, if $b_2 = 0$ the polynomial $b_0 + b_1 y + b_2 y^2$ has one real root, namely $y_1 = -b_0/b_1$ if $b_1 \neq 0$. And for $b_1 = 0$ we get $\dot{y} = b_0$. Therefore we consider two different cases, namely either $b_2 \neq 0$ or $b_2 = 0$.

Case 1: $b_2 \neq 0$. Then the differential equation $\dot{y}$ in system (5) is a Riccati differential equation. It is well known that a Riccati differential equation can have at most two limit cycles, see [3, 7].

Moreover if $y_2, y_3 \in (0,1)$ and y_2, y_3 are not roots of the polynomial $a_0 + a_1 y + a_2 y^2$, then the circles $y = y_2$ and $y = y_3$ are limit cycles on the cylinder. Besides the limit cycle in $y = y_2$ is stable (resp. unstable) either $b_2 > 0, y_2 < y_3$ or $b_2 < 0$, $y_3 < y_2$ (resp. $b_2 > 0, y_3 < y_2$ or $b_2 < 0, y_2 < y_3$). And the limit cycle in $y = y_3$ is stable (resp. unstable) either $b_2 > 0, y_3 < y_2$ or $b_2 < 0$, $y_2 < y_3$ (resp. $b_2 > 0, y_2 < y_3$ or $b_2 < 0, y_3 < y_2$). The limit cycle $y = y^*$ with $y^* \in \{y_2, y_3\}$, travels clockwise (resp. counterclockwise) sense either $a_2 > 0$ and either $x_2 < y^* < x_3$ or $x_3 < y^* < x_2$; or $a_2 < 0$ and either $x_2 < x_3, y^* < x_2, y^* > x_3$ or $x_3 < x_2, y^* < x_3, y^* > x_2$; or $a_2 = 0$ and either $a_1 > 0, y^* < x_1$ or $a_1 < 0, y^* > x_1$ (resp. $a_2 > 0$ and either $x_2 < x_3, y^* < x_2, y^* > x_3$ or $x_3 < x_2, y^* < x_3, y^* > x_2$; or $a_2 < 0$ and either $x_2 < y^* < x_3$ or $x_3 < y^* < x_2$; or $a_2 = 0$ and either $a_1 > 0, y^* > x_1$ or $a_1 < 0, y^* < x_1$).

If y^* is root of the polynomial $a_0 + a_1 y + a_2 y^2$ for $y^* \in \{y_2, y_3\}$, then the circle $y = y^*$ is filled with equilibrium points of system (5). Moreover the Jacobian matrix of the system (5) evaluated at these equilibrium points has the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = \pm\Delta \neq 0$. Therefore from Theorem 2.1 the circle filled with equilibrium points is normally hyperbolic, and it is stable (resp. unstable) if $\lambda_2 = -\Delta$ (resp. $\lambda_2 = \Delta$).

Case 2: $b_2 = 0$. Since the system is quadratic $a_2 \neq 0$ and $\dot{y} = b_0 + b_1 y$. Then we have two different subcases either $b_1 \neq 0$ or $b_1 = 0$.

Subcase 2.1: $b_1 \neq 0$. Then the polynomial $b_0 + b_1 y$ has the real solution $y_1 = -b_0/b_1$. The solution of system (5) satisfying $x_0 = 0$ and $y_0 \in (0,1)$ is

$$\begin{aligned} x(t,0,y_0) =& \frac{1}{2b_1^3}\Big(a_2(3b_0^2 + 2b_0^2 b_1 t + 2b_0 b_1 y_0 - b_1^2 y_0^2 - 4b_0 e^{b_1 t}(b_0 + b_1 y_0) + e^{2b_1 t}(b_0 + b_1 y_0)^2) \\ &+ 2b_1(a_0 b_1^2 t + a_1 e^{b_1 t}(b_0 + b_1 y_0) - a_1(b_0 + b_0 b_1 t + b_1 y_0)))\Big), \\ y(t,0,y_0) =& \frac{-b_0 + e^{b_1 t}(b_0 + b_1 y_0)}{b_1}. \end{aligned}$$

If there is a $T > 0$ such that $x(T,0,y_0) = 1$, then $\Pi(y_0) = (-1 + e^{b_1 T})(b_0 + b_1 y_0)/b_1$. So the equation $\Pi(y_0) = 0$ has a unique solution $y_0 = y_1 = -b_0/b_1$ if $y_1 \in (0,1)$.

