Universitat Autònoma de Barcelona

# Limit cycles of small amplitude in polynomial and piecewise polynomial planar vector fields

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Bellaterra

December 2019

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Certifico que aquesta memòria ha estat realitzada per en Luiz Fernando da Silva Gouveia sota la meva supervisió i que constitueix la seva tesi per a aspirar al grau de Doctor en Matemàtiques per la Universitat Autònoma de Barcelona.

Bellaterra, Desembre 2019

Joan Torregrosa Arús

Dedico aos meus pais Alfredo Gouveia e Maria Teresa

# Contents

# Acknowledgments

Primeiramente agradeço as pessoas mais importantes na minha vida, meu pai Alfredo Gouveia e minha mãe Maria Teresa, vocês são os pilares de nossa família, tudo o que temos de bom, veio de vocês. Não menos importante, agradeço aos meus irmão Paulo e Rui por todo o apoio e conselho nas horas de pânico e por sempre estarem ao meu lado, amo muito todos vocês. Agradeço a minha cunhada Marina e a minha cunhada Talita. Não poderia deixar de fora as duas joias da minha vida, Duda e Amaya, o tio ama vocês demais e ficar longe de vocês todo esse tempo foi difícil. Um obrigado muito especial a minha namorada Ana Livia. Muito obrigado por segurar a barra tanto tempo no relacionamento a distancia, obrigado pelo amor, carinho, amizade, companherismo. Muito obrigado por tudo. Obrigado família.

Um obrigado muito especial a minha tia Lucinda, que sempre se preocupou comigo e a Mariana que conheci esse ano e é gente boa demais.

Um obrigado também para o meu Padrinho, a minha Madrinha, a outra Madrinha, ao Joãozinho, a Rose, ao Ulisses, a minha vó e a Letícia.

Um obrigado especial a família Rodero e aos defumados Rodero que me acolheu muito bem e me alimentou com comidas gostosas. Obrigado Dona Ana, seu Pedro, Gabriel, Lord Caju e Lori.

Um obrigado também a meus amigos da época de escola, Guilherme e Domingos.

Durante essa caminhada, várias pessoas foram de alguma forma importantes. Entre elas, o pessoal do grupo UVA vs Stars, Gabriel, Fat, Kaio, Giordano e Julio. Valeu galera pelas brigas e pelas risadas.

Um agradecimento muito especial para dois caras que são de suma importância na minha vida: Daniel e Kamael. Vocês me ajudaram demais a segurar a barra aqui em Barcelona, sem vocês também, eu não teria conseguido. Muito obrigado por tudo.

Agradeço também ao grupo dos Gordinhos, Pedro Bolota, Giane, Ana Maria japonesa por parte de irmã, Monisse e Dona Redonda que está sempre pre. . . parada. Em especial, agradeço ao corinthiano maloqueiro sofredor Leonardo. Há anos trilhamos juntos os caminhos dinâmicos, obrigado por tudo.

Não poderia deixar de fora as melhores companheiras de casa Julia, Gabi e Tainã. Com vocês, a vida em Barcelona era muito mais leve. Espero que vocês tenham aprendido a cozinhar temperando o fogão também, que é algo essencial. Em especial, queria agradecer a Tainã que por 3 anos foi uma big sister para mim.

Quero agradecer também aos amigos que fiz aqui. Alex, Natalia, Gustavo, e a prima Bruna, pois temos uma tia em comum em Paris. Um obrigado especial para o Murilo que neste final tem me dado muito suporte.

Grupos de whatssap não faltam, mais o melhor grupo dinâmico do mundo é a Bagunça Dinâmica. Obrigado Kamilinha, Anna, Thaisinha e Otavio, no qual espero um dia ter uma discussão acadêmica que voe cadeiras hahahahaha.

Um obrigado muito especial aos melhores jogadores de Pubg do mundo: André, André the boot, Lady Carol Rayovac, Kucheatermon, Jessica e Will. Valeu também Junior, Pão, Otavio e Helo. Em especial, obrigado Lucas por todo o apoio e preocupação que você tem comigo, pode ter certeza que você me ajudou em muitos momentos tensos.

Não poderia deixar de fora dois caras sensacionais, Jarne e Yagor. Mano, vocês são os caras, muito obrigado por tudo, em especial ao chupeta do Yagor que por um ano inteiro me aguentou e foi parceiro demais.

Quero deixar um agradecimento especial, ao Prof. Dr. Cladio A. Buzzi. Desde a graduação, me orientando e ajudando sempre que podia. Se hoje estou terminando o doutorado, muito é por causa do senhor.

Mesmo não merecendo ultimamente, quero agradecer ao São Paulo por ter me dado muitas alegrias. Você é o meu time do coração e por mais que eu diga que não vou assistir mais aos jogos, eu não aguento, sou louco por ti São Paulo.

Um muito obrigado aos computadores do departamento chamados carinhosamente de antz, sem vocês minha tese não existiria.

Um obrigado também a minha psiquiatra Dra. Natalie Bifano e a minha psicologa Meire...vocês me ajudaram demais nessa etapa. Sem vocês, dificilmente eu teria conseguido concluir essa etapa.

No puc evitar donar les gràcies a una de les persones d'enorme importància de la meva vida que és el Prof. Dr. Joan Torregrosa. Vostè és molt més que un tutor de tesi per a mi, ets un amic i un mentor amb el que espero compartir molts bons moments al llarg de les nostres vides. Moltes gràcies pels ensenyaments, consells i amistat.

També vull agraïr als professors del grup de sistemes dinàmics, especialment a Prof. Dr. Jaume Giné, Prof. Dr. Armengol Gasull i Prof. Dr. Francesc Mañosas, que hagin discutit diverses hores amb nosaltres ajudant-nos en el desenvolupament de la tesi.

Moltes gràcies també al personal de la secretaria, i de neteja, sou persones molt importants que faciliten molt la nostra feina.

Por fim, gostaria de agradecer ao CNPq 200484/2015-0 por todo o apoio financeiro nestes últimos 4 anos.

## Summary

David Hilbert in the year 1900, in the International Congress of Mathematics proposed 23 problems that in his opinion would motivate advances in mathematics during the 20th century. Among these problems, one is linked with the study of ordinary differential equations. The 16th Hilbert problem, whose second part asking about the maximum number and the relative position of the isolated periodic orbits, also called limit cycles, of a planar polynomial system in function of its degree $n$. Until nowadays, the 16th Hilbert problem remain unsolved. Over the years and without one solution, weaker versions began to emerge to 16th Hilbert problem. We are interested here in one of them, that consist in to provide the maximum number $M(n)$ of small-amplitude limit cycles bifurcating from an elementary center or an elementary weak-focus.

In order to help to solve this problem, our contribution in this thesis is offer a mechanism that simplifies the calculation of the Taylor developments of the Lyapunov constants and to present a theory that help us to use the constants obtained for classical differential system to the study of lower bounds for the value $M(n)$. We dedicate part of this work to study the same problem to piecewise systems. In this work, we consider fixed vector fields and we present the parallelization tool that will help us to calculate high order Taylor developments of Lyapunov constants near a center different from the linear one and get some results about how to obtain limit cycles using these developments. Moreover, we consider a family of vector fields and we present a result that allows us to get $k$ extra limit cycles if the unperturbed system has a center having $k$ free parameters. For piecewise systems, we consider again fixed vector fields and using parallelization, we were able to calculate the necessary Lyapunov constants for cubic and quartic systems to improve lower bounds of limit cycles. We prove that $M(3)$ and $M(4)$ are bigger than or equal to 12 and 21, respectively. Moreover, we prove that if an analytic piecewise system has weak-focus or order $2n+1$, we can unfold the total number of limit cycles perturbing in the analytic piecewise class. This result is a natural extension of the classical result showed by Andronov for analytic systems. Moreover, using the equivalence among Lyapunov constants and Melnikov functions, we improve also the lower bounds for the known values of the local cyclicity for sextic vector fields.

## Resum

David Hilbert l'any 1900, al Congrés Internacional de Matemàtiques va proposar 23 problemes que, segons el seu parer, motivarien els avenços en matemàtiques durant el segle XX. Entre aquests problemes, n'hi ha un apareix en l'estudi de les equacions diferencials ordinàries. El 16è problema de Hilbert, la segona part del qual pregunta pel nombre màxim i la posició relativa de les òrbites periòdiques aïllades, també anomenats cicles límit, d'un sistema polinòmial al pla en funció del seu grau $n$. Fins a l'actualitat, el 16è problema de Hilbert segueix essent un problema obert. Amb els anys i sense una solució, van començar a aparèixer versions més dèbils al 16è problema de Hilbert. Aquí ens interessa una d'elles, que consisteix en proporcionar el màxim nombre $M(n)$ de cicles límit d'amplitud petita que es bifurquen des d'un centre o focus dèbil elementals.

Per ajudar a resoldre aquest problema, la nostra contribució en aquesta tesi és oferir un mecanisme que simplifiqui el càlcul dels desenvolupaments de Taylor de les constants de Lyapunov i presentar una teoria que ens ajudi a utilitzar les constants obtingudes per un sistema diferencial clàssic per estudiar cotes inferiors del valor de $M(n)$. Dediquem part d'aquest treball a estudiar el mateix problema als sistemes definits a trossos. En aquest treball, considerem els camps vectorials fixos i presentem l'eina de paral·lelització que ens ajudarà a calcular els desenvolupaments de Taylor d'ordre alt de les constants de Lyapunov prop d'un centre no lineal i obtenir resultats sobre com obtenir cicles límit mitjançant aquests desenvolupaments. A més, considerem una família de camps vectorials i presentem un resultat que ens permet obtenir addicionalment $k$ cicles límit si el sistema no perturbat té un centre amb k paràmetres lliures. Per als sistemes a trossos, es consideren de nou els camps vectorials fixos i mitjançant la paral·lelització, s'han pogut calcular les constants Lyapunov necessàries de tal forma que per a sistemes cúbics i quàrics es milloren les cotes inferiors pel nuúmero de cicles límit d'amplitud petita. Provem que $M(3)$ i $M(4)$ són més grans o iguals a 12 i 21, respectivament. A més a més, demostrem que si un sistema analític a trossos té un focus feble d'ordre $2n+1$, podem desplegar el nombre total de cicles límit pertorbant dins la classe de camps analítics definits a trossos. Aquest resultat és una extensió natural del resultat clàssic mostrat per Andronov per als sistemes analítics. A més, utilitzant l'equivalència entre les constants de Lyapunov i les funcions de Melnikov, millorem també la cota inferior de la ciclicitat local per a camps vectorials de grau sis.

# Resumo

David Hilbert, no ano de 1900, no Congresso Internacional de Matemática propôs 23 problemas que, em sua opinião, motivariam os avanços da matemática durante o século XX. Entre esses problemas, um está relacionado ao estudo de equações diferenciais ordinárias. O 16º problema de Hilbert, cuja segunda parte consiste em determinar o número máximo e a posição das órbitas periódicas isoladas, também denominadas de ciclos limites, de um sistema polinomial planar em função de seu grau $n$. Até hoje, o 16º problema de Hilbert permanece sem solução. Ao longo dos anos e sem uma solução, versões mais fracas começaram a surgir. Estamos interessados aqui em uma delas, que consiste em fornecer o número máximo $M(n)$ de ciclos limites de pequena amplitude que se bifurcam a partir de um centro elementar ou de um foco fraco elementar.

Para ajudar a resolver esse problema, nossa contribuição nesta tese é oferecer um mecanismo que simplifique o cálculo das expansões de Taylor das constantes de Lyapunov e apresentar uma teoria que nos ajude a usar as constantes obtidas para obter os ciclos limite. Dedicamos parte deste trabalho ao estudo do mesmo problema em sistemas por partes. Neste trabalho, consideramos campos vetoriais fixos e apresentamos a ferramenta de paralelização que nos ajudará a calcular as constantes Lyapunov. Além disso, consideramos uma família de campos vetoriais e apresentamos um resultado que nos permite obter $k$ ciclos-limite extras se o sistema não perturbado tiver um centro com $k$ parâmetros livres e mostramos que $M(3) \geq 12$ e $M(4) \geq 21$ . Para sistemas por partes, consideramos novamente os campos vetoriais fixos e, usando a paralelização, conseguimos calcular as constantes Lyapunov necessárias para sistemas cúbicos e quárticos para melhorar a cota inferior de ciclos limites para campos de grau 3 e 4. Além disso, provamos que, se um sistema analítico por partes tem um foco fraco ou ordem $2n+1$, podemos obter ao menos $2n+1$ ciclos limites. Este resultado é uma extensão natural do resultado clássico mostrado por Andronov para sistemas analíticos. Além disso, usando a equivalência entre constantes de Lyapunov e funções de Melnikov, mostramos que $M(6) \geq 44$.

## Resumen

David Hilbert en el año 1900, en el Congreso Internacional de Matemáticas, propuso 23 problemas que, en su opinión, motivarían los avances en matemáticas durante el siglo XX. Entre estos problemas, uno está relacionado con el estudio de las ecuaciones diferenciales ordinarias. El problema 16 de Hilbert, cuya segunda parte pregunta por el número máximo y la posición relativa de las órbitas periódicas aisladas, también conocidas como ciclos límite, de un sistema polinomial plano en función de su grado $n$. A día de hoy, el problema número 16 de Hilbert sigue sin resolverse. Con el paso de los años y sin una solución, han surgido versiones más débiles. Aquí estamos interesados en una de ellas, que consiste en proporcionar el número máximo $M(n)$ de ciclos límite de amplitud pequeña que bifurcan desde un centro o un foco débil elementales.

Para ayudar a resolver este problema, nuestra contribución en esta tesis es ofrecer un mecanismo que simplifique el cálculo de los desarrollos de Taylor de las constantes de Lyapunov y presentar una teoría que nos ayude a usar las constantes obtenidas para el sistema diferencial clásico para estudiar nuevas cotas inferiores para $M(n)$. Dedicamos parte de este trabajo a estudiar el mismo problema en los sistemas polinomials definidos a trozos. En este trabajo, consideramos campos de vectores fijos y presentamos la herramienta de paralelización que nos ayudará a calcular desarrollos de Taylor de alto orden para las constantes de Lyapunov cerca de un centro no lineal y obtener algunos resultados sobre cómo obtener ciclos límite utilizando estos desarrollos. Además, para una familia de campos vectoriales, presentamos un resultado que nos permite obtener $k$ ciclos límite adicionales si el sistema no perturbado tiene un centro que tiene $k$ parámetros libres. Para los sistemas definidos a trozos, consideramos nuevamente campos vectoriales fijos y, usando la paralelización, podemos calcular las constantes de Lyapunov necesarias para, en los sistemas cúbicos y cuárticos, mejorar las cotas inferiores conocidas para el número de ciclos límite de pequeña amplitud. Probamos que $M(3)$ y $M(4)$ son mayores o iguales que 12 y 21, respectivamente. Además, demostramos que si un sistema analítico a trozos tiene un foco débil de orden $2n+1$, podemos desplegar el número total de ciclos límite perturbando en la clase de campos analíticos definidos en dos zonas. Este resultado es una extensión natural del resultado clásico mostrado por Andronov para sistemas analíticos. Además, utilizando la equivalencia entre las constantes de Lyapunov y

las funciones de Melnikov, mejoramos también las cotas inferiores para la ciclicidad local en los campos polinomiales de grado seis.

