TREBALL DE RECERCA

# STRUCTURAL STABILITY OF VECTOR FIELDS IN DIMENSION TWO

per

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# INTRODUCTION

In this "**Treball de recerca**" we deal essentially with the structural stability of planar vector fields. Many authors consider this problem for classes of vector fields on 2-dimensional manifolds.

First definition of structural stability for planar vector fields goes back to Andronov and Pontrjagin [**AP**], who in 1937 studied the structural stability for analytic vector fields on the closed 2-dimensional disk. Rouhgly speaking, we say that a vector field $X$ is structurally stable if its phase portrait is topologically equivalent (via a homeomorphism near to the identity map, and called equivalence homeomorphism) at the phase portrait of all its neighbours in a suitable topology.

In 1962 Peixoto [**P2**] extended this result characterizing the $C^1$-vector fields defined on a compact differentiable 2-manifold without boundary. Also, he showed under his assumptions that the requeriment that the equivalence homeomorphism lies in a pre-assigned neighborhood of identity is redundant. In 1982 Kotus, Krich and Nitecki [**KKN**] gave sufficient conditions in order that a $C^1$-vector field in an open differentiable 2-manifold, $N^2$, to be structurally stable. Furthermore they proved that these conditions are necessary if $N^2 = \mathbb{R}^2$. Unfortunately, they gave an example which shows that when the 2-manifold is open the requeriment that the equivalence homeomorphism lies near the identity is not redundant.

In 1987 Shafer [**S1**] considered the set of polynomial vector fields of degree $\leq n$ on $\mathbb{R}^2$ and gave sufficient conditions for structural stability when only polynomial perturbations are allowed. He proved that these conditions are necessary with an exception related with the hyperbolicity of limit cycles. In comparing Shafer's work with anothers on structural stability of polynomial vector fields (see for instance [**So2**]), it should be noted that, while he uses the Poincaré compactification of $\mathbb{R}^2$ he does not require that the compactified Poincaré vector fields $p(X)$ and $p(Y)$ (see Section 2.1) be equivalent on $S^2$ in order that $X$ and $Y$ to be equivalent.

Shafer in [**S2**] characterizes the planar gradient polynomial vector fields which are structurally stable with respect to perturbations in the set of all $C^r$ planar vector fields and in the set of all planar polynomial vector fields. Also he presents sufficient conditions for structural stability in the set of all planar gradient polynomial vector fields.

In this context we began, in 1990, the study of structural stability for planar Hamiltonian vector fields. First, we considered planar Hamiltonian polynomial vector fields and characterized who are structurally stable with respect to perturbations in the set of all planar $C^r$-vector fields (see Theorem 3.3.1), in the set of all planar polynomial vector fields (see Theorem 3.3.2) and in the set of all planar Hamiltonian polynomial vector fields (see Theorem 3.3.12). Second, we characterized the structural stability of planar $C^1$-Hamiltonian vector fields with respect to perturbations in the set of all planar $C^1$-vector fields (see Theorem 3.5.2), and using an idea of Andronov, Vitt and Khaikin [**AVK**] we can extended this characterization to all $C^1$-integrable planar vector fields (Corollary 3.5.3).

When one knows what are the necessary and sufficient conditions for structural stability of a class of vector fields on a 2-manifold it is important to know if these conditions are satisfied for a "big" subset of vector fields or are not (here "big" means open and dense). This problem is called the generic problem. We state the main results on genericity with respect structural stability on 2-manifolds and prove a generic theorem for the class of planar Hamiltonian polynomial vector fields (see Theorem 4.3.1).

This work is organized as follows. In the first chapter we give the principal theorems about structural stability of smooth (at least $C^1$) vector fields on 2-manifolds. We distinguish two different kind of 2-manifolds; closed 2-manifolds (that is, compact differentiable 2-manifolds without boundary, see Section 1.1) and open 2-manifolds (that is, open differentiable 2-manifolds without boundary, see Section 1.2) and we examine which are the fundamental differences.

In the second chapter we deal with the structural stability problem in the polynomial case. Here we introduce the Poincaré compactification which allows the study of the structural stability at infinity (Section 2.1). After we give theorems on the structural stability of planar polynomial vector fields when only polynomial perturbations are allowed (Section 2.2). In the last part of this chapter we present theorems for the structural stability of planar gradient polynomial vector fields with respect different kind of perturbations (Section 2.3).

In the first two chapters we do not give all the proofs of the theorems that we state, but we indicate the more important steps for proving the main results.

In Chapter 3 we study the structural stability of planar

Hamiltonian vector fields. First we prove some basic results about planar Hamiltonian vector fields that we will need later for studying their structural stability (Sections 3.1 and 3.2). In Section 3.3 we show three theorems for the structural stability of planar Hamiltonian polynomial vector fields; first with respect to planar smooth perturbations, second with respect to planar polynomial perturbations and third with respect to planar Hamiltonian polynomial perturbations. In this section and in Section 3.4 we also discused about the definition of structural stability. By using Theorem 3.3.12 we have a classification of the non-equivalent topological "canonical region" of planar Hamiltonian polynomial vector fields structurally stable with respect to perturbations in the set of all planar Hamiltonian polynomial vector fields, but this result is not included in this work. Finally, in Section 3.5 we prove a theorem of structural stability for planar $C^1$-Hamiltonian vector fields with respect to planar $C^1$ perturbations and generalize this one to $C^1$-planar integrable vector fields.

In the last chapter we deal with the generic problem. We state the main results and prove that the set of structurally stable planar Hamiltonian vector fields with respect to perturbations in the set of all planar Hamiltonian polynomial vector fields is generic.

Voldria agrair molt sincerament a tots aquells que m'han ajudat en aquest treball en especial a la Cori per la seva paciència. Tambe voldria agrair a en Jaume el seu esforç i dedicació sense el qual aquest treball no hagués estat possible.