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24 crossing limit cycles in only one nest for piecewise cubic systems

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ABSTRACT

In this work, we are interested in crossing limit cycles surrounding only one equilibrium point or a sliding segment. The studied systems are piecewise cubic polynomial defined in two zones separated by a straight line. In this class, we get at least 24 crossing limit cycles, all of them in only one nest, bifurcating from a cubic polynomial center. The computations use a parallelization algorithm.

1. Introduction

Andronov, in [1], started the study of piecewise linear systems. In last years, this subject has been widely studied, since many problems of engineering, physics, and biology can be modeled by such systems, see [2,3]. Most of usual models propose piecewise differential systems defined in two half planes separated by a straight line. Recently, the study of the number of limit cycles have received a special attention, see for example [4–6]. In particular, it can be considered as a generalization of the 16th-Hilbert problem, see more details on this classical problem in [7].

In this paper, we are interested in the study of isolated periodic orbits, the so-called limit cycles, for piecewise differential equations of the form

$$\begin{cases} (x',y')=(P^{+}(x,y,\lambda),Q^{+}(x,y,\lambda)), \text{ when } y>0, \\ (x',y')=(P^{-}(x,y,\lambda),Q^{-}(x,y,\lambda)), \text{ when } y<0, \end{cases} \tag{1}$$

where $P^{\pm}(x,y,\lambda)$ and $Q^{\pm}(x,y,\lambda)$ are polynomials of degree n in (x,y) and $\lambda\in\mathbb{R}^K$, where K is the total number of parameters. It is not restrictive to consider the straight line $\Sigma=\{y=0\}$, it divides the real plane in two half-planes $\Sigma^{\pm}=\{(x,y):\pm y>0\}$, and the trajectories on Σ are defined following the Filippov convention, see [3]. We will consider only limit cycles of crossing type, that is, when both vector fields point out in the same direction in the intersection points with the discontinuity line Σ. We can define

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$H_p^c(n)$ as the maximum (when it is finite) number of crossing limit cycles of (1). When all crossing limit cycles bifurcate from a singular point we will denote such maximum by $M_p^c(n)$. Both definitions are the piecewise generalizations of the known as global and local Hilbert numbers $H(n)$ and $M(n)$.

It is well-known that linear systems have no limit cycles, so $H(1) = M(1) = 0$. This is not the case for piecewise linear systems defined in two zones separated by a straight line. Huan and Yang in [8] firstly showed a numerical evidence that $H_p^c(1) \geq 3$. In [6] Llibre and Ponce provide an analytical proof of this fact. Later, using the averaging bifurcation mechanism this lower bound was reobtained, [9]. Recently, also the same number was obtained in [10]. The three limit cycles in [10] are explained studying the full return map, two appear near the origin and the other one far from it. In fact, these two limit cycles, appearing from an equilibrium point, provide the lower bound $M_p^c(1) \geq 2$. This value can be proved with the results in [11]. We will show this in the next section.

For quadratic vector fields is also well known that $H(2) \geq 4$, see [12]. But for piecewise quadratic systems there are few works providing good lower bounds. Using averaging theory of order five, and perturbing the linear center, Llibre and Tang in [13] proved that $H_p^c(2) \geq 8$. Recently, da Cruz et al. in [14] provide a better lower bound, $H_p^c(2) \geq 16$. These limit cycles appear using also the averaging method but at order two and perturbing some quadratic isochronous systems. The new lower bound is quite surprising because is higher than what it can be expected a priori, that is doubling (because we have two vector fields) the value 4 obtained for usual quadratic vector fields.

The best known lower bound for the number of limit cycles in cubic systems is $H(3) \geq 13$, see [15]. For piecewise cubic polynomial vector fields a very recent work, see [16], provides $H_p^c(3) \geq 18$ in two nests of nine limit cycles each.