# Introduction

From the moment that the human started to be aware of the natural events around him, the humanity sought to understand such events. In addition, it also find a way to predict them. Perhaps the most basic problem representing these phenomena is the rain cycle. Trying to understand the period of greatest rainfall would be useful to have a better and bigger planting. Taking this into account, the mathematics is, without any doubt, the basic language that describes natural events. For those who have faith, the mathematics is the language that God used to create the universe and its laws. There is evidence that mathematics started around 1900 BC and until nowadays there is no signs that it is near its end. In the 17th century, Isaac Newton and Gottfried Leibniz introduced the differential calculus. With this new approach, some phenomena of nature started to gain greater understanding, because it proved to be an important tool to model, in an abstract language, what occurs in the real world over of time. This created the pillars of what would be the study of ordinary differential equations.

We can write ordinary differential equations in the form

$$F(t, x, x', x'', \dots, x^{(n)}) = 0, \tag{1}$$

where $x^{(n)}$ denote the $n-$th derivative of $x$ with respect to $t$. When $F$ not depends of $t$, we say that system is autonomous. If $x$ is a vector instead of a real function, equation (1) is called a differential system. Many problems can be modeled by ordinary differential equations. We can cite the problem of $n$-bodies that was modeled by Newton, the problem prey-predator modeled by Vito Volterra and Alfred Lotka in 1925.

Almost two centuries later, the study of differential system gets a new approach with Henry Poincaré in "Mémoire sur les courbes définies par une équation différentielle". Here, Poincaré introduces a more qualitative study on ordinary differential equations. Using geometric and topological techniques, Poincaré was able to investigate qualitative properties of the solutions of a differential equation without such solutions having to be determined explicitly. Among the contributions of Poincaré, we can mention the concept of phase portrait, the concepts such as return map or the Annular Region Theorem, which are fundamental for classifying orbits with particular behaviors. These results would be the pillars of **Qualitative Theory of Differential Equations**.

The notion of limit cycle was also introduced by Poincaré that defines a limit cycle as a periodic orbit such that at least one trajectory of the vector field, approaches in positive or negative time. Usually, when the vector field is of class $\mathcal{C}^1$ an alternative definition is given. A limit cycle is a closed orbit isolated from the other periodic orbits. Years later, in the early twentieth century, a swedish mathematician named Ivar Otto Bendixson presents a result showing that the principal solutions are called singular or minimal sets (critical points, periodic orbits and separatrix) defined a differential equation on a compact set has the property that the other solution goes to a singular solution. This results would come to be known as Poincaré–Bendixon Theorem. Stimulate by this result, Lyapunov studied the behavior of solutions in a neighborhood of an equilibrium position. Because of his work, Lyapunov is will know as the founder the modern theory of stability of motion.

In this work, let is consider a first-order autonomous planar differential systems in the form

$$\begin{cases} \dot{x} &= X(x(t), y(t)), \\ \dot{y} &= Y(x(t), y(t)), \end{cases} \tag{2}$$

where $x(t), y(t), X(x, y)$ and $Y(x, y)$ are real functions and the dot means the derivative with respect to the time $t$.

David Hilbert in the year 1900, in the International Congress of Mathematics proposed 23 problems that in his opinion would motivate advances in mathematics during the 20th century. Among these problems, one is linked to the study of differential equations. The 16th Hilbert problem, whose second part asking about the maximum number (by convention this number is called $H(n)$) and the position of the limit cycles of a polynomial planar system in function of its degree, that is, a system like (2) with $X$ and $Y$ polynomials of degree $n$. Until nowadays, the 16th Hilbert problem remain unsolved, even for the simplest case $n = 2$.

Henri Dulac, in 1923 took the first steps in the direction of 16th Hilbert problem. His work goes in the direction of proving the finitude of the number of limit cycles in a polynomial vector field in the plane. In 1970, Yulij Ilyashenko observed that the proof given by Dulac was false. Some years later and independently, Ilyashenko and Écalle provided a correct proof. Although the proof given by Dulac was wrong, the ideas given by him were very fruitful and generated results like the classical Dulac Theorem and its generalization, known as the Bendixon–Dulac Theorem.

During the last decades many mathematicians have contributed to better understand 16th Hilbert problem. We highlight the works of A. Andronov, C. Christopher, F. Dumortier, J. Écalle, J.P. Françoise, A. Gasull, J. Giné, Y. Ilyashenko, J. Llibre C. Li, M. Peixoto, R. Roussarie, J. Sotomayor, J. Torregrosa, A. Varchenko, Y. Ye, Z. Zhang, H. Zoladek.

Over the years and without one solution, weaker versions began to emerge to 16th Hilbert problem. One of them is the so-called Arnold–Hilbert problem, however it is still unsolved. Arnold–Hilbert problem says that if $H$, $P$ and $Q$ be polynomials of degree $n$ and $V$ an inverse integrating factor, given $\Gamma(h)$ a level curve $\{H(x,y)=h\}$ of the system

$$\begin{cases} \dot{x} &=& -\dfrac{\partial H}{\partial y} + \varepsilon P(x,y,\varepsilon,\lambda), \\ \dot{y} &=& \dfrac{\partial H}{\partial x} + \varepsilon Q(x,y,\varepsilon,\lambda), \end{cases}$$

and given

$$\mathcal{M}(h) = \int_{\Gamma(h)} \frac{Q(x,y,0,\lambda)dx - P(x,y,0,\lambda)dy}{V(x,y)},$$

what is the number of zeros of $\mathcal{M}(h)$? The function $\mathcal{M}(h)$ is known as Abelian integral or Melnikov's function. The maximum number of simple zeros of $\mathcal{M}(h)$ is also closed to two related problems: the highest multiplicity of a weak-focus and the maximal cyclicity (the maximum number $M(n)$ of small limit that we get from an equilibrium point by a given polynomial perturbation) of an equilibrium point. Clearly $M(n) \leq H(n)$. In this work, we are interested in this version of the problem. For $n=2$, Bautin proved that $M(2)=3$. Sibirskii proved that for cubic systems without quadratic terms there are no more than five limit cycles bifurcating from one critical point. In fact these are the unique general families for which this local number is completely determined. The first evidence that $M(3) \geq 11$ was presented by Zoladek in 1995. Recently, Giné, conjectures that $M(n) = n^2 + 3n - 7$. This suggests a high value for $M(n)$ for polynomial vector fields of lower degree. For degree $n = 5, 7, 8, 9$ the best lower bounds for $M(n)$ until now were obtained by Liang and Torregrosa providing examples exhibiting 28, 54, 70, and 88 limit cycles of small amplitude, respectively.

For the reader to get an idea of the difficulty to solve 16th Hilbert problem, there is another version more restricted, which consists in determining the number $H(n)$ but for the Liénard family

$$\begin{cases} \dot{x} &=& y - F(x), \\ \dot{y} &=& -x, \end{cases}$$

where $F$ is a real polynomial of degree $n$ and $F(0)=0$. This weaker version is still unsolved.

One way to approach Arnold–Hilbert problem is using Lyapunov constants. From the study of the return map, Liapunov consider the importance of the terms of the series expansion of this application. The problem with this approach, is the difficult of calculations. In order to help to solve this problem, our contribution in this thesis is offer a mechanism that simplifies the calculation of the Taylor developments of the Lyapunov constants and to present a theory that help us to use the constants obtained for classical differential system to study lower bounds for $M(n)$. Moreover, we improve the known values of $M(n)$ for $3 \leq n \leq 9$.

We dedicate a part of this work to study the Arnold–Hilbert problem to piecewise systems. The study of piecewise linear systems started by Andronov and has been widely studied in the last years, since many problems of engineering, physics, economy and biology can be modeled by such systems. One of the most studied problem is given by two vector fields defined in two hapf-planes separated by a straight line. Moreover, a large set of classical theorems are not satisfied by the piecewise systems. Among others, we can cite the Existence and Uniqueness Theorem and the Poincaré–Bendixson Theorem.

In this work, we are interested in the study of limit cycles of small amplitude bifurcating from the origin, for piecewise differential equations of the form

$$\begin{cases} (x', y') = (P^+(x,y,\lambda), Q^+(x,y,\lambda)), \text{ when } y \geq 0, \\ (x', y') = (P^-(x,y,\lambda), Q^-(x,y,\lambda)), \text{ when } y < 0, \end{cases}$$

with $P^{\pm}(x,y,\lambda)$ and $Q^{\pm}(x,y,\lambda)$ are polynomials. The straight line $\Sigma = \{y = 0\}$ divides the plane in two half-planes $\Sigma^{\pm} = \{(x,y) : \pm y > 0\}$ and the trajectories on $\Sigma$ are defined following the Filippov convention. We call of $M_p^c(n)$ the maximum number of limit cycles bifurcating from a monodromic singular point and $H_p^c(n)$ the maximum number of limit cycles of polynomial piecewise systems of degree $n$. Clearly $M_p^c(n) \leq H_p^c(n)$. It is well-know that linear systems have no limit cycles, so $H(1) = M(1) = 0$. This is not the case for piecewise linear systems defined in two zones separated by a straight line. There are works showing $H_p^c(1) \geq 3$. For quadratic vector fields is also well known that $H(2) \geq 4$. But for piecewise quadratic systems there are few works providing good lower bounds. Using averaging theory of order five, and perturbing the linear center, Llibre and Tang in proved that $H_p^c(2) \geq M_p^c(2) \geq 8$. Recently, da Cruz, Novaes and Torregrosa provide a better lower bound, $H_p^c(2) \geq M_p^c(2) \geq 16$. The best known lower bound for the number of limit cycles in cubic systems is $H(3) \geq 13$,. For piecewise cubics a recent work provides $H_p^c(3) \geq 18$ in two nests of nine limit cycles each.

The work has been developed in collaboration with Joan Torregrosa, and it is structured in an introduction and then four chapters where the results and proofs are developed. As it is explained in the title, the main results are concerning to limit cycles of small amplitude for differential and piecewise differential systems in the plane.

In Chapter 1, considering fixed vector fields, we present the concept of Lyapunov constants, the Parallelization tool that will help us to calculate high order Taylor developments of Lyapunov constants near a center different from the linear one and get some results about how to obtain limit cycles using these equations. With these tools, we present a new cubic system having also 11 limit cycles of small amplitude and we have improved the valuee for $M(n)$ for $4, 5, 7, 8$ and 9.

In Chapter 2, considering a family of vector fields, we present a theorem that allows us to get $k$ extra limit cycles if the unperturbed system has a center having $k$ free parameters. Using this result, we show that $M(3) \geq 12$ and $M(4) \geq 21$. We present also two new families of cubic vector fields such that $M(3) \geq 11$. This chapter has been done in collaboration with Jaume Giné.

In Chapter 3, again considering fixed vector fields, we dedicate our effort to cyclicity but in piecewise systems. Using also Parallelization, we were able to calculate the necessary Lyapunov constants for cubic and quartic systems to show that $M_p^c(3) \geq 26$ and $M_p^c(4) \geq 40$. Moreover, we prove that if an analytic piecewise system has weak-focus or order $2n+1$, we can unfold the total number of limit cycles perturbing in the analytic piecewise class. This result is a natural extension of the classical result showed by Andronov for polynomial systems.

In Chapter 4, using the equivalence among Lyapunov constants and Melnikov functions, we improve that $M(n)$ for $n = 6$. Moreover, we also extend this result to piecewise systems.

Finally, we dedicate the last chapter to conclusions of this work and future works.

We notice that all our calculations were made using the Computer Algebra System MAPLE on a cluster with 9 machines that have 128 CPUs with 725 MB of ram memory.

CHAPTER 1

# Lower bounds for the local cyclicity of centers

In this chapter, we are interested in small-amplitude isolated periodic orbits, so called limit cycles, surrounding only one equilibrium point. We develop a parallelization technique to study higher order developments, with respect to the parameters, of the return map near the origin. This technique is useful to study lower bounds for the local cyclicity of centers. We denote by $M(n)$ the maximum number of limit cycles bifurcating from the origin via a degenerate Hopf bifurcation for a polynomial vector field of degree $n$. We get lower bounds for the local cyclicity of some known cubic centers and we prove that $M(4) \geq 20$, $M(5) \geq 33$, $M(7) \geq 61$, $M(8) \geq 76$, and $M(9) \geq 88$.

## 1.1. Introduction

Hilbert early last century presented a list of problems that almost all of them are solved. One problem that remains opened is the second part of the 16th Hilbert's problem: It consists in determine the maximal number $H(n)$ of limit cycles, and their relative positions, of planar polynomial vector fields of degree $n$. In last years have been proposed other related problems. In 1977, Arnold in [**4**] proposed a weakened version, focused on the study of the number of limit cycles bifurcating from the period annulus of Hamiltonians systems. We are interested here in another local version, that consist in to provide the maximum number $M(n)$ of small-amplitude limit cycles bifurcating from an elementary center or an elementary focus, clearly $M(n) \leq H(n)$. In other words, $M(n)$ is an upper bound of the cyclicity of such equilibrium points. For more details, we refer to [**54**]. For $n = 2$, Bautin proved in [**6**] that $M(2) = 3$. Sibirskii in [**57**] proved that for cubic systems without quadratic terms there are no more than five limit cycles bifurcating from one critical point. In fact these are the unique general families for which this local number is completely determined. The first evidence that $M(3) \geq 11$ was presented by Zoladek in [**64**]. Providing a center with very high local cyclicity. This problem was recently revisited by himself in [**66**]. The first proof of this fact was done by Christopher in [**17**], studying first order perturbations of another cubic center also provided by Zoladek in [**65**]. Basically the used technique consists in to choose a point on the center variety and at this point consider the linear terms, $L_i^1$, of the *Lyapunov constants*. If the point is chosen on a component of the center variety of codimension $r$, then the first $r$ linear terms of the Lyapunov constants are independent, that is, there exist perturbations which can produce $r-1$ limit cycles, and this number is generically the maximum. Apart from the fact that the solution of the center's problem for vector fields of degree $n$ is unknown, the main problem is how to compute the codimension of each component of the center variety. Usually, technique has been used to provide lower bounds for $M(n)$. The idea to study only linear developments, with respect to the parameters, near centers appear previously in [**14**] and also in [**36**].

Giné, in [**32, 33**], conjectures that $M(n) = n^2 + 3n - 7$. This suggests a high value for $M(n)$ for polynomial vector fields of low degree. This lower bound for $n = 4$ says that $M(4) \geq 21$. This problem was studied in [**32**] using only developments of the Lyapunov constants of order 2. But, as we will explain in Section 1.5, order 2 is not enough to prove such result. For degree $n = 5, 7, 8, 9$ the best lower bounds for $M(n)$ were obtained in [**45**] providing examples exhibiting 28, 54, 70, and 88 limit cycles, respectively. Next result improves all these values increasing the known lower bounds for $M(n)$ for these degrees. In particular, the conjecture of Giné for $n = 5$ suggests $M(5) \geq 33$, our result get such lower bound, studying general quintic perturbations of a quintic center with homogeneous nonlinearities appearing in [**31**]. The best result for degree $n$ vector fields is $M(n) \geq n^2 - 2$, see [**51**].