When $y_1 \in (0,1)$ and y_1 is not a root of the polynomial $a_0 + a_1 y + a_2 y^2$, the circle $y = y_1$ is the unique periodic orbit of system (5) on the cylinder. For $b_1 < 0$ (resp. $b_1 > 0$) the limit cycle is stable (resp. unstable). The analysis to know the sense of this limit cycle is similar to Case 1.

If y_1 is a root of the polynomial $a_0 + a_1 y + a_2 y^2$, then the circle $y = y_1$ is filled with equilibrium points of system (5). Since the Jacobian matrix of the system evaluated at these equilibrium points has the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = b_1 \neq 0$. From Theorem 2.1 the circle filled with equilibrium points is normally hyperbolic, and it is stable (resp. unstable) if $b_1 < 0$ (resp. $b_1 > 0$).

Subcase 2.2: $b_1 = 0$. Then the system (5) has the solution $y(t, 0, y_0) = b_0 t + y_0$. If there is $T > 0$ such that $x(T, 0, y_0) = 1$, then $\Pi(y_0) = b_0 T$. When $b_0 \neq 0$ system (5) does not have periodic orbits. For $b_0 = 0$ the system (5) has a continuum of periodic orbits on the cylinder. The analysis about the sense of these periodic orbits is similar to Case 1.

This completes the proof of Theorem 1.2. □

Proof of Proposition 1.2. In what follows we provide quadratic differential systems on the cylinder without periodic orbits, or having either only one limit cycle, or two limit cycles, or one circle filled with equilibium points, or two circles filled with equilibium points, or one limit cycle and one circle filled with equilibrium points, or a continuum of periodic orbits.

We consider the quadratic differential system on the cylinder

$$\dot{x} = \frac{5}{2} - 2y - 4y^2, \qquad \dot{y} = -1 - 2y. \tag{11}$$

Then the solution with initial conditions $x_0 = 0$ and $y_0 \in (0,1)$ is

$$\begin{aligned} x(t,0,y_0) =& \frac{e^{-4t}}{4}\big(- 2e^{2t}(1+2y_0) + (1+2y_0)^2 + e^{4t}(1+10t-4y_0^2)\big),\\ y(y,0,y_0) =& -1 + e^{-2t}(1+2y_0). \end{aligned}$$

If there exists $T > 0$ such that $x(T, 0, y_0) = 1$, then $\Pi(y_0) = 1/2((-1+e^{-2T})(1+2y_0))$. The unique solution of $\Pi(y_0) = 0$is $y_0 = -1/2$, but this $y_0 \neq (0,1)$. Consequently the quadratic differential system (11) has no periodic orbits on the cylinder. See Figure 5.

Now we consider the quadratic differential system on the cylinder

$$\dot{x} = \frac{1}{2} + 2y - 3y^2, \qquad \dot{y} = \frac{43}{50} - \frac{7}{2}y. \tag{12}$$

The roots of the polynomials $1/2 + 2y - 3y^2$ and $43/50 - 7y/2$ are $x_{2,3} = 1/6(2 \pm \sqrt{10})$ and $y_1 = 43/175$, respectively. Moreover the solution of system (12) with initial conditions $x_0 = 0$ and $y_0 \in (0,1)$ is

$$\begin{aligned} x(t,0,y_0) =& \frac{e^{-7t}}{428750}\big(6(43-175y_0)^2 - 368e^{7t/2}(-43+175y_0)\\ &+ e^{7t}(347417t - 2(-43+175y_0)(-313+525y_0))\big),\\ y(t,0,y_0) =& \frac{1}{175}\big(- 43 + e^{-7t/2}(-43+175y_0)\big). \end{aligned}$$

From this solution we obtain from $x(T, 0, y_0) = 1$ that $T = 61250/49631$, and after that $\Pi(y_0) = (-1 + e^{-7t/2})(-43/175 + y_0)$. Therefore $\Pi(y_0) = 0$ if and only if $y_0 = 43/175$. Hence, since $y_0 = y_1 = 43/175 \in (0,1)$ and $y_1 \neq x_{2,3}$, the circle $y = y_1$ is the unique limit cycle of the system (12). Furthermore, as $b_1 = -7/2 < 0$, this limit cycle is stable and travels counterclockwise, because $a_2 = -3 < 0$ and $x_2 < y_1 < x_3$. See Figure 6.