In this work we will prove that $H_p^c(3) \geq M_p^c(3) \geq 24$. Furthermore, all the crossing limit cycles are in the same nest, all surrounding a small enough sliding segment. Our approach is based in the degenerate Hopf bifurcation, studying small limit cycles appearing from an equilibrium point of center-focus type, see [17,18]. That is, through the computation of the coefficients (Lyapunov quantities) of the return map near the origin but for piecewise differential systems. As we consider the perturbation of centers, we need to compute only the linear parts of the Lyapunov quantities. This idea, using the Implicit Function Theorem, was stated by Chicone and Jacobs in [19] for an equivalent problem. Also [20,21] use such approach. In this paper we propose to extend it for piecewise vector fields. However, as the computations are quite hard, we use also the parallelization ideas introduced in [22]. But with more computation time this is not essential. Our main result is the following.

Theorem 1.1. *There exists a piecewise cubic polynomial vector field, defined in two zones separated by a straight line, exhibiting 24 small amplitude crossing limit cycles. That is $H_p^c(3) \geq M_p^c(3) \geq 24$.*

2. Degenerated Hopf and pseudo Hopf bifurcations

Let us consider the differential equation (1) in the usual polar coordinates, $(x, y) = (r\cos\theta, r\sin\theta)$. We will assume that, when $\lambda = 0$, system (1) has a non-degenerate center at the origin. That is, in polar coordinates the linear part of the Jacobian matrix has zero trace and positive determinant. Then, using the Filippov convention, we can write (1) as

$$\begin{cases} \dot{r} = R^+(r,\theta,\lambda), & \theta \in [0,\pi], \\ \dot{r} = R^-(r,\theta,\lambda), & \theta \in [\pi,2\pi], \end{cases} \tag{2}$$

where the dot represents the derivative with respect to θ. When the Taylor series of $R^\pm$ in (2), with respect to r, have terms of order higher or equal than 2, we can write the solution, satisfying $r^+(0,\rho,\lambda) = \rho$, as $r^+(\theta,\rho,\lambda) = \rho + \sum_{k=2}^{\infty} v_k^+(\theta,\lambda)\rho^k$. It is defined in $\theta \in [0,\pi]$. Equivalently, we can write $r^-(\theta,\rho,\lambda) =$

$\rho+\sum_{k=2}^{\infty} v_k^-(\theta,\lambda)\rho^k$ for the solution of (2) in $\theta \in [\pi, 2\pi]$ such that $r^-(0,\rho,\lambda)=\rho$. Consequently, the Poincaré half-return maps will be $\Pi^{\pm}(\rho,\lambda)=r^{\pm}(\pm\pi,\rho,\lambda)$. As usual, instead of the complete Poincaré map obtained by composition, we define it by the difference map

$$\Delta(\rho,\lambda)=\Pi^+(\rho,\lambda)-\Pi^-(\rho,\lambda)=\sum_{k=2}^{\infty} L_k(\lambda)\rho^k. \tag{3}$$

The coefficients L_k are known as the Lyapunov quantities associated to system (2). Under the above conditions, when we take $\lambda = 0$ the unperturbed system has a center and $L_k(0) = 0$ for every k. Consequently, for the perturbed system, the first nonvanishing L_K provides the stability of the origin. In this case we say that the origin is a generalized weak focus of order K. We have followed the classical Lyapunov algorithm scheme. For more details we refer the reader to [17]. As the usual Lyapunov quantities, see [23], to solve the analytic center-focus problem for non degenerate centers, L_k are polynomials in the parameters λ with rational coefficients. Moreover, they are also defined when the previous vanish. The main difference in (3) between the analytic study versus the piecewise one is the fact that in the first, the Lyapunov quantities with even indices are zero while in the second not. In both cases, as we are perturbing a center, the expressions of L_K are polynomials that vanish at $\lambda = 0$. Then, we compute the Taylor series of L_k of order 1 with respect to λ, $L_k^{(1)}$, and we consider $A_K(\lambda)$ the matrix of coefficients of $L_k^{(1)}$, for $k = 2,\dots,K$. Then, if the rank of A_K is $K-1$, the Implicit Function Theorem provides $K-1$ hyperbolic limit cycles. Because, the first $K-1$ coefficients in (3) are independent.