Theorem 1.1. *The number of limit cycles bifurcating from a singular monodromic point for vector fields of degree four, five, seven, eight and nine is at least* $M(4) \geq 20$, $M(5) \geq 33$, $M(7) \geq 61$, $M(8) \geq 76$ *and* $M(9) \geq 88$.

Among the above result we study lower bounds for the local cyclicity of some cubic Darboux polynomial systems, including all Darboux cubic polynomial systems with codimension 12 given by Zoladek in [**65**] having a real center. More concretely, we show in almost all cases that always 11 limit cycles bifurcate from each studied center. The proofs show that the study of developments of higher order are necessary.

To prove all results first we extend, in Section 1.3, the parallelization technique introduced by Liang and Torregrosa in [**45**] to higher order developments. Second, following the ideas in [**17**] to study higher developments, we uses a blow-up procedure to get a complete unfolding of the return map near a polynomial center perturbing with polynomials of the same degree. We remark that the parallelization technique drastically reduces the computation time. In particular, in the proof of Theorem 1.1 only developments of order two are necessary to be obtained, but instead of more than one month of computation time we need one hour. We have used a cluster of computers with ninety processors simultaneously.

This chapter is structured as follows. In Section 1.2, we recall the necessary definitions and algorithms to get the coefficients of the return map, the so called Lyapunov constants among other preliminary results as *Poincaré–Miranda Theorem* ([**43**]) and the *Gershgorin Theorem* ([**30**]) about localization of eigenvalues of a matrix. In Section 1.3, we present and prove the parallelization results. The study of cubic Darboux centers is done in Section 1.4. Finally, in Sections 1.5 and 1.6 we prove Theorem 1.1.

## 1.2. Degenerated Hopf bifurcation

In this section we recall how to obtain the Lyapunov constants or focal values, that is, the coefficients of the return map near an equilibrium point with differential of non-degenerate center type. As usual it is not restrictive to assume that the equilibrium point is located at the origin. See more details in [**2**].

Consider the system

$$\begin{cases} \dot{x} &= a\,x - y + \displaystyle\sum_{k=2}^{n} P_k(x,y), \\ \dot{y} &= x + a\,y + \displaystyle\sum_{k=2}^{n} Q_k(x,y), \end{cases} \tag{3}$$

with $P_k$ and $Q_k$ homogeneous polynomials of degree $k$ in the variables $x, y$. Writing the above system in polar coordinates, $(r, \theta) = (r\cos\theta, r\sin\theta)$, we have

$$\frac{dr}{d\theta} = a\,r + \sum_{k=2}^{\infty} S_k(\theta) r^k, \tag{4}$$

with $S_k(\theta)$ trigonometric polynomials of degree $k$ in the variables $\sin\theta, \cos\theta$. Let $r(\theta,\rho)$ be the solution of system (4) such that $r(0,\rho) = \rho$. The stability of the origin is clearly stated, using Hartman-Grobman Theorem, when $a \neq 0$. When $a = 0$ the stability problem is known as the center-focus problem and there are some classical tools to distinguish when the origin is stable or unstable. So, for $a = 0$ and close to $\rho = 0$, we can develop this solution in series with respect to $\rho$,

$$r(\theta,\rho) = \rho + \sum_{k=2}^{\infty} r_k(\theta)\rho^k,$$

where $r_k(0) = 0$ for all $k \geq 2$. Then, the *Poincaré return map,* $\Pi(\rho)$, can be obtained evaluating the above expression at $\theta = 2\pi$,

$$\Pi(\rho) = r(2\pi,\rho) = \rho + \sum_{k=2}^{\infty} r_k(2\pi)\rho^k. \tag{5}$$

When $V_{\tilde{K}} = r_{\tilde{K}}(2\pi) \neq 0$ for some value $\tilde{K}$ we say that the origin of system (3) is a *weak focus* otherwise we say that the origin is a *center*. In this context, it is well known that the first nonzero value, when it exits, corresponds to an odd subindex $\tilde{K} = 2K + 1$, see [**2, 13, 24**], and consequently the $K$-Lyapunov constant is defined as $L_K = V_{2K+1}$ when $L_1 = \cdots = L_{K-1} = 0$. Then, we say that the origin is a *weak focus of order* $K$ when $L_K \neq 0$ and $L_1 = \cdots = L_{K-1} = 0$. These constants are polynomials in the coefficients of $P_k$ and $Q_k$ defined in (3). See more details in [**18**].

When the first Lyapunov constant is negative (positive), $L_1 \neq 0$, a small-amplitude stable (unstable) limit cycle bifurcates from the equilibrium when the trace parameter $\lambda_0$ moves from zero to positive (negative). Because the stability of the equilibrium point changes and the limit cycles appear by using the Poincaré-Bendixson Theorem. This phenomenon is known as the the classical Hopf bifurcation. The results in this chapter are dealing with the $K$-degenerated Hopf bifurcation. That is when $K$ small-amplitude limit cycles bifurcate from a weak-focus of order $K$. More informations about Hopf bifurcation, see [**16**]. According Roussarie in [**54**], at most $K$ limit cycles can bifurcate from a weak-focus of order $K$ under analytic perturbations. In the context of (3), the main difficulty is how can we ensure the existence of polynomial perturbations such that the $K$ limit cycles appear from a weak-focus of order $K$. In order to unify notation, we denote the trace parameter in (3) by $L_0 = \lambda_0$. The fact that the coefficients of the return map (5) corresponding to monomials of even degree does not play any role in the bifurcation phenomenon is due to the property that the Bautin ideal is generated only by the coefficients of odd degree $\mathbf{B} = \langle L_1, \ldots, L_n, \ldots \rangle = \langle V_3, V_4, \ldots, V_{2n}, V_{2n+1}, \ldots \rangle$. See more details in [**54**]. This property has been revisited recently in [**19**].

Another equivalent procedure to study the center-focus problem is to propose, for $a = 0$, a function $H(x, y) = x^2 + y^2 + O(||(x, y)||^3)$ such that, using (3),

$$\dot{H} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} = h_4 r^4 + h_6 r^6 + \cdots + h_{2k} r^{2k} + \cdots\ ,$$

with $r^2 = x^2 + y^2$. Then, the first nonvanishing coefficient of the above derivative, which always has an even integer, determines the stability of the origin of (3) being $H$ a Lyapunov function. In fact, both coefficients, $h_{2K+2}$ and $L_{2K+1}$, differs on a multiplicative nonzero constant. Here the center property reads as $h_{2k} = 0$ for all $k$. In complex variables, via the change of variables $z = x + iy$ and for $a = 0$, system (3) writes as

$$\dot{z} = R(z, \bar{z}) = iz + \sum_{k=2}^{n} R_k(z, \bar{z}),$$

where $R_k(z, \bar{z})$ is a homogeneous polynomial of degree $k$ in $(z, \bar{z})$. Consequently, the above function $H = z\bar{z} + O_3(z, \bar{z})$ satisfies

$$\frac{\partial H}{\partial z}\dot{z} + \frac{\partial H}{\partial \bar{z}}\dot{\bar{z}} = \sum_{k=4}^{\infty} g_k (z\bar{z})^k,$$

being $g_k$ the $k$-Lyapunov constant.

To find good lower bounds for $M(n)$, instead of to study the local cyclicity of the origin of system (3), we consider perturbations of a fixed center. This is done considering $P_c, Q_c, P, Q$ be polynomials of degree $n$ in $(x, y)$ and studying the perturbed system

$$\left\{ \begin{array}{rcl} \dot{x} & = & P_c(x, y) + P(x, y, \lambda), \\ \dot{y} & = & Q_c(x, y) + Q(x, y, \lambda), \end{array} \right. \tag{6}$$

with $P, Q$ having only monomials of degree higher or equal than two. When $\lambda = 0$, the unperturbed system has a center at the origin of type (3) with $a = 0$. Then we are interested in finding limit cycles of small amplitude bifurcating of the origin of system (6).

In most cases, the explicit computation of the Lyapunov constants for system (3) is very hard and impossible to do by hand. So a Computer Algebra System is necessary to be used. Moreover, sometimes more specific algorithms can be developed to decrease the computation time. Here, in order to reduce not only the computation time but also the total necessary memory, we extend, in next section, the parallelization algorithm proposed by Liang and Torregrosa in [**45**] to higher order developments but for systems of type (6).

Next two results, which are proved in [**17**], provide conditions to show the bifurcation of small limit cycles near a polynomial center. The first uses only the linear developments of the Lyapunov constants. As we have commented before, similar versions of this result can be found in [**14**] and [**36**]. The second uses higher order developments but all with the same order. In the following sections we will show how we can use higher developments with different orders simultaneously.

THEOREM 1.2 ([**17**]). *Suppose that $s$ is a point on the center variety and that the first $k$ Lyapunov constants, $L_1,\ldots,L_k$, have independent linear parts (with respect to the expansion of $L_i$ about $s$), then $s$ lies on a component of the center variety of codimension at least $k$ and there are bifurcations which produce $k$ limit cycles locally from the center corresponding to the parameter value $s$. If, furthermore, we know that $s$ lies on a component of the center variety of codimension $k$, then $s$ is smooth point of the variety, and the cyclicity of the center for the parameter value $s$ is exactly $k$. In the latter case, $k$ is also the cyclicity of a generic point on this component of the center variety.*

The scheme of the proof is as follows. Under the hypotheses of the above result, there exists a change of variables such that the first Lyapunov constants write as

$$L_i = u_i + O_2(u_1,\ldots,u_k,\ldots,u_m), i = 1,\ldots,k, \tag{7}$$

assuming that we have $m \geq k$ bifurcation parameters. Using the Implicit Function Theorem it is clear that we can write $L_i = v_i$, for $i = 1,\ldots,k$. Then the first coefficients of the return map (5) are independent. It is clear that, when $a = 0$ in (3), we get only $k-1$ limit cycles. But adding the parameter $a$ we have an extra limit cycle by the classical Hopf bifurcation, obtaining in total $k$ as the above result ensures. In fact, this proves the existence of a curve, in the parameter space, of weak-foci of order $2k+1$ that unfolds $k$ limit cycles. Using the Weierstrass Preparation Theorem this is the maximal number near such curve.

When the linear parts of the next Lyapunov constants are linear combination of the first $k$ we can use the higher developments to obtain more limit cycles. This is the aim of the next result also proved by Christopher in [**17**, Theorem 3.1].

THEOREM 1.3. *Suppose that, after a change of variables if necessary, $L_0 = L_1 = \cdots = L_k = 0$ and the next Lyapunov constants $L_i = h_i(u) + O_{m+1}(u)$, $i = k+1,\ldots,k+l$, where $h_i$ are homogeneous polynomials of degree $m \geq 2$ and $u = (u_{k+1},\ldots,u_{k+l})$. If there exists a line $\ell$, in the parameter space, such that $h_i(\ell) = 0$, $i = k+1,\ldots,k+l-1$, the hypersurfaces $h_i = 0$ intersect transversally along $\ell$ for $i = k+1,\ldots,k+l-1$, and $h_{k+l}(\ell) \neq 0$, then there are perturbations of the center which can produce $k+l$ limit cycles.*

The above result is not written exactly as in the original Christopher paper because we have adapted to include also the conclusion of Theorem 1.2. We have added an alternative proof.

PROOF OF THEOREM 1.3. We start taking the blow-up change of variable $u_j = v_j u_{k+l}$, for $j = k+1,\ldots,k+l-1$. Then $v = (v_{k+1},\ldots,v_{k+l-1})$ and write $h_i(u) =$

$u_{k+l}^m \widehat{h}_i(v)$, for $i = k+1, \ldots, k+l$. Consequently,

$$L_i(u) = u_{k+l}^m \widetilde{L}_i(v) = u_{k+l}^m \left( \widehat{h}_i(v) + \sum_{j=1}^{\infty} g_{ij}(v) u_{k+l}^j \right),$$

The existence of a line $\ell$ as in the statement gets $v^*$ such that $\widehat{h}_i(v^*) = 0$, for $i = k+1, \ldots, k+l-1$, $\widehat{h}_{k+l}(v^*) \neq 0$, and the determinant of the Jacobian matrix of $(\widehat{h}_{k+1}, \ldots, \widehat{h}_{k+l-1})$ with respect to $v$ does not vanish at $v^*$, then the Implicit Function Theorem applies, in a neighborhood of $v^*$, and the change of variables $w_i = \widetilde{L}_i(v)$ is well defined. The proof follows the same scheme explained in the comments before to state this result but changing $u_j = w_j u_{k+l}^m$. Because, now $u_{k+l}^m$ is a common factor of the complete return and the polynomial provided by the Weierstrass Preparation Theorem has $\widehat{h}_{k+l}(v^*) \neq 0$ as the coefficient of maximal degree monomial and the other are the independent coefficients $w_j$, for $j = 1, \ldots, k+l-1$. Finally, as above we can use the trace parameter to get the value of $k+l$ limit cycles. $\square$

We remark that, as in the previous result, this maximal value of limit cycles is obtained only near the weak-foci curve. Then, the previous results provide only lower bounds for the local cyclicity problem. The proofs suggest that we can study, restricting the parameter space if necessary, how is the intersection of the polynomial varieties $\mathcal{S}_L = \{L_1(u) = L_2(u) = \cdots = L_m(u) = 0\}$ near $u = 0$. This is equivalent to know if the function $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$

$$(u_1, \ldots, u_n) \mapsto (L_1, L_2, \ldots, L_n)$$

is locally surjective at the origin, see [**54**, Page 71]. From the singularities theory, see [**5**], some properties used to compute the local multiplicity of a function in a point, $\mu_0[f]$, are useful to study the local intersection $\mathcal{S}_L$.

Proposition 1.4 ([**5**]). *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a finite map germ. Defining $f = (f_1, \ldots, f_n)$ and $\mathcal{S}_f = \{f_1 = f_2 = \cdots = f_n = 0\}$ the next properties hold,*

*(i) if $f_i = f_i^{k_i} +$ higher order terms. Then $\mu_0[f] \geq \prod_{i=1}^n k_i$ and $\mu_0[f] = \prod_{i=1}^n k_i$ if and only if the system $f_i^{k_i} = 0$, $i = 1, \ldots, n$, has only the trivial solution in $\mathbb{C}^n$.*

*(ii) if $g_i = f_i + \sum_{j<i} A_j^i f_j$, then $\mu_0[f] = \mu_0[g]$ and $\mathcal{S}_f = \mathcal{S}_g$.*

From the second property of the above result, we can use the same Gauss algorithm to triangularize a matrix to convert the local intersection of the set $\mathcal{S}_L$ in, for example, (7). This simple mechanism helps also to reduce the total computation time.

As we will see in the proofs of the results of the next sections, sometimes the application of Theorem 1.3 is not so simple. Because it depends on finding explicitly the intersection of some manifolds and if it is transversal. Although this intersection point can be obtained numerically, we use a computer assisted proof to prove analytically the existence of such point. This is done using Poincaré–Miranda's Theorem together with the results of the appendix. For the transversality property we can use the Circles of Gershgorin Theorem. For completeness we add here.

THEOREM 1.5 ([**43**], Poincaré–Miranda). *Let* $\boldsymbol{c}$ *be a positive real number and* $S = [-c, c]^n$ *a n-dimensional cube. Consider* $f = (f_1, \ldots, f_n) : S \to \mathbb{R}^n$ *a continuous function such* $f_i(S_i^-) < 0$ *and* $f_i(S_i^+) > 0$ *for each* $i \leq n$*, where* $S^\pm = \{(x_1, \ldots, x_n) \in S : x_i = \pm b\}$*. So, there exists a* $\boldsymbol{d} \in S$ *such that* $f(d) = 0$*.*

THEOREM 1.6 ([**30**], Circles of Gershgorin). *Let* $A = (a_{i,j}) \in \mathbb{C}^{n\times n}$ *and* $\alpha_k$ *your eigenvalues. Consider for each* $i = 1, \ldots, n$

$$D_i = \{z \in \mathbb{C} : |z - a_{i,i}| \leq r_i,$$

*where* $r_i = \sum_{i\neq j} |a_{i,j}|$*. So, for all* $k$*, each* $\alpha_k \in D_i$ *for some* $i$*.*

The Poincaré–Miranda's Theorem was conjectured by Poincaré in the 19th century and proved by Miranda in last century. Note that this result is a generalization of the Bolzano's Theorem for higher dimensions. The reader can get more details on Gershgorin Circle Theorem in [**34**].

## 1.3. Parallelization

In [**45**], Liang and Torregrosa present a parallelization mechanism to compute the linear parts of the Lyapunov constants. In this section we extend this result to compute also the terms of higher degree. We start recalling the linearization result for completeness.

THEOREM 1.7 ([**45**]). *Let* $p(z, \bar{z})$ *and* $Q_j(z, \bar{z})$*,* $j = 1, \ldots, s$ *be polynomials with monomials of degree higher or equal than two such that the origin of* $\dot{z} = iz + p(z, \bar{z})$ *is a center. If* $L_{k,j}^{(1)}$ *denotes the linear part, with respect to* $\lambda_j \in \mathbb{R}$*, of the k-Lyapunov constant of equation*

$$\dot{z} = iz + p(z, \bar{z}) + \lambda_j Q_j(z, \bar{z}), \quad j = 1, \ldots, s,$$

*then the linear part of the k-Lyapunov constants, with respect to* $\Lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N$*, of equation*

$$\dot{z} = iz + p(z, \bar{z}) + \sum_{j=1}^{N} \lambda_j Q_j(z, \bar{z}),$$

*are* $L_k^{(1)} = \sum_{j=1}^{N} L_{k,j}^{(1)}$*.*

In the following result we show how to get the terms order $\ell$ of the Lyapunov constants. The main idea is to decompose the global computation problem in a collection of simpler problems. The simplicity of the previous result is that each perturbation parameter $\lambda$ appears in only one simpler problem because the authors were interested only in the linear approximation. Now, because of we study order $\ell$ developments, we need to decompose the global problem in simpler problems having exactly $\ell$ parameters or monomials. But, as many parameters appear in some of the simple perturbation problems we need to correct the obtained coefficients of the developments of order $\ell$. To further clarify the main idea of parallelization, we present the next proposition where we perturbate a quadratic center with four parameters and we calculate the first three Lyapunov constants.

PROPOSITION 1.8. *Consider the quadratic perturbed equation*

$$\dot{z} = i\,z + z^2 + (a_{20} + ib_{20})z^2 + (a_{11} + ib_{11})z\bar{z}. \tag{8}$$

*Then the developments of order* 2 *of the Lyapunov constants of the above system are*

$$\begin{aligned}
L_1^{(2)} &= -2a_{20}b_{11} - 2b_{20}a_{11} - 2b_{11},\\
L_2^{(2)} &= 36a_{20}b_{11} + 12b_{20}a_{11} + 32a_{11}b_{11} + 8b_{11}^2 + 12b_{11},\\
L_3^{(2)} &= -540a_{20}b_{11} - 108b_{20}a_{11} - 582a_{11}b_{11} - 192b_{11}^2 - 108b_{11}.
\end{aligned}$$

PROOF. We have the perturbed equation (8) with four real parameters, $a_{20}, b_{20}$, $a_{11}, b_{11}$, then for the parallelization mechanism to compute the Lyapunov constants of order two we consider six pairs, that are

$$S = \{(a_{20}, b_{20}), (a_{20}, a_{11}), (a_{20}, b_{11}), (b_{20}, a_{11}), (b_{20}, b_{11}), (a_{11}, b_{11})\}.$$

Because they are the combination (without repetition) of 4 elements taken 2 by 2. Next, we consider 6 equations, one for each element in $S$, such that in (8) the parameters that are not in the chosen pair are zero. For example, the corresponding equation to $S_3 = (a_{20}, b_{11})$ is

$$\dot{z} = i\,z + z^2 + a_{20}z^2 + ib_{11}z\bar{z}.$$

For each equation $S_j$ we compute, with the mechanism described in Section 1.2, the first three Lyapunov constants. We denote by $L_{k,j}$ the $k$-Lyapunov constant of equation $S_j$. Straightforward computations show that $L_{k,j}^{(2)} = 0$, for $k = 1, 2, 3$ and $j = 1, 2$ and

$$\begin{aligned}
L_{1,3}^{(2)} &= -2a_{20}b_{11} - 2b_{11}, & L_{1,5}^{(2)} &= -2b_{11},\\
L_{2,3}^{(2)} &= 36a_{20}b_{11} + 8b_{11}^2 + 12b_{11}, & L_{2,5}^{(2)} &= 8b_{11}^2 + 12b_{11},,\\
L_{3,3}^{(2)} &= -540a_{20}b_{11} - 192b_{11}^2 - 108b_{11}, & L_{3,5}^{(2)} &= -192b_{11}^2 - 108b_{11},\\
L_{1,4}^{(2)} &= -2a_{11}b_{20}, & L_{1,6}^{(2)} &= -2b_{11},\\
L_{2,4}^{(2)} &= 12a_{11}b_{20}, & L_{2,6}^{(2)} &= 32a_{11}b_{11} + 8b_{11}^2 + 12b_{11},\\
L_{3,4}^{(2)} &= -108a_{11}b_{20}, & L_{3,6}^{(2)} &= -582a_{11}b_{11} - 192b_{11}^2 - 108b_{11}.
\end{aligned}$$

We notice that we can not compute $L_k^{(2)} = \sum_{j=1}^{6} L_{k,j}^{(2)}$ as in the linear case, because it can be repeated terms. The monomials having only one parameter, $a_{kl}, a_{kl}^2, b_{kl}, b_{kl}^2$, appear more than one time, while the monomials having two, $a_{kl}b_{kl}$, only ones. For example, the monomials $a_{20}$ and $a_{20}^2$ appear in $S_1$, $S_2$, and $S_3$, that is, exactly $3 = \binom{3}{1}$ times. Then, these monomial will be divided by 3. But, as the monomial $a_{11}b_{20}$ only appears in $S_4$ the corresponding coefficient is the same. So, we need to add a correction factor which depends on the number of repetitions. As we have four perturbating parameters, we should divide each repeated term by 3 because is the number of times that they appear in $S$.

Denoting by $\hat{L}_{k,j}^{(2)}$ the corrected $k$-Lyapunov constant corresponding to the pair $S_j$, we can obtain $L_k^{(2)} = \sum_{j=1}^{6} \hat{L}_{k,j}^{(2)}$. The statement follows because, in our case, we have $\hat{L}_{k,j}^{(2)} = 0$, for $k = 1, 2, 3$ and $j = 1, 2$ and

$$
\begin{aligned}
\hat{L}_{1,3}^{(2)} &= -2a_{20}b_{11} - \frac{2}{3}b_{11}, & \hat{L}_{1,5}^{(2)} &= -\frac{2}{3}b_{11},\\
\hat{L}_{2,3}^{(2)} &= 36a_{20}b_{11} + \frac{8}{3}b_{11}^2 + 4b_{11}, & \hat{L}_{2,5}^{(2)} &= \frac{8}{3}b_{11}^2 + 4b_{11},,\\
\hat{L}_{3,3}^{(2)} &= -540a_{20}b_{11} - 64b_{11}^2 - 36b_{11}, & \hat{L}_{3,5}^{(2)} &= -64b_{11}^2 - 36b_{11},\\
\hat{L}_{1,4}^{(2)} &= -2a_{11}b_{20}, & \hat{L}_{1,6}^{(2)} &= -\frac{2}{3}b_{11},\\
\hat{L}_{2,4}^{(2)} &= 12a_{11}b_{20}, & \hat{L}_{2,6}^{(2)} &= 32a_{11}b_{11} + \frac{8}{3}b_{11}^2 + 4b_{11},\\
\hat{L}_{3,4}^{(2)} &= -108a_{11}b_{20}, & \hat{L}_{3,6}^{(2)} &= -582a_{11}b_{11} - 64b_{11}^2 - 36b_{11}.
\end{aligned}
$$

□

Now we can state the main result of this section, the computation in a parallelized form of the Lyapunov constants of order $\ell$.

THEOREM 1.9. *Let $p(z,\bar{z})$ and $Q_j(z,\bar{z})$, $j = 1,\dots,N$ be polynomials with monomials of degree higher or equal than two such that the origin of $\dot{z} = iz + p(z,\bar{z})$ is a center. For $\ell \le N$, we denote by $L_k^{(\ell)}$ the k-Lyapunov constant of order $\ell$ of equation*

$$\dot{z} = iz + p(z,\bar{z}) + \sum_{j=1}^{N} \lambda_j Q_j(z,\bar{z}), \tag{9}$$

*with $\Lambda = (\lambda_1,\dots,\lambda_N) \in \mathbb{R}^N$. Let $S$ be the set of all combinations of the components of $\Lambda$ taken $\ell$ at a time. That is, $S = \{(\lambda_1,\dots,\lambda_\ell), (\lambda_2,\dots,\lambda_{\ell+1}),\dots,(\lambda_{\ell-N},\dots,\lambda_N)\}$ and it has $\binom{N}{\ell}$ elements. For each element $S_j = (\lambda_{\sigma(j,1)},\dots,\lambda_{\sigma(j,\ell)})$ in $S$, $j = 1,\dots,\binom{N}{\ell}$, we denote by $L_{k,j}^{(\ell)}$ the k-Lyapunov constant of order $\ell$ with respect to $S_j$*

*of equation*

$$\dot{z} = iz + p(z,\bar{z}) + \sum_{l=1}^{\ell} \lambda_{\sigma(j,l)} Q_{\sigma(j,l)}(z,\bar{z}). \tag{10}$$

*Then*

$$L_k^{(\ell)} = \sum_{l=1}^{N} \hat{L}_{k,j}^{(\ell)},$$

*where* $\hat{L}_{k,j}^{(\ell)} = \sum_p \frac{\mu_{k,j,p}}{\binom{N-s(p)}{\ell-s(p)}} \Lambda_j^p$, *for* $\Lambda_j^p = \lambda_{\sigma(j,1)}^{p_1} \lambda_{\sigma(j,2)}^{p_2} \cdots \lambda_{\sigma(j,\ell)}^{p_\ell}$ *and* $p = (p_1, \ldots, p_\ell)$ *writing* $L_{k,j}^{(\ell)} = \sum_p \mu_{k,j,p} \Lambda_j^p$ *with* $s(p) = \sum_{l=1}^{\ell} \operatorname{sgn}(p_l)$ *where* $\operatorname{sgn}(x) = \begin{cases} 1, & \textit{if } x > 0, \\ 0, & \textit{if } x = 0. \end{cases}$

Proof. It is well known that the $k-$Lyapunov constant, $L_k$, of a differential equation (9) is a polynomial in the parameters $\Lambda = (\lambda_1, \ldots, \lambda_N)$, see [**18**]. Moreover, in our case $L_k$ vanishes when $\Lambda = 0$. Consequently, $L_k^{(\ell)}$ is the Taylor polynomial of $L_k$ of degree $\ell$ at $\Lambda = 0$ and $L_k^{(\ell)}(0) = 0$. We write it as

$$L_k^{(\ell)} = \sum_p \mu_{k,p} \Lambda^p, \tag{11}$$

where $\Lambda^p = \lambda_1^{p_1} \lambda_2^{p_2} \cdots \lambda_N^{p_N}$, $p = (p_1, \ldots, p_N)$, and $\sum_{l=1}^{N} p_l \leq \ell$. This last condition allow us to decompose the total sum (11) in partial sums of $\ell$ parameters in $\Lambda$. Each partial sum is in fact the $k$-Lyapunov constant of order $\ell$ of equation (10). As each monomial can appear more than ones in each partial sum we need to correct the corresponding coefficient with a factor that controls how many times this monomial appear. This factor depends on the number of different $\lambda_i$ appearing in each monomial besides the total number $N$ and the order $\ell$, in fact the number of times that it appear in the partial sums, that is the combinatorial number $\binom{N-s(p)}{\ell-s(p)}$.

□

## 1.4. Applications to cubic centers

In this section we study, using the results and procedure described in Sections 1.2 and 1.3, lower bounds for the local cyclicity of some Darboux cubic centers, in particular the codimension 12 ones in [**65**] having a real center. In all the results of this section and for $n = 3$, moving the center point to the origin and rescaling variables and time if necessary, we consider the perturbed system (3), that writes in complex coordinates as

$$\dot{z} = (i + \lambda_0) z + p(z,\bar{z}) + \sum_{k+l=2}^{3} (a_{k,l} + i\, b_{k,l}) z^k \bar{z}^l. \tag{12}$$

Moreover, the corresponding unperturbed system $\dot{z} = iz + p(z,\bar{z})$ has a cubic Darboux center and we have, in general and among the trace parameter $\lambda_0$, 14 real

parameters
$$(a_{20}, a_{11}, a_{02}, \ldots, b_{20}, b_{11}, b_{02}, \ldots) \in \mathbb{R}^{14}.$$
In the proofs, we have denoted by $\lambda = (u_1, u_2, \ldots, u_{11}) \in \mathbb{R}^{11}$ the essential parameters and there are three that have been chosen as zero to simplify the computations. Using Theorems 1.2 and 1.3 we first study the number of limit cycles appearing with the essential parameters and then we add an extra one using the trace $\lambda_0$. In each result, we have detailed why, the unperturbed system, is a center and which is the labeled name following the notation in [**65**]. In some cases we have described also what is known up to now. Most of the considered centers are 1-parameter families depending on a special parameter $a$. Following the ideas of Theorem 1.2, see also [**17**], most probably, the cyclicity results are generic. That is, are valid for almost every value $a$, but for simplicity, we have choice only one value for it.

PROPOSITION 1.10. *There exist cubic polynomial perturbations such that from the origin of system*
$$\begin{cases} \dot{x} = -\frac{343}{576}\,x^3 - \frac{7}{72}\,x^2 - \frac{49}{36}\,xy - 2\,y, \\ \dot{y} = \frac{343}{72}\,x^3 - \frac{343}{96}\,x^2y + \frac{49}{24}\,x^2 + \frac{259}{36}\,xy - \frac{49}{9}\,y^2 + x, \end{cases} \tag{13}$$
*bifurcate at least* 11 *limit cycles of small amplitude.*

This family is labeled as $CD_{11}^{12}$ in [**65**] and as $CD45$ in [**66**]. System (13) has
$$H = -\frac{81}{245}\,\frac{\left(\frac{5764801}{23887872}x^5 + \frac{588245}{331776}x^4 + \frac{132055}{20736}x^3 + \frac{20335}{2592}x^2 + \frac{1715}{648}xy + \frac{245}{72}x + \frac{35}{9}y + 1\right)^4}{\left(\frac{117649}{331776}x^4 + \frac{2401}{1152}x^3 + \frac{1715}{288}x^2 + \frac{49}{18}x + \frac{28}{9}y + 1\right)^5} \tag{14}$$
as a first integral. In both previous works, Melnikov theory or order 2 and 3, respectively, is used to prove the same statement. The first integral proposed in [**66**] depends on one parameter, named $a$, that it is fixed to zero. For this value the corresponding system has a real saddle point at the origin but as the computations done by Zoladek are all in complex he comment that, generically, the result for real cyclicity is also valid. Recently, in [**58**] this special parameter value is fixed to $a = -3$ where the system has a real center at the point $(3/2, -11/4)$. The proof uses developments of seventh order of the Lyapunov constants. The function (14) is obtained from the one proposed in [**58**] but moving the equilibrium point and doing an affine change of coordinates that writes the linear part simpler. This expression reduces the computation time. Our proof, which is different, shows that order 7 is the minimum necessary to unfold 11 limit cycles but we have used developments of the Lyapunov constants of order 10 for a better understanding of this bifurcation phenomenon.

Proof of Proposition 1.10. Clearly system (13) has a center at the origin because (14) is a first integral well defined at the origin and the corresponding level curves in a neighborhood of the origin are ovals. In fact $H = 1 + x^2 + 2y^2 + \cdots$.

The first step is the computation of the Taylor developments of the Lyapunov constants, $L_k$, corresponding to a cubic perturbation of system (13) having only quadratic and cubic terms. We will study which are the principal parts, near the origin in the parameters space, of each Lyapunov constant when the previous vanish.

The second step is the study of linear parts of $L_k$. Unfortunately we get only 9 linearly independent. More concretely the 10th and the 11th are linearly dependent with respect to the first nine. Consequently, Theorem 1.2, adding the trace parameter, provides an unfolding with 9 limit cycles. To obtain the remaining two as it is stated, we need to look at higher order developments. After a linear change of variables and doing the necessary transformations following the scheme detailed in Section 1.2 we get

$$\begin{aligned} L_k &= u_k + O_2(\lambda), \text{ for } k = 1, \ldots, 9, \\ L_{10} &= O_2(\lambda), \\ L_{11} &= O_2(\lambda). \end{aligned}$$

Here $\lambda = (u_1, u_2, \ldots, u_{11}) \in \mathbb{R}^{11}$ denotes the essential parameters, the other are zero, and $O_k(\lambda)$ contains all monomials of degree at least $k$ in $\lambda$.

The third step continues studying the higher order developments, in particular the ones corresponding to $L_{10}$ and $L_{11}$. The Implicit Function Theorem ensures that there exists an analytical local change of variables in the parameter space, well defined in a neighborhood of the origin, such that $L_k = v_k$, for $k = 1, \ldots, 9$. With this change, doing an affine change of variables, and following the scheme detailed in Section 1.2 we can write the last two Lyapunov constants depending only on two parameters $(v_{10}, v_{11})$, except positive multiplicative constants, as

$$\begin{aligned} L_{10} &= v_{10}^3 + O_4(v_{10}, v_{11}), \\ L_{11} &= -v_{10}^3 + O_4(v_{10}, v_{11}). \end{aligned}$$

Here, taking $v_{11} = 0$, it is clear that we have a curve, in the parameters space, of weak-foci of order 10, because $L_{10} \neq 0$, that unfolds (using the lineality of $v_1, \ldots, v_9$ and the trace parameter) 10 limit cycles. This is also true taking $v_{10} = 0$, but with a higher development, because $L_{10} \neq 0$. In fact, except a positive multiplicative constant, $L_{10} = v_{11}^6 + O_7(v_{11})$. It is evident that, only with a third order development, we can not unfold the eleventh limit cycle.

The next step is the study, if they exist, of the different real branches near the origin that has the algebraic curve $L_{10} = 0$ and if there exist one such that $L_{11} \neq 0$. This is done having more terms in the development of $L_{10}$ and $L_{11}$. We can follow [**11**] to use the Newton-Puiseux algorithm with different weights for the variables

$(v_{10}, v_{11})$. With weights $(2,1)$ the principal parts are

$$L_{10} = v_{10}^3 + \alpha_{22} v_{10}^2 v_{11}^2 + \alpha_{14} v_{10} v_{11}^4 + \alpha_{06} v_{11}^6 + O_7^{(2,1)}(v_{10}, v_{11}),$$
$$L_{11} = -(v_{10}^3 + \alpha_{22} v_{10}^2 v_{11}^2 + \alpha_{14} v_{10} v_{11}^4 + \alpha_{06} v_{11}^6) + O_7^{(2,1)}(v_{10}, v_{11}),$$

where $O_k^{(2,1)}$ denotes the monomials with degree higher or equal than $k$ with respect to the weight $(2,1)$. In fact, the principal part of $L_{10}$, with this specific choice of weights, decompose as a product of two factors, one simple and one double,

$$L_{10}(v_{10}, v_{11}) = (v_{10} - a_2 v_{11}^2)\left(v_{10} - b_2 v_{11}^2\right)^2 + O_7^{(2,1)}(v_{10}, v_{11}), \tag{15}$$

with

$$a_2 = -\frac{660160595890746}{37506906889} \text{ and } b_2 = -\frac{487045680336990}{37506906889}.$$

The study of the different branches , $v_{10}^{[j]}(v_{11})$, of the curve $L_{10}(v_{10}, v_{11}) = 0$ near the origin is done using the weighted blow-up $v_{10} = v_{11}^2 w_{10}$. Then the function (15), collecting in $v_{11}$, can be written as polynomials in $v_{11}$ of degree $k$ with coefficients polynomials in $w_{10}$ of degree $[k/2]$, for $k \geq 6$, where $[\cdot]$ denotes the integer part function. Consequently, dividing by $v_{11}^6$, (15) writes as

$$\widetilde{L}_{10}(w_{10}, v_{11}) = (w_{10} - a_2)(w_{10} - b_2)^2 + \sum_{k \geq 1}^{\infty} W_k(w_{10}) v_{11}^k,$$

with $W_k$ polynomials of degree $[(k+6)/2]$. An equivalent expression, $\widetilde{L}_{11}(w_{10}, v_{11})$ can be obtained for $L_{11}$ but with different functions $W_k(w_{10})$.

Now, we can write $\widetilde{L}_{10} = \widetilde{w}_{10}$ in a neighborhood of $(w_{10}, v_{11}) = (a_2, 0)$, using the Implicit Function Theorem, because $\widetilde{L}_{10}(a_2, 0) = 0$ and the partial derivative

$$\left.\frac{\partial L_{10}}{\partial w_{10}}\right|_{(a_2,0)} = (a_2 - b_2)^2 \neq 0.$$

Clearly, when $\widetilde{w}_{10} = 0$, there exists an analytic branch, $w_{10} = \omega^{[1]}(v_{11}) = a_2 + O_1(v_{11})$, such that $\widetilde{L}_{10}$ and also $L_{10}$ vanish on it. Clearly, $v_{10} = v_{11}^2 \omega^{[1]}(v_{11}) = a_2 v_{11}^2 + O_3(v_{11})$. With this change of variables, after multiplying by $v_{11}^6$, we write (15) as $L_{10} = v_{11}^6 \widetilde{w}_{10}$. Additionally, except a positive multiplicative constant, we get

$$L_{11}|_{L_{10}=0} = L_{11}|_{\widetilde{w}_{10}=0} = -v_{11}^7 + O_8(v_{11}) \neq 0,$$

for $v_{11} \neq 0$ small enough, and $L_{11} = -v_{11}^7(1 + O_1(\widetilde{w}_{10}, v_{11}))$. We notice that the neighborhood of $(a_2, 0)$ for the variables $(w_{10}, v_{10})$ moves to a neighborhood of the origin for the variables $(\widetilde{w}_{10}, v_{11})$.

This proves that there exists a curve, in the parameters space, for $v_{11}$ small enough, of weak-foci of order 11 that birth from the origin. The cubic perturbation mechanism described in Section 1.2 following the scheme of Roussarie, see [**54**], proves that only 11 limit cycles can bifurcate from the origin of the system (13). This is because, from the Weierstrass Preparation Theorem, see [**60**], the return is a polynomial with coefficients the Lyapunov constants, that write, after all the changes of variables, as $L_k = u_k$, for $k = 1, \ldots, 9$, $L_{10} = v_{11}^6 \widetilde{w}_{10}$ and $L_{11} = -v_{11}^7(1 + O_1(\widetilde{w}_{10}, v_{11}))$. Finally, using the weighted blowup $\{u_1 = z^7 z_1, u_2 = z^7 z_2, \ldots, u_9 = z^7 z_9, \widetilde{w}_{10} = z z_{10}, u_{11} = z\}$, after dividing by $z^7$, it is clear that we have constructed a versal unfolding, obtaining the maximal number of limit cycles. As $v_{11}$ can vanish this upper bound is in fact a lower bound for the cyclicity of the center as we wanted to prove. $\square$

We remark that the above proof is a generalization of the result provided by Christopher in [**17**], see Theorem 1.3, where the higher order Lyapunov constants have all the same order. Here we have different orders and the transversal straight line $\ell$ is now an analytic curve.

We notice that as we have proved that curve $L_{10} = 0$ has a real branch associated to the simple factor, using the Weierstrass Preparation and Division Theorems, [**60**], we can write (15), except a non vanishing function, as

$$(v_{10} - v_{11}^2 \omega^{[1]}(v_{11}))(v_{10}^2 + v_{11}^2 \phi_1(v_{11}) v_{10} + v_{11}^4 \phi_0(v_{11})), \tag{16}$$

where $\phi_0$ and $\phi_1$ are analytic functions that vanish at zero. As we have computed the Lyapunov constants up to order 10, using the Puiseux series (see [**11**]), we can compute four extra terms of this analytic simple branch $v_{10} = v_{11}^2 \omega^{[1]}(v_{11}) = \sum_{k\geq 2}^{\infty} a_k v_{11}^k$. Straightforward computations show that

$$
\begin{aligned}
a_3 &= \frac{161045709458196921216382226351}{25209283713691672597112320},\\
a_4 &= \frac{386258251571578220793485476718239267056083732367}{123628442038275561958744770186426115741450240},\\
a_5 &= -\frac{7344527305232752838312438300220617202784745335855878366073914837}{1080216991377709756831953767622443166786863443981081614745600 0},\\
a_6 &= -\frac{104066771941021272704828260898 4\cdots 19508240344604534104987984310 7}{29136111074477920359454647525 72\cdots 84481482053532499004358656000 0}.
\end{aligned}
$$

Using the above coefficients and the order 10 development of $L_{10}$ we can compute the first terms of the Taylor series of the functions $\phi_0$ and $\phi_1$ in (16). It can be checked that they provide the double factor appearing in (15) and that the discriminant, with respect to $v_{10}$, writes as $\phi_1^2(v_{11}) - 4\phi_0(v_{11}) = A v_{11}^8 + \cdots$ with $A > 0$. This proves that there exist another two real branches, tangent to the double factor in (15), $v_{10} = v_{11}^2 \omega^{[j]}(v_{11}) = \sum_{k\geq 2}^{\infty} b_k^{[j]} v_{11}^k$. In the above proof we have done the first coefficient, that coincides for both branches, $b_2^{[0]} = b_2^{[1]} = b_2$. In fact, the second also coincides,

$$b_3^{[0]} = b_3^{[1]} = b_3 = -\frac{3083409250744624645028983 2}{98473764506608096082 47},$$

because the discriminant starts with order 8. The next coefficients $b_4^{[0]}$ and $b_4^{[1]}$, which are different, are the zeros of the quadratic equation $\beta^2+\beta_1\beta+\beta_0=0$, where

$$\beta_1 = \frac{374029705710452551852715772722429520}{369344631635866288497876795513583},$$

$$\beta_0 = \frac{85667774170854557031750775271805944351962031047773686502665550988333822 08}{33421786944967271392447900695765062189771656497217124612800686982805}.$$

We observe that we have needed order 10 to distinguish the other two real branches. Over them, we have checked that $L_{11}$ vanishes up to this order.

Proposition 1.11. *There exists cubic polynomial perturbations such that from the origin of system*

$$\begin{cases} \dot{x} &= \dfrac{5\,x^3}{32} - \dfrac{15\,x^2y}{64} - \dfrac{5\,xy^2}{32} - y, \\[2ex] \dot{y} &= \dfrac{15\,x^3}{64} + \dfrac{35\,x^2y}{32} - \dfrac{15\,xy^2}{16} - \dfrac{15\,y^3}{32} + x, \end{cases} \tag{17}$$

*bifurcate at least* 11 *limit cycles of small amplitude.*

This is the case $CD_{12}^{12}$ in [**65**]. Christopher studies in [**17**] the local cyclicity for this family for $a=2$. He also needs order two, but with a high computational effort using Grobner Basis. The above system is obtained taking $a=3/5$ but doing an adequate affine change of coordinates. The computations are simpler because the linear part at the origin is in normal form of a non-degenerate center point.

Proof of Proposition 1.11. System (17) has a center at the origin because it has the rational first integral

$$H = -\frac{-36(625x^4+1920x^2+2560xy+4096)^3}{5(78125x^6+360000x^4+480000x^3y+1044480x^2+737280xy+491520y^2+786432)^2},$$

which is well defined at the origin and with Taylor series $-4/5+x^2+y^2+\cdots$.

Computing the Lyapunov constants, following the scheme explained in Section 1.3 for order 1, corresponding to the perturbed system (12), we have that the first five linear terms have rank five with respect to the parameters. Doing a linear change of coordinates we have that they write $L_k^{(1)}=u_k$, for $k=1,\ldots,5$ Using the properties detailed in Proposition 1.4 we can simplify the next Lyapunov constants to get $L_k^{(1)}=0$, for $k=6,\ldots,11$. Consequently, up to order 1 only five limit cycles bifurcate from the origin.

The second step is the computation of the second order terms. Using again Proposition 1.4 to eliminate the parameters $u_k$, $k=1,\ldots,5$, we obtain 6 homogeneous polynomials of degree 2 for the second order terms of $L_k$ for $k=6,\ldots,11$.

$$
\begin{aligned}
L_6^{(2)} = {} & \frac{65234375}{2491416576}a_{11}^2 - \frac{5695234375}{226718908416}a_{02}a_{11} + \frac{187109375}{32388415488}a_{11}a_{20} + \frac{1842265625}{75572969472}a_{11}b_{02} \\
& - \frac{250390625}{5813305344}a_{11}b_{11} + \frac{23046875}{1937768448}a_{11}b_{20} + \frac{390625}{899678208}b_{11}^2 - \frac{1466328125}{75572969472}a_{02}b_{11} \\
& - \frac{250390625}{75572969472}a_{20}b_{11} - \frac{950546875}{75572969472}b_{02}b_{11} + \frac{266796875}{25190989824}b_{11}b_{20} - \frac{141015625}{18893242368}a_{02}^2 \\
& + \frac{21171875}{4359979008}a_{02}a_{20} - \frac{20703125}{25190989824}a_{02}b_{02} + \frac{708828125}{75572969472}a_{02}b_{20} + \frac{390625}{1349517312}a_{20}^2 \\
& - \frac{204921875}{75572969472}a_{20}b_{02} + \frac{23046875}{25190989824}a_{20}b_{20} + \frac{241796875}{75572969472}b_{02}^2 + \frac{47421875}{37786484736}b_{02}b_{20} \\
& - \frac{11328125}{3598712832}b_{20}^2, \\
L_7^{(2)} = {} & -\frac{804296875}{53150220288}a_{11}^2 + \frac{62870546875}{4836670046208}a_{02}a_{11} - \frac{229296875}{76772540416}a_{11}a_{20} - \frac{488059296875}{43530030415872}a_{11}b_{02} \\
& + \frac{221214453125}{14510010138624}a_{11}b_{11} - \frac{240866796875}{43530030415872}b_{20}a_{11} - \frac{94140625}{345476431872}b_{11}^2 + \frac{406132109375}{43530030415872}a_{02}b_{11} \\
& + \frac{8782421875}{3348463878144}a_{20}b_{11} + \frac{3529140625}{537407782912}b_{02}b_{11} - \frac{26304296875}{4836670046208}b_{11}b_{20} + \frac{9333203125}{3627502534656}a_{02}^2 \\
& - \frac{1203359375}{329772957696}a_{02}a_{20} + \frac{11367578125}{43530030415872}a_{02}b_{02} - \frac{163425546875}{43530030415872}a_{02}b_{20} - \frac{12109375}{86369107968}a_{20}^2 \\
& + \frac{39773359375}{14510010138624}a_{20}b_{02} - \frac{5510546875}{4836670046208}a_{20}b_{20} + \frac{126171875}{76772540416}b_{20}^2 - \frac{10987578125}{7255005069312}b_{02}b_{20} \\
& - \frac{8826953125}{14510010138624}b_{02}^2, \\
L_8^{(2)} = {} & \frac{205818359375}{68341519613952}a_{11}^2 - \frac{27010282421875}{15786891030822912}a_{02}a_{11} + \frac{566685546875}{2255270147260416}a_{11}a_{20} \\
& + \frac{6561098046875}{15786891030822912}a_{11}b_{02} + \frac{49375697265625}{15786891030822912}a_{11}b_{11} + \frac{2022599609375}{15786891030822912}b_{20}a_{11} \\
& + \frac{471892578125}{5920084136558592}b_{11}^2 - \frac{6456421484375}{15786891030822912}a_{02}b_{11} - \frac{4613353515625}{5262297010274304}a_{20}b_{11} \\
& - \frac{47627121484375}{47360673092468736}b_{02}b_{11} + \frac{33662533203125}{47360673092468736}b_{11}b_{20} + \frac{733501953125}{1315574252568576}a_{02}^2 \\
& + \frac{1488928515625}{1315574252568576}a_{02}a_{20} + \frac{3438419921875}{15786891030822912}a_{02}b_{02} - \frac{5852162890625}{15786891030822912}a_{02}b_{20} \\
& - \frac{5248046875}{140954384203776}a_{20}^2 - \frac{121922265625}{103182294319104}a_{20}b_{02} + \frac{30685546875}{53154515255296}a_{20}b_{20} \\
& - \frac{3643357421875}{15786891030822912}b_{20}^2 + \frac{509573828125}{607188116570112}b_{02}b_{20} - \frac{3731908203125}{5262297010274304}b_{02}^2, \\
L_9^{(2)} = {} & \frac{4251095908203125}{274240849906866585 6}a_{11}^2 - \frac{1670155486328125}{914136166356221952}a_{02}a_{11} + \frac{120403642578125}{210954499928358912}a_{11}a_{20} \\
& + \frac{1678778216796875}{747929590655090688}a_{11}b_{02} - \frac{43118063095703125}{8227225497205997568}a_{11}b_{11} + \frac{9782838056640625}{8227225497205997568}b_{20}a_{11} \\
& + \frac{144507529296875}{959842974674033049 6}b_{11}^2 - \frac{9930148478515625}{52355071345856348 16}a_{02}b_{11} - \frac{240089833984375}{2742408499068665856}a_{20}b_{11} \\
& - \frac{757186115234375}{914136166356221952}b_{02}b_{11} + \frac{14849983759765625}{19196859493480660992}b_{11}b_{20} - \frac{377087236328125}{436292261215469568}a_{02}^2 \\
& + \frac{143162263671875}{685602124767166464}a_{02}a_{20} - \frac{1169718330078125}{44300444984955371 52}a_{02}b_{02} + \frac{62189059427734375}{57590578480441982976}a_{02}b_{20} \\
& + \frac{2179873046875}{28566755198631936}a_{20}^2 + \frac{9000880859375}{210954499928358912}a_{20}b_{02} - \frac{210343720703125}{2742408499068665856}a_{20}b_{20} \\
& - \frac{1433194228515625}{6398953164493553664}b_{20}^2 - \frac{1701270740234375}{9598429746740330496}b_{02}b_{20} + \frac{1006989326171875}{17451690448618782 72}b_{02}^2,
\end{aligned}
$$

$$\begin{aligned}
L_{10}^{(2)} = &-\frac{16403503182763671875}{1105739106824486073139 2}a_{11}^2 + \frac{6019489875693359375}{4553043381042001477632}a_{02}a_{11}\\
&- \frac{4110241175146484375}{1105739106824486073139 2}a_{11}a_{20} - \frac{2264107737402343 75}{1804236304841818767 36}a_{11}b_{02}\\
&+ \frac{1537815485986328125}{781835732098121465856}a_{11}b_{11} - \frac{18534506355957031 25}{2866731017693112041472}b_{20}a_{11}\\
&- \frac{543782850048828125}{19350434369428506279936}b_{11}^2 + \frac{2782840775895507812 5}{25800579159238008373248}a_{02}b_{11}\\
&+ \frac{719020527197265625}{2866731017693112041472}a_{20}b_{11} + \frac{5092581035735351562 5}{77401737477714025119744}b_{02}b_{11}\\
&- \frac{4357958081596679687 5}{77401737477714025119744}b_{11}b_{20} + \frac{1929508851220703125}{6450144789809502093312}a_{02}^2\\
&- \frac{763898642225585937 5}{19350434369428506279936}a_{02}a_{20} + \frac{54847838720703125}{5058937090046668308 48}a_{02}b_{02}\\
&- \frac{103268798154296875}{2168115895734286417 92}a_{02}b_{20} - \frac{17619605615234375}{4607246278435358638 08}a_{20}^2\\
&+ \frac{6869360360732421875}{25800579159238008373248}a_{20}b_{02} - \frac{2794670756347656 25}{2866731017693112041472}a_{20}b_{20}\\
&+ \frac{4340212115087890625}{25800579159238008373248}b_{20}^2 - \frac{1595675557617187 5}{1592628343162840023 04}b_{02}b_{20}\\
&- \frac{3536245476904296875}{25800579159238008373248}b_{02}^2,\\
L_{11}^{(2)} = &\frac{42718448752197265625}{951840915348329531965 44}a_{11}^2 - \frac{171788232009716796875}{602832579720608703578112}a_{02}a_{11}\\
&+ \frac{6210508852783203125}{951840915348329531965 44}a_{11}a_{20} + \frac{525938230895356445312 5}{3797845252239834832542 1056}a_{11}b_{02}\\
&+ \frac{2996301505219970703125}{1265948417413278277514 0352}a_{11}b_{11} + \frac{2217310253376220703125}{3797845252239834832542 1056}b_{20}a_{11}\\
&+ \frac{794951144899902343 75}{632974208706639138757 0176}b_{11}^2 - \frac{452616734432524414062 5}{3797845252239834832542 1056}a_{02}b_{11}\\
&- \frac{1501926097403076171875}{1265948417413278277514 0352}a_{20}b_{11} - \frac{2113627250105517578125}{1265948417413278277514 0352}b_{02}b_{11}\\
&+ \frac{8402376020751953125}{676977763322608704552 96}b_{11}b_{20} + \frac{170019216537353515625}{3164871043533195693785088}a_{02}^2\\
&+ \frac{77972993629345703125}{452124434790456527683584}a_{02}a_{20} + \frac{378160348073486328125}{3797845252239834832542 1056}a_{02}b_{02}\\
&- \frac{34473371226674804687 5}{3797845252239834832542 1056}a_{02}b_{20} + \frac{5821488256835937 5}{322946024850326091202 56}a_{20}^2\\
&- \frac{2056445968159619140625}{1265948417413278277514 0352}a_{20}b_{02} + \frac{56233442866943359375}{744675539654869575008256}a_{20}b_{20}\\
&- \frac{371336987060546875}{9568771106676328628224}b_{20}^2 + \frac{66655457287260742187 5}{632974208706639138757 0176}b_{02}b_{20}\\
&- \frac{93035501024731445312 5}{1265948417413278277514 0352}b_{02}^2.
\end{aligned}$$

We consider now the systems $\{L_6^{(2)} = \cdots = L_{10}^{(2)} = 0\}$. Doing, for example, the blowup $a_{02} = zv_1, a_{11} = zv_2, a_{20} = zv_3, b_{02} = z, b_{11} = zv_4, b_{20} = zv_5$, we solve a system of five equations of degree 2 with respect to 5 variables. Using a computer algebra system we get that $v_k = p_k(\alpha)/q(\alpha)$ with $p_k$ and $q$ polynomials with rational coefficients of degree 27 and $\alpha$ being a solution of a given polynomial, $Q(\alpha)$, also with rational coefficients, of degree 28. This polynomial have 20 simple real solutions,

$$\{ -1.460571830, -0.6718444255, -0.6670390163, -0.5158935998, -0.4874999611,$$
$$-0.3970369469, -0.3874233159, -0.2401990480, -0.02992848475, 0.02384205186,$$

$$0.03979267840, 0.08015376288, 0.2087598950, 0.2131172755, 0.2232320471,$$
$$0.2463926997, 0.2995004189, 0.3312032992, 0.3788127882, 1.397031032\}.$$

The next step is to check that the Jacobian of the five equations with respect to $\{v_1, \ldots, v_5\}$ is different from zero. The size of the polynomials does not make possible to compute directly the determinant of the Jacobian matrix in terms of $\alpha$. Then, we do a Gauss elimination to get a triangular matrix. The diagonal elements are now of the form $J_{11} = q_1(\alpha)/q(\alpha)$ and $J_{kk} = q_k(\alpha)/(q_{k-1}(\alpha)q(\alpha))$, for $k = 2, \ldots, 4$, where all polynomials are also with rational coefficients and degree 27 in $\alpha$, in particular $q_1 = p_1$. Consequently, the determinant is $q_5(\alpha)/q^5(\alpha)$. Moreover, it can be checked also that $L_{11}^{(2)} = p_6(\alpha)/q^2(\alpha)$. The last step is the computation of the resultants of all polynomials $p_k$, $q_k$ with $Q$ with respect to $\alpha$, checking that all are different from zero. So all the variables $v_k$, among of the determinant and the value of $L_{11}^{(2)}$ are nonzero real numbers.

It is clear that with the proposed blowup, after dividing by $z^2$ we can apply the Implicit Function Theorem to find an analytic curve $v_k = V_k(z)$ where the varieties $\{L_6^{(2)}, \ldots, L_{10}^{(2)}\}$ intersect transversally and $L_{11}^{(2)}$ is non zero. Then 6 extra limit cycles appear and the statement follows.

The scheme of the proof is the same than the proof of Proposition 1.10. But, in fact, we could apply Theorem 1.3. But the work to find the line $\ell$ is the same.

We notice that all the computations in this proof have been made with a personal computer in few minutes. We have not shown the polynomials because of the size of them. The coefficients are rational numbers with more than 200 digits. $\square$

Remark 1.12. *An alternative proof of the above result can be done computing numerically an approximation of a solution*

$$v_1 = 0.19244538335484 29, \quad v_2 = 0.2384205185830677,$$
$$v_3 = 0.1490077870024313, \quad v_4 = 0.7626068770346651,$$
$$v_5 = 1.5437258992144801,$$

*with enough digits to ensure that* $L_6^{(2)}/z^2 = 6.80504529555082 \cdot 10^{-9}$ *and the Jacobian,* $-4.45823335837756 \cdot 10^{-10}$ *are non zero real numbers. This can be done using a computer assisted proof with the Poincaré–Miranda Theorem, as we will do in some of the following results.*

The following result shows the difficulties to get more than 10 limit cycles. Our computations do not provide a better result for the cyclicity of the next cubic polynomial system. The unperturbed system is labeled as $CD_{29}^{12}$ in [**65**] but we have not considered it directly. To simplify the computations, we have made an affine change of coordinates.

PROPOSITION 1.13. *There exist cubic polynomial perturbations such that the origin of system*

$$\begin{cases} \dot{x} &=& 8x^3 - 40x^2y + 2x^2 - 30xy - 5y, \\ \dot{y} &=& \dfrac{24}{5}x^3 + 24x^2y - 80xy^2 + 4x^2 + 10xy - 10y^2 + x, \end{cases} \tag{18}$$

*has cyclicity at least* 10.

PROOF. System (18) has a center at the origin because it has the rational first integral

$$\frac{(64x^3 + 72x^2 + 120xy + 30x + 30y + 5)^4}{(128x^4 + 192x^3 + 320x^2y + 128x^2 + 240xy + 40x + 40y + 5)^3},$$

which is well defined at the origin and the level curves are topologically circumferences.

Doing a first order analysis of the Lyapunov constants, only the first 6 are linearly independent. Then, as the previous studies, we can write $L_k^{(1)} = u_k$, for $k = 1, \ldots, 6$. Applying the simplification algorithm described previously, using Proposition 1.4, we get

$$\begin{aligned} L_7^{(2)} &= A_7u_7u_8, \\ L_8^{(3)} &= A_8u_7u_9u_{10}, \\ L_9^{(3)} &= u_7u_9(A_9u_7 + B_9u_{10}), \\ L_{10}^{(3)} &= u_7u_9(A_{10}u_7 + B_{10}u_{10}), \\ L_{11}^{(3)} &= u_7u_9(A_{10}u_7 + B_{10}u_{10}), \end{aligned}$$

where $A_k, B_k$ are non vanishing rational numbers. We notice that $L_{11}^{(3)} = L_{10}^{(3)}$. To study how is the local intersection of the varieties $L_k$, for $k = 7, \ldots, 11$ we need to do an adequate weighted blow-up using a privileged parameter. Here we have chosen $u_7 = z$, $u_8 = z^2z_8$, $u_9 = zz_9$, $u_{10} = zz_{10}$, $u_{11} = zz_{11}$, the other three parameters are taken as zero. The Taylor development with respect to $z$, dividing by a nonzero rational, number we get

$$\begin{aligned} L_7 &= z^3(z_8 + p_2(z_9, z_{10})) + \sum_{j\geq 4} W_{7j}(z_8, z_9, z_{10}, z_{11})z^j, \\ L_8 &= z^3z_9z_{10} + \sum_{j\geq 4} W_{8j}(z_8, z_9, z_{10}, z_{11})z^j, \\ L_9 &= z^3z_9p_1(z_9) + \sum_{j\geq 4} W_{9j}(z_8, z_9, z_{10}, z_{11})z^j, \\ L_{10} &= z^3z_9^2 + \sum_{j\geq 4} W_{10j}(z_8, z_9, z_{10}, z_{11})z^j, \end{aligned}$$

where $p_1$ and $p_2$ are polynomials of degree 1 and 2, respectively, $p_1(0) \neq 0$, and $W_{k,j}$ are also polynomials.

The statement follows as in the previous studies because there exists a transversal intersection of the varieties $\{(z_8 + p_2(z_9, z_{10})) = 0, z_9 z_{10} = 0, z_9 p_1(z_9) = 0\}$ with $z_9 \neq 0$. Then, clearly, $L_{10}$ is non vanishing for $z$ small enough. □

REMARK 1.14. *We remark that we have tried to improve the above result unsuccessfully. We have used different weighted blow-ups, higher order developments and the study of the Newton polyhedron.*

PROPOSITION 1.15. *There exists cubic polynomial perturbations such that from the origin of system*

$$\begin{cases} \dot{x} &= -\frac{338}{441}x^3 + \frac{4394}{9261}x^2y - \frac{338}{147}xy^2 + \frac{338}{147}y^3 - \frac{26}{21}x^2 - \frac{793}{441}xy + \frac{65}{21}y^2 + y, \\ \dot{y} &= \frac{338}{3087}x^3 - \frac{338}{343}x^2y - \frac{338}{1323}xy^2 - \frac{338}{1029}y^3 - \frac{65}{147}x^2 - \frac{247}{147}xy - \frac{26}{147}y^2 - x, \end{cases} \tag{19}$$

*bifurcate at least* 11 *limit cycles of small amplitude.*

PROOF. System (19) has a center at the origin because it has the rational first integral

$$H = \frac{\left(\frac{2704\,x^2}{3087} - \frac{5408\,xy}{3087} + \frac{2704\,y^2}{1323} - \frac{208\,x}{147} + \frac{416\,y}{147} + 1\right)\left(\frac{104\,x}{441} + \frac{104\,y}{147} + 1\right)^3}{\left(-\frac{2704\,x^2}{15435} + \frac{2704\,y^2}{1715} - \frac{52\,x}{147} + \frac{52\,y}{21} + 1\right)^2}$$

which is well defined at the origin, $H = 1 + 86528(x^2 + y^2)/324135 + \cdots$.

We consider the perturbed system (12) where the center is (19). Following the scheme explained in Section 1.3, the order 1 developments of the first 8 Lyapunov constants are linearly independent. Then, after a linear change of the perturbation parameters, we have that they write as $L_k^{(1)} = u_k$, for $k = 1, \ldots, 8$. Using the properties detailed in Proposition 1.4 we can simplify the next Lyapunov constants to get $L_k^{(1)} = 0$, for $k = 9, \ldots, 11$. At this point we assume, to simplify computations, that $b_{03} = b_{12} = b_{30} = 0$. Then, we can write

$$\begin{aligned} L_9^{(2)} &= -Au_9u_{10}, \\ L_{10}^{(2)} &= Bu_9u_{10}, \\ L_{11}^{(2)} &= -Cu_9u_{10}, \end{aligned}$$

where $A, B$ and $C$ are rational numbers with between 48 and 71 digits in numerator and between 68 and 86 digits in denominator. So, we see clearly here that we have at least 9 limit cycles using the trace parameter together with $u_k$, for $k = 1, \ldots, 8$. In fact, at most 9 with only order 2 developments.

Hence, if we want more limit cycles, we should compute up to order 4 developments. Because, after using again the simplification with $L_9$ and Proposition 1.4, $L_{10}^{(3)} = L_{11}^{(3)} = 0$.

Doing the blow-up $u_9 = z, u_{10} = zz_{10}$, and $u_{11} = zz_{11}$, we can divide $L_9$ by $z^2$ and we can use the Implicit Function Theorem to write $z_{10}$ as a function of $z$ and

$z_{11}$. Then, $L_{10}^{(4)} = z^4 p_4(z_{11})$ and $L_{11}^{(4)} = z^4 q_4(z_{11})$ where $p_4$ and $q_4$ are polynomial with rational coefficients of degree 4 having both 4 different real roots. Moreover, $z^4$ is a common factor of the complete $L_{10}$ and $L_{11}$ and the resultant of $p_4$ and $q_4$ with respect to $z_{11}$ is different from zero. Therefore, applying also the Implicit Function Theorem near the simple zeros of $p_4$, there exists values of the parameters, for small enough $z$, such that $L_{10} = 0$ and $L_{11} \neq 0$.

The statement follows because we have proved the existence of an analytic curve of weak-foci of order 11 such that, as in the previous proofs, 11 limit cycles of small amplitude bifurcate from the origin. $\square$

PROPOSITION 1.16. *There exists cubic polynomial perturbations such that from the origin of system*

$$\begin{cases} \dot{x} &= \frac{2809}{32946}\,x^3 - \frac{22472}{49419}\,yx^2 + \frac{5618}{16473}\,xy^2 + \frac{69695}{98838}\,x^2 - \frac{66992}{16473}\,xy + \frac{636}{289}\,y^2 + x - \frac{2112}{289}\,y, \\ \dot{y} &= \frac{22472}{247095}\,xy^2 - \frac{2809}{16473}\,y^3 + \frac{86549}{197676}\,xy - \frac{20988}{27455}\,y^2 + \frac{151447}{247095}\,x - y, \end{cases} \tag{20}$$

*bifurcate at least* 11 *limit cycles of small amplitude.*

This is the case $CD_{31}^{12}$ in [**65**]. Christopher in [**17**] provides the first analytic proof that 11 limit cycles of small amplitude exit for a cubic polynomial vector field. Here we add it for completeness.

PROOF. Proof of Proposition 1.16 The corresponding first integral of (20), which is well defined at the origin, is

$$H = \frac{\left(xy^2 + \dfrac{280\,xy}{53} + \dfrac{342\,y^2}{53} + \dfrac{22409\,x}{2809} + \dfrac{95760\,y}{2809} + \dfrac{7812755}{148877}\right)^5}{\left(x + \dfrac{342}{53}\right)^3 F_6^2(x,y)},$$

where

$$\begin{aligned} F_6(x,y) = {} & xy^5 + \frac{700}{53}\,xy^4 + \frac{342}{53}\,y^5 + \frac{406045}{5618}\,xy^3 + \frac{239400}{2809}\,y^4 + \frac{30389450}{148877}\,xy^2 \\ & + \frac{139611775}{297754}\,y^3 + \frac{18788141215}{63123848}\,xy + \frac{10549512750}{7890481}\,y^2 \\ & + \frac{150246782525}{836390986}\,x + \frac{826646189040}{418195493}\,y + \frac{26977377387858}{22164361129}. \end{aligned}$$

The proof of the above proposition follows computing the linear terms of the Lyapunov constants and then using Theorem 1.2 to provide the complete unfolding of 11 limit cycles. $\square$

## 1.5. Order one studies to get lower bounds for $M(8)$ and $M(9)$

This section is devoted to prove the statement of Theorem 1.1 corresponding to local cyclicity of polynomial vector fields of degrees 8 and 9, using only linear developments. The proofs follows computing the order 1 developments and using Theorem 1.2. In both results the unperturbed systems are centers of degrees 7 and 8 with a straight line of equilibrium points, $(1-x-y)$.

We notice that the parallelization procedure described in Section 1.3 is indispensable to get the results. The total computation time, in both cases, is less than one hour.

Proposition 1.17. *Consider the perturbed system* (6) *of degree* $n=8$ *with the center* $(\dot{x},\dot{y})=(P_c(x,y),Q_c(x,y))$,

$$\begin{cases} \dot{x} &= (1-x-y)(-\frac{2527}{3}x^6y-\frac{2968}{3}x^5y^2-\frac{4186}{3}x^4y^3-\frac{2800}{3}x^3y^4-553x^2y^5 \\ &\quad +56xy^6+\frac{184}{3}x^3y+\frac{88}{3}x^2y^2+48xy^3-y), \\ \dot{y} &= (1-x-y)(672x^7+1484x^6y+\frac{2219}{3}x^5y^2+\frac{5684}{3}x^4y^3-\frac{742}{3}x^3y^4+\frac{1148}{3}x^2y^5 \\ &\quad -315y^6-28y^7-58x^4-44x^3y-\frac{104}{3}x^2y^2-\frac{44}{3}xy^3+10y^4+x). \end{cases}$$

*There are perturbation parameters* $\lambda$ *such that at least* 76 *limit cycles of small amplitude bifurcate from the origin.*

The above system, without the straight line of equilibrium points, is a center because it has the first integral $H(x(x^2+y^2),y(x^2+y^2))$ where

$$H(x,y)=\frac{(42x-7y-1)^3\,f_3(x,y)}{(448x^2+336xy+63y^2-44x-12y+1)^3\,(1183x^2-68x+1)} \tag{21}$$

and $f_3(x,y)=(10752x^3+29568x^2y+17640xy^2+3024y^3-1600x^2-2760xy-576y^2+74x+57y-1)$. The rational first integral (21) corresponds to the cubic polynomial center provided by Bondar and Sadovski in [**7**]. They prove that the cubic perturbations provide also 11 limit cycles using only order 1 developments as system (20).

Proposition 1.18. *Consider the perturbed system* (6) *of degree* $n=9$ *with the center* $(\dot{x},\dot{y})=(P_c(x,y),Q_c(x,y))$,

$$\begin{cases} \dot{x} &= (1-x-y)(\frac{54}{175}x^8+\frac{18}{35}x^7y-\frac{54}{175}x^6y^2+\frac{894}{175}x^5y^3-2x^4y^4+\frac{66}{25}x^3y^5 \\ &\quad -\frac{26}{35}x^2y^6-\frac{342}{175}xy^7+\frac{16}{25}y^8-y), \\ \dot{y} &= (1-x-y)(-\frac{198}{175}x^7y-\frac{1254}{175}x^6y^2-\frac{586}{175}x^5y^3-\frac{258}{35}x^4y^4-\frac{22}{5}x^3y^5 \\ &\quad +\frac{18}{25}x^2y^6-\frac{382}{175}xy^7+\frac{162}{175}y^8+x). \end{cases}$$

*There are perturbation parameters* $\lambda$ *such that at least* 88 *limit cycles of small amplitude bifurcate from the origin.*

The proof that the above system, without the straight line of equilibrium points, is a center follows from an idea of Giné in [**32**]. We consider the center with homogeneous quartic nonlinearities given in [**32**, System (6), Pag. 8857] taking $c=4/5$ and

$s = 3/5$. The change of variables $(x, y) = r^{3/7}(\cos\theta, \sin\theta)$ in such quartic system gets a system of degree 8 having also a center at the origin.

We notice that for other degrees, $n = 3, \ldots, 7$, adding a straight line of equilibria to a center of degree $n-1$, we have not obtained higher lower bounds for the local cyclicity than the ones obtained previously or the best given in the results of the next section using higher order Taylor series. For example, the best cubic system of Section 1.4 adding such curve provides a quartic system with only 19 limit cycles up to first and second order studies. For degree 6, we have not found any system to improve the highest value found in [**45**], $M(6) \geq 40$.

## 1.6. Higher order studies to get lower bounds for $M(4)$, $M(5)$, and $M(7)$

This section is devoted to prove the statement of Theorem 1.1 for the local cyclicity of polynomial vector fields of degrees 4, 5, and 7. In the proofs we will use higher order developments.

PROPOSITION 1.19. *Consider the perturbed system* (6) *of degree $n = 4$ with the center $(\dot{x}, \dot{y}) = (P_c(x, y), Q_c(x, y))$,*

$$\begin{cases} \dot{x} &= \dfrac{1}{25}(32x^4 - 168x^2y^2 + 32xy^3 + 24y^4) - y, \\ \dot{y} &= \dfrac{1}{25}(-32x^3y - 192x^2y^2 - 24xy^3 + 64y^4) + x. \end{cases} \tag{22}$$

*There are perturbation parameters $\lambda$ such that at least* 20 *limit cycles of small amplitude bifurcate from the origin.*

PROOF. The system in the statement is presented in [**32, 33**]. In [**32**] it is proved that the origin is a center because it has the inverse integrating factor

$$\begin{aligned} V(x,y) =& \frac{1}{3125}(8xy - 4y^2 + 10y - 5)(64x^3 - 96x^2y + 48xy^2 - 8y^3 - 25)\times \\ & (64x^2y^2 - 64xy^3 + 16y^4 - 80xy^2 + 40y^3 + 40xy + 80y^2 + 50y + 25) \end{aligned}$$

First, we compute the Taylor developments of the Lyapunov constants up to order 4. Because we will see that with order 3 is not enough to prove the statement. As we have detailed in the proofs of Propositions 1.10 and 1.11, simplifying with Proposition 1.4 and up to multiplicative nonzero constants, we have

$$\begin{aligned} &L_k = u_k + O_2(\lambda), \text{ for } k = 1, \ldots, 16, && L_{19} = O_3(\lambda), \\ &L_{17} = -u_{17}u_{19} + u_{18}^2 + O_3(\lambda), && L_{20} = O_3(\lambda), \\ &L_{18} = u_{17}u_{18} + O_3(\lambda), && L_{21} = u_{20}^2 + O_3(\lambda). \end{aligned} \tag{23}$$

Here $\lambda$ denotes the perturbation parameters and $O_k(\lambda)$ are the monomials of degree at least $k$ in $\lambda$.

Second, using the Implicit Function Theorem, we can consider only $L_k$, for $k = 17, \ldots, 21$ depending on $(u_{17}, \ldots, u_{24})$.

The third step continues studying the higher order developments. Here an adequate blow-up is

$$
\begin{aligned}
u_{17} &= z, & u_{18} &= z^2 w_1, & u_{19} &= z^2 w_2, & u_{20} &= z w_3,\\
u_{21} &= z w_4, & u_{22} &= z w_5, & u_{23} &= z w_6, & u_{24} &= z w_7.
\end{aligned}
$$

Then, doing a Taylor expansion in $z$ of order 3, and after dividing by $z^3$, we have

$$
\begin{aligned}
\widetilde{L}_{17}^{(3)} &= w_2 + \frac{1}{3402000} w_3^2 + \frac{79}{4838400} w_6^2 + \frac{1}{241920} w_3 w_6,\\
\widetilde{L}_{18}^{(3)} &= w_1 - \frac{387}{39040} w_6 w_7 + \frac{9}{2440} w_4 w_6 + \frac{1}{1220} w_5 w_6 - \frac{1}{1600} w_3 w_7\\
&\quad - \frac{227971304366744606815875047421098088169074843984539306177 69}{2259939501758800074182560369244655286878167559053615525151 3600} w_6\\
&\quad - \frac{99996445400250450505317073169188260741337096265876115 29}{21707922980521260958464655186135905035330759084147177815875 0} w_3,\\
\widetilde{L}_{19}^{(3)} &= \frac{54197\ldots 00000}{866211\ldots 28447} w_3 + \frac{362945\ldots 0000}{28873\ldots 76149} w_7,\\
\widetilde{L}_{20}^{(3)} &= 0.
\end{aligned}
$$

As the above non vanishing three terms have rank 3 with respect to $w_1, w_2$, and $w_3$, we can use also the Implicit Function Theorem to solve $\widetilde{L}_k^{(3)} = z_k$, with respect to $w_1, w_2, w_3$. Therefore,

$$
L_{17}^{(3)} = z^3 z_{17}, \quad L_{18}^{(3)} = z^3 z_{18}, \quad L_{19}^{(3)} = z^3 z_{19}, \quad L_{20}^{(4)} = z^4 w_6^2.
$$

From the above description it is clear the existence of a curve of weak-foci of order 20 that unfolds 20 limit cycles of small amplitude at it is indicated in the statement. The proof finishes as the proofs of Propositions 1.10 and 1.11. $\square$

Remark 1.20. *In* [**32**], *it is studied the homogeneous nonlinearities perturbation of degree* 4 *of* (22). *It is proved, with order two,* 7 *limit cycles exists. This proves that there exists a curve of weak-foci of order* 21. *This fact can be seen in* (23) *because, in the homogeneous case,* $L_{21}$ *is non vanishing. But it is not possible to find a complete unfolding of the* 21 *limit cycles with order* 2 *because there are no free parameters. Because* $L_{19}^{(2)} = L_{20}^{(2)} = 0$. *In the proof is clear that with order three we get only* 19 *limit cycles and with order* 4 *we get* 20. *The problem about the existence of a complete unfolding of* 21 *remains open. The computations to go further in the order is very hard.*

Proposition 1.21. *Consider the perturbed system* (6) *of degree* $n = 5$ *with the center* $(\dot{x}, \dot{y}) = (P_c(x, y), Q_c(x, y))$,

$$
\left\{
\begin{array}{rcl}
\dot{x} &=& \frac{1}{25}(42x^5 - 12x^4 y - 476x^3 y^2 - 68x^2 y^3 + 266xy^4 + 56y^5) - y,\\
\dot{y} &=& \frac{1}{25}(-8x^5 - 26x^4 y - 28x^3 y^2 - 4x^2 y^3 - 132xy^4 + 6y^5) + x.
\end{array}
\right. \tag{24}
$$

*There are perturbation parameters $\lambda$ such that at least* 33 *limit cycles of small amplitude bifurcate from the origin.*

Proof. As the previous result, system (24) appear also in [**32, 33**]. The system has a center because

$$\begin{aligned} V(x,y) =&(64x^8+1600x^7y+13456x^6y^2+32000x^5y^3-99616x^4y^4-380800x^3y^5\\ &+320656x^2y^6+548800xy^7+153664y^8+20000x^4+85000x^3y\\ &-10000x^2y^2-155000xy^3-100000y^4+15625)\times\\ &(-64x^4+192x^3y-16x^2y^2-192xy^3-64y^4+25)^{\frac{1}{4}} \end{aligned}$$

is an inverse integrating factor.

Computing the Lyapunov constants up to order 2 we can check that the first 17 linear parts are linearly independent. Then up to a linear change of coordinates in the parameter space we can write $L_k^{(1)} = u_k$, for $k = 1, \ldots, 17$, and $L_k^{(1)} = 0$, for $k = 18, \ldots, 33$. Then using the scheme of the proof of Proposition 1.11, simplifying also with the properties of Proposition 1.4, and using the Implicit Function Theorem we can restrict our study to see the intersection of 16 homogeneous polynomials of degree 2,

$$L_k^{(2)} = P_k(\hat{\lambda}), \text{ for } k = 18, \ldots, 33, \tag{25}$$

with $\hat{\lambda} = (u_{18}, u_{19}, \ldots, u_{33})$ choosing three perturbation parameters as zero. We recall that for degree 5 perturbation we have 36 parameters, but here only 33 will be essential.

The next step is to consider the blow-up $u_k = zz_k$, for $k = 18, \ldots, 32$ and $u_{33} = z$ in (25), writing $P_k(\hat{\lambda}) = z^2 P_k(\hat{z})$ with $\hat{z} = (z_{18}, z_{19}, \ldots, z_{32})$. Then, we need to show that this system of 15 equations of degree 2 with respect to 15 variables has at least a transversal intersection real point, $\hat{z}^*$. Moreover, we should check that $P_{33}(\hat{z}^*)$ is non vanishing. The proof finishes applying Theorem 1.3 to provide the complete unfolding of 33 limit cycles of small amplitude. The main difference with respect to the proof of Proposition 1.11 is that here we can not obtain the explicit solution in terms of polynomials in one privileged variable. Because of the high number of variables and the size of the coefficients of the polynomials $P_k$.

Numerically, we can get an approximate solution

$$\begin{aligned} &z_{18}^* \approx 0.414467055443, && z_{19}^* \approx 0.977703106281, && z_{20}^* \approx 0.831273897080,\\ &z_{21}^* \approx 10.87232453671, && z_{22}^* \approx 0.089114602089, && z_{23}^* \approx 5.803007782422,\\ &z_{24}^* \approx -13.46886905316, && z_{25}^* \approx -2.653100632593, && z_{26}^* \approx 0.071920750628,\\ &z_{27}^* \approx 1.279070836650, && z_{28}^* \approx -0.963042490919, && z_{29}^* \approx -7.708796674748,\\ &z_{30}^* \approx -0.27853535522, && z_{31}^* \approx -7.245147157590, && z_{32}^* \approx 2.513953010283. \end{aligned}$$

Then $L_{33}(z^*) = -5.28936073528$ and the Jacobian matrix of $(P_{18}, \dots, P_{32})$ with respect to $\hat{z}$ at $\hat{z}^*$ is $1.2572040284 \cdot 10^{14}$. We have solved numerically with different number of Digits (up to 1000 digits) to ensure the convergence of the numerical solution. The above numerical approximation is shown with only 12 digits.

Then we will use the Poincaré–Miranda Theorem, to prove analytically the existence of the point $z^*$. This will be doing an interval analysis to apply Theorem 1.5. Finally, we will check that the Jacobian and $L_{33}^{(2)}$ are nonvanishing at $z^*$, also with an accurate interval analysis. This is done as a *Computer Assisted Proof* mechanism.

The first step is to convert the above approximate solution to its rational expression

$$
\begin{aligned}
z_{18}^* &\approx \frac{8289341108806467679}{20000000000000000000}, & z_{19}^* &\approx \frac{97770310628309363091}{100000000000000000000},\\
z_{20}^* &\approx \frac{83127389707517527887}{100000000000000000000}, & z_{21}^* &\approx \frac{10872324536703550641}{1000000000000000000},\\
z_{22}^* &\approx \frac{22278650522188043761}{250000000000000000000}, & z_{23}^* &\approx \frac{14507519456029022899}{2500000000000000000},\\
z_{24}^* &\approx -\frac{6734434526579186771}{500000000000000000}, & z_{25}^* &\approx -\frac{26531006325905618217}{10000000000000000000},\\
z_{26}^* &\approx \frac{35960375313628249487}{500000000000000000000}, & z_{27}^* &\approx \frac{12790708366491310147}{10000000000000000000},\\
z_{28}^* &\approx -\frac{240760622728448541}{250000000000000000}, & z_{29}^* &\approx -\frac{38543983373719519357}{5000000000000000000},\\
z_{30}^* &\approx -\frac{1740845970112931499}{6250000000000000000}, & z_{31}^* &\approx -\frac{14490294315170358141}{2000000000000000000},\\
z_{32}^* &\approx \frac{25139530102793502293}{10000000000000000000}.
\end{aligned}
$$

Then, we consider an affine change of parameters such that the linear part of $f = (L_{18}, \dots, L_{32})$ will be the new variables. Then, the Jacobian matrix at this point will be near the identity matrix. Next, we apply Theorem 1.5 with $n = 15$, $c = 10^{-9}$ to $f = (L_{18}, \dots, L_{32})$. The conditions about the sign of the components of $f$ on the faces $S_i^{\pm}$ is obtained from Lemmas 1.23 and 1.24. The existence of $z^*$ is guaranteed because $f_i(S_i^-) \subset [-2.01 \cdot 10^{-9}, -1.99 \cdot 10^{-9}]$ and $f_i(S_i^-) \subset [1.99 \cdot 10^{-9}, 2.01 \cdot 10^{-9}]$. We notice that the numerator and denominators of $L_i, i = 18, \dots, 32$ are integer numbers between 138 to 152 digits each. Moreover, $L_{33} \in [-5.482536, -5.0966185]$ its numerator and denominator has more than 123 digits, so we have $L_{33} < 0$. Finally, we must show that the determinant of the Jacobian matrix, $Jf(z)$, of $f$ with respect to $z$ does not vanish. This determinant, as it is a $15 \times 15$ matrix, needs a very high computational cost. Alternatively, we can use the Theorem of Gershgorin to show that its eigenvalues are in a ball centered at 1 with radius $10^{-3}$. That is $|\lambda - 1| < 10^{-3}$. Calling $J_{i,j}$ the $i, j$-element of $Jf(z)$, using Lemmas (1.23) and (1.24), we can check that $J_{i,i} \in [0.999919, 1.000009]$ and for $i \neq j$, $J_{i,j} \in [0.000018, 000212]$ or $J_{i,j} \in [-5.7414588 \times 10^{-7}, -0.0001480, ]$ if $J_{i,j}$ is positive or negative, respectively. Therefore, all eigenvalues, see Theorem 1.6, do not vanish and, consequently, the determinant $Jf(z)$ and the result follows. $\square$

PROPOSITION 1.22. *Consider the perturbed system* (6) *of degree* $n = 7$ *with the center* $(\dot{x}, \dot{y}) = (P_c(x, y), Q_c(x, y))$,

$$\left\{\begin{array}{rcl} \dot{x} &=& (1-x)\Big(-y+\dfrac{8}{45}(2x-y)(24x^5-12x^4y-32x^3y^2+12x^2y^3-42xy^4-5y^5)\Big), \\ \dot{y} &=& (1-x)\Big(2x-\dfrac{16}{45}y(2x-y)(28x^4+66x^3y+6x^2y^2+19xy^3+6y^4)\Big). \end{array}\right. \tag{26}$$

*There are perturbation parameters* $\lambda$ *such that at least* 61 *limit cycles of small amplitude bifurcate from the origin.*

PROOF OF PROPOSITION 1.22. A similar system as the unperturbed one, without the straight line of equilibria $1 - x = 0$, appears in [**32**] in the study of local cyclicity for homogeneous nonlinearities perturbation. Giné proposes to start with a quartic system as (22) having a center at the origin because it has an integrating factor. Then we should change to polar coordinates and transform variable $r$ to a new radial variable $R^{3/5}$. With these changes the new system has degree 6. With this mechanism we get system (26) and we know that it has a center at the origin. In fact, it has the next inverse integrating factor

$$\begin{aligned} V(x,y) =& (2x^2+y^2)^{-2/3}(128x^5-192x^4y+160x^3y^2-112x^2y^3+48xy^4-8y^5-9)\times \\ & (2048x^7y^3-3072x^6y^4+3584x^5y^5-3328x^4y^6+2048x^3y^7-1024x^2y^8 \\ & +384xy^9-64y^{10}+864x^3y^2+432xy^4-27). \end{aligned}$$

Using only the order 1 developments of the Lyapunov constants, see Theorem 1.2, we get only 58 limit cycles of small amplitude because there exists a linear change of variables in the parameter space such that $L_k^{(1)} = u_k$, for $k = 1, \ldots, 58$ and $L_k^{(1)} = 0$, for $k = 59, 60, 61$.

Computing the second order developments of the Lyapunov constants and doing the simplifications as the previous proofs, using again Proposition 1.4, we can remove $u_k$ for $k = 1, \ldots, 58$ from $L_k^{(2)}$, for $k = 59, 60, 61$. Vanishing the non essential parameters and doing the blow-up $u_{59} = zz_1$ $u_{60} = zz_2$ and $u_{61} = z$ we get

$$L_{59}^{(2)} = A_{59}z^2\mathcal{L}_1(z_1, z_2), \quad L_{60}^{(2)} = A_{59}z^2\mathcal{L}_2(z_1, z_2), \quad L_{61}^{(2)} = A_{59}z^2\mathcal{L}_3(z_1, z_2),$$

where $\mathcal{L}_k$ are polynomials of degree 2 and $A_k$ rational nonvanishing numbers. These polynomials have rational coefficients with numerators and denominators of around

1900 digits each. Approximately they write

$$\mathcal{L}_1 \approx z_1^2 + 77.576637 z_1 z_2 + 22.493284 z_2^2 + 107.76288 z_1 + 1038.9032 z_2 + 1265.0912,$$
$$\mathcal{L}_2 \approx z_1^2 - 2.6270001 z_1 z_2 + 0.27770877 z_2^2 + 25.446941 z_1 - 35.950489 z_2 + 160.06265,$$
$$\mathcal{L}_3 \approx z_1^2 + 3.4484543 z_1 z_2 + 2.9923181 z_2^2 + 32.128183 z_1 + 44.721607 z_2 + 248.24137.$$

The last step is to show that there exists at least a transversal real solution, $z^* = (z_1^*, z_2^*)$, of $\{\mathcal{L}_1 = 0, \mathcal{L}_2 = 0\}$ such that $\mathcal{L}_3(z^*)$ is non vanishing. Then Theorem 1.3 applies and the proof follows.

With an algebraic manipulator we can find the solution of $\{\mathcal{L}_1 = 0, \mathcal{L}_2 = 0\}$. It writes as $(z_1^*, z_2^*) = (p_3(\alpha), \alpha)$ where $p_3(\alpha)$ is a polynomial of degree 3 with rational coefficients and $\alpha$ is a real root of a given polynomial of degree 4, $p_4(\alpha)$. The polynomials $p_3$ and $p_4$ have rational coefficients with numerators and denominators of around 6000 and 4000 digits each, respectively. Approximately they write

$$p_4(\alpha) \approx \alpha^4 + 0.87600811\alpha^