We consider the following quadratic differential system on the cylinder

$$\dot{x} = -\frac{1}{2} + y + 3y^2, \qquad \dot{y} = \frac{7}{10} - \frac{9}{2}y + 5y^2. \tag{13}$$

The polynomials $-1/2+y+3y^2$ and $7/10-9y/2+5y^2$ have the roots $x_2 = 1/6(-1-\sqrt{7}) < x_3 = 1/6(-1+\sqrt{7})$ and $y_2 = 1/5 < y_3 = 7/10$, respectively.

Doing the change of variables

$$x = u, \qquad y = v = \frac{1}{y - y_2},$$

system (13) becomes

$$\dot{u} = -\frac{1}{2} + v + 3v^2, \qquad \dot{v} = -5 + \frac{5}{2}v. \tag{14}$$

Moreover the solution given by the circle $y = y_3$ of system (13) becomes the solution given by the circle $v = v_1 = 1/(y_3 - y_2) = 2$ for the system (14). The solution of system (14) with initial conditions $u_0 = 0$ and $v(0) = v_0$ is

$$\begin{aligned} u(t,0,v_0) =& \frac{1}{10}\big(135t + 52e^{5t/2}(-2+v_0) + 6e^{5t}(-2+v_0)^2 - 2(-2+v_0)(20+3\rho)\big),\\ v(t,0,v_0) =& 2 + e^{5t/2}(-2+v_0). \end{aligned}$$

From this solution solving $u(T,0,v_0) = 1$ we get $T = 2/27$ and $\Pi(v_0) = (-1+e^{5t/2})(-2+v_0)$. Thus the unique real solution of $\Pi(v_0)$ is $v_0 = 2$. As $y_2, y_3 \in (0,1)$ and $y_k \neq x_l$ for $k,l \in \{2,3\}$, system (13) has two limit cycles on the cylinder generated by the circles $y = y_2$ and $y = y_3$. Furthermore as $b_2 = 5 > 0$ and $y_2 < y_3$, the limit cycle generates by $y = y_2$ (resp. $y = y_3$) is stable (resp. unstable). Since $a_2 = 3 > 0$ and $x_2 < y_2 < x_3 < y_3$, the limit cycle $y = y_2$ (resp. $y = y_3$) travels in clockwise (resp. counterclockwise) sense. See Figure 7.

If we consider the quadratic differential system on the cylinder

$$\dot{x} = \frac{8}{5} - \frac{4}{5}y - \frac{37}{10}y^2, \qquad \dot{y} = -\frac{8}{111} + \frac{8\sqrt{38}}{111} - \frac{2}{3}y. \tag{15}$$

Using the previous notations we obtain that $x_{2,3} = 4/37(-1 \pm \sqrt{38})$ and $y_1 = 4/37(-1+\sqrt{38})$. So $y_1 = x_2$, and as $y_1 \in (0,1)$, system (15) has the circle $y = y_1$ filled with equilibrium points, by Theorem 2.1 this circle is stable because $b_1 = -2/3 < 0$. See Figure 8.

We consider the quadratic differential system on the cylinder

$$\dot{x} = -\frac{1}{10} + \frac{3}{4}y - \frac{5}{4}y^2, \qquad \dot{y} = \frac{2}{5} - 3y + 5y^2. \tag{16}$$

Then $y_2 = 1/5 = x_3$ and $y_3 = 2/5 = x_2$. As $y_{2,3} \in (0,1)$, system (16) has the two circles $y = y_2$ and $y = y_3$ filled with equilibrium points. Since the Jacobian matrix of the system at the equilibrium points of the circle $y = y_2$ has the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -1$, this circle is stable. While the Jacobian matrix of the system at the equilibrium points of the circle $y = y_3$ has the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 1$, then this circle is unstable. See Figure 9.