Usually, the computation of the Lyapunov quantities needs a hard effort and high memory computers. In our approach, only the expressions of their linear parts are necessary to be computed. As in the analytic scenario, a parallelization algorithm can be used. We have used the procedure developed by Liang and Torregrosa in [22]. Without it, the involved computations are impossible to be obtained.

For analytic vector fields with a weak focus of order 1 at the origin only one limit cycle bifurcate from the origin using the trace parameter. This phenomenon is the classical Hopf bifurcation. See more details also in [17]. Next result is the generalization of this property for piecewise analytic, where two crossing limit cycles appear. This result follows from the study of the return map near the origin given in [11].

Proposition 2.1. *Consider the perturbed system*

$$\begin{cases} \dot{x} = -(1+c^2)y + \sum_{k+\ell=2}^{3} a_{k\ell}^+ x^k y^\ell, \\ \dot{y} = x + 2cy + \sum_{k+\ell=2}^{3} b_{k\ell}^+ x^k y^\ell, \end{cases} \qquad \begin{cases} \dot{x} = -y + \sum_{k+\ell=2}^{3} a_{k\ell}^- x^k y^\ell, \\ \dot{y} = d + x + \sum_{k+\ell=2}^{3} b_{k\ell}^- x^k y^\ell, \end{cases} \tag{4}$$

for $y \geq 0$ and $y < 0$, respectively. If $a_{11}^+ - a_{11}^- + 2(b_{02}^+ - b_{02}^-) + b_{20}^+ - b_{20}^- \neq 0$ then there exist c and d small enough such that two crossing limit cycles bifurcate from the origin.

Proof. When $c = d = 0$, from the condition given in the statement, the origin is stable or unstable because the first nonvanishing coefficient in (3) is

$$L_2 = \frac{2}{3}\left(a_{11}^+ - a_{11}^- + 2(b_{02}^+ - b_{02}^-) + b_{20}^+ - b_{20}^-\right).$$

So, for c, d small enough, computing the return map as in the proof of Proposition 7.3 of [11] we can write, for $x \geq 0$,

$$\Delta(x) = \Pi(x) - x = \Delta_0(c,d) + \Delta_1(c,d)x + \Delta_2(c,d)x^2 + \cdots,$$

where $\Delta_0(c,d) = d$, $\Delta_1(c,0) = e^{\pi c} - 1$, and $\Delta_2(0,0) = L_2$. As c and d are arbitrary parameters, two crossing limit cycles bifurcate from the origin. □

An alternative way to get the same bifurcation, see [14], is to obtain a first crossing limit cycle assuming that there is no sliding segment, $\Delta(0)=0$, computing the Δ_1 and Δ_2 and checking that they have opposite sign. The second can be obtained adding a sliding segment with an adequate stability. The first bifurcation was showed in [24] where the return map of a focus-focus point is studied without the existence of a sliding segment. The second mechanism is presented as a pseudo-Hopf bifurcation in [25].

3. Degenerated Hopf bifurcation in piecewise cubic systems

This section is devoted to prove our main theorem.

Theorem 3.1. *Consider the perturbed system of the form (1)*

$$\begin{cases} (\dot{x},\dot{y}) = \left(P_c(x,y) + \sum_{k+\ell=2}^{3} a_{k\ell}^{+}\, x^k y^\ell, Q_c(x,y) + \sum_{k+\ell=2}^{3} b_{k\ell}^{+}\, x^k y^\ell\right) \text{ for } y>0, \\ (\dot{x},\dot{y}) = \left(P_c(x,y) + \sum_{k+\ell=2}^{3} a_{k\ell}^{-}\, x^k y^\ell, Q_c(x,y) + \sum_{k+\ell=2}^{3} b_{k\ell}^{-}\, x^k y^\ell\right) \text{ for } y<0, \end{cases} \tag{5}$$

where $P_c(x,y)=-y(1-68x+1183x^2)$ and $Q_c(x,y)=x-58x^2-44xy+30y^2+672x^3+1484x^2y-945xy^2-84y^3$. Then, there exist small enough values of the parameters $a_{k\ell}^{\pm}, b_{k\ell}^{\pm}$ such that the above system has at least 22 hyperbolic crossing limit cycles bifurcating from the origin.

