Proceedings of the Edinburgh Mathematical Society (2022) **65**, 356–375
doi:10.1017/S0013091522000128

THE LOCAL CYCLICITY PROBLEM: MELNIKOV METHOD USING LYAPUNOV CONSTANTS

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(Received 24 March 2020; first published online 19 April 2022)

Abstract In 1991, Chicone and Jacobs showed the equivalence between the computation of the first-order Taylor developments of the Lyapunov constants and the developments of the first Melnikov function near a non-degenerate monodromic equilibrium point, in the study of limit cycles of small-amplitude bifurcating from a quadratic centre. We show that their proof is also valid for polynomial vector fields of any degree. This equivalence is used to provide a new lower bound for the local cyclicity of degree six polynomial vector fields, so $\mathcal{M}(6) \geq 44$. Moreover, we extend this equivalence to the piecewise polynomial class. Finally, we prove that $\mathcal{M}_p^c(4) \geq 43$ and $\mathcal{M}_p^c(5) \geq 65$.

Keywords: Melnikov theory; Lyapunov constants; local cyclicity

2020 Mathematics subject classification: Primary 34C07;
Secondary 34C23; 37C27

1. Introduction

In the last century, Hilbert presented a list of problems that almost all of them are solved. One problem that remains open consists in determining the maximal number $\mathcal{H}(n)$ of limit cycles, and their relative positions, of planar polynomial vector fields of degree n. This problem is known as the second part of the 16th Hilbert's problem. In the year of 1977, Arnol'd in [4] proposed a weakened version, focused on the study of the number of limit cycles bifurcating from the period annulus of Hamiltonian systems.

In this work, we are interested in another local version, that consists in providing the maximum number $\mathcal{M}(n)$ of small amplitude limit cycles bifurcating from an elementary centre or an elementary focus, clearly $\mathcal{M}(n) \leq \mathcal{H}(n)$. In other words, $\mathcal{M}(n)$ is an upper bound of the (local) cyclicity of such equilibrium points. For more details, see [49]. For $n = 2$, Bautin proved that $\mathcal{M}(2) = 3$, see [5]. For $n = 3$, the family of cubic systems