If we consider the quadratic differential system on the cylinder

$$\dot{x} = -\frac{19}{180} + \frac{1}{18\sqrt{10}} + \frac{y}{2} - \frac{y^2}{2}, \qquad \dot{y} = \frac{13}{10} - 4y + 3y^2. \tag{17}$$

Then $y_{2,3} = 1/30(20 \mp \sqrt{10})$ and $x_2 = 1/30(20 + \sqrt{10})$, $x_3 = 1/30(10 - \sqrt{10})$. Doing the change of variables $u = x$ and $y = v = 1/(y - y_3)$, we obtain the equivalent system

$$\dot{u} = -\frac{19}{180} + \frac{1}{18\sqrt{10}} + \frac{v}{2} - \frac{v^2}{2}, \qquad \dot{v} = -3 - \sqrt{\frac{2}{5}}v. \tag{18}$$

Hence the solution given by the circle $y = y_2$ of system (17) becomes the solution given by the circle $v = v_1 = 1/(y_2 - y_3) = -3\sqrt{5/2}$ for the system (18). The solution of system (18) with initial conditions $u_0 = 0, v(0) = v_0$ is

$$\begin{aligned} u(t,0,v_0) =& \frac{1}{720}\Bigg(675(4+9\sqrt{10}) - 8(1022+67\sqrt{10})t + 180(15+\sqrt{10})v_0 + 45e^{-2\sqrt{2/5}t} \\ & \Bigg(45\sqrt{10} + 60v_0 - 4e^{\sqrt{2/5}t}\Bigg(15 + 45\sqrt{10} + (30+\sqrt{10})v_0 + \sqrt{10})v_0^2 \\ & \sinh\left(\sqrt{\frac{2}{5}}t\right)\Bigg)\Bigg)\Bigg), \\ v(t,0,v_0) =& -3\sqrt{\frac{5}{2}} + e^{-\sqrt{2/5}t}\left(3\sqrt{\frac{5}{2}} + v_0\right). \end{aligned}$$

For this system we have that $\Pi(v_0) = (-1 + e^{\sqrt{2/5}t})(3\sqrt{5/2} + v_0)$. The unique real solution of $\Pi(v_0) = 0$ is $v_0 = -3\sqrt{5/2} = v_1$. As $y_2, y_3 \in (0,1)$, and $y_3 = x_2$, $y_2 \neq x_3$, system (17) has one limit cycle on the cylinder given by the circle $y = y_2$, and the circle $y = y_3$ is filled with equilibrium points. Furthermore as $b_2 = 3 > 0$ and $y_2 < y_3$, the limit cycle is stable, and it travels in counterclockwise sense because $a_2 = -1/2 < 0$ and $x_3 < y_2 < x_2$. Moreover the circle $y = y_3$ is unstable, because the Jacobian matrix of the system on the equilibrium points of this circle has the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = \sqrt{2/5}$. See Figure 10.

Finally we consider the following quadratic system on the cylinder

$$\dot{x} = -\frac{1}{2} + y + 2y^2, \qquad \dot{y} = 0. \tag{19}$$

We observed that $\dot{y} = 0$, therefore $\Pi(y_0) = 0$ for all $y_0 \in (0,1)$. Hence for $y_0 \in (0,1)$ every circle $y = y_0$ generates a periodic orbit on the cylinder. Hence system (19) has a continuum of periodic orbits on the cylinder. As $a_2 = 2 > 0$ and $x_2 < x_3$, we have the periodic orbit generates by $y = y_0$ travels clockwise (resp. counterclockwise) sense if $x_2 < y_0 < x_3$ (resp. $y_0 > x_3$). See Figure 11.

In summary the proof of the proposition is done. □

Acknowledgements

This work is partially supported by the Agencia Estatal de Investigación grant PID2019-104658GB-I00, the H2020 European Research Council grant MSCA-RISE-2017-777911, the Generalitat de Catalunya grant 2021 SGR 00113, and by the Acadèmia de Ciències i Arts de Barcelona.

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[1] Universidade Federal do Oeste da Bahia, 47600000 Bom Jesus da Lapa, Bahia, Brazil

Email address: `jjohanajimenez@gmail.com`

[2] Departament de Matematiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

Email address: `jaume.llibre@uab.cat`