Proof. The origin of the unperturbed system (5) is a Darboux center with the rational first integral

$$H(x,y)=\frac{(42x-7y-1)^3 h(x,y)}{(448x^2+336xy+63y^2-44x-12y+1)^3(1183x^2-68x+1)},$$

where $h(x,y)=10752x^3+29568x^2y+17640xy^2+3024y^3-1600x^2-2760xy-576y^2+74x+57y-1$. Note that it is well defined at the origin. With the perturbation in system (5) the difference map near the origin writes as (3). Straightforward computations show that the linear terms of the first 23 Lyapunov quantities, denoted by $L_2^{(1)},\ldots,L_{24}^{(1)}$, are linearly independent, then, using the Implicit Function Theorem, there are small enough values of the parameters in (5) such that the difference map (3) has at least 22 simple zeros. They correspond to 22 hyperbolic crossing limit cycles for (5) and the statement follows. The expressions of the first linear parts are

$$\begin{aligned} L_2^{(1)} &= -\frac{2}{3}((a_{11}^+-a_{11}^-)+2(b_{02}^+-b_{02}^-)+(b_{20}^+-b_{20}^-)),\\ L_3^{(1)} &= -\frac{\pi}{8}(128(a_{02}^++a_{02}^-)+184(a_{20}^++a_{20}^-)+(a_{12}^++a_{12}^-)+3(a_{30}^++a_{30}^-)\\ &\quad +44(b_{02}^++b_{02}^-)+28(b_{11}^++b_{11}^-)+44(b_{20}^++b_{20}^-)+(b_{21}^++b_{21}^-)+3(b_{03}^++b_{03}^-)),\\ L_4^{(1)} &= \frac{2}{45}(4856(a_{02}^+-a_{02}^-)-109135(a_{11}^+-a_{11}^-)-10796(a_{20}^+-a_{20}^-)-768(a_{03}^+-a_{03}^-)\\ &\quad -372(a_{21}^+-a_{21}^-)+88(a_{12}^+-a_{12}^-)-132(a_{30}^+-a_{30}^-)-200862(b_{02}^+-b_{02}^-)\\ &\quad -3920(b_{11}^+-b_{11}^-)-108128(b_{20}^+-b_{20}^-)+12(b_{12}^+-b_{12}^-)-176(b_{21}^+-b_{21}^-)-102(b_{30}^+-b_{30}^-)). \end{aligned}$$

We do not show here the complete expressions because of their size. In fact, the rational coefficients appearing in the used expressions have numerators and denominators with more than one hundred digits each. □

Proof of Theorem 1.1. The perturbation considered in (5) has no constant nor linear terms, then the 22 hyperbolic crossing limit cycles, using Theorem 1.1, bifurcate from the origin as the difference map $\Delta(x)$ starts with terms of order at least 2. In fact, after these limit cycles bifurcate we have that $L_2 \neq 0$, Then, as the perturbation (4) add linear and constant terms and the 22 limit cycles are hyperbolic, the 2 limit cycles bifurcating, using Proposition 2.1, from the origin are different, having in total 24. This finishes the proof. □

We observe that using only linear parts we have no more crossing limit cycles because the next Lyapunov quantities are linearly dependent with respect to the first. The difficulties in the computations do not allow us to study higher order developments to get more limit cycles, if they exist.

As we have mentioned above, the described approach to find a lower bound for the cyclicity of the origin from the fact that linear parts provide a complete unfolding of K small limit cycles, adding the trace parameter, when the rank is K, follows from [19–21]. This approach, in the analytic scenario and for the unperturbed center (5), was followed in [26] to provide another explicit example with 11 small limit cycles in polynomial cubic vector fields. This is because in (3) only odd terms appear in the unfolding. What is relevant here is that the same center provides, in the piecewise scenario, more than double crossing limit cycles, in fact $2(K+1)$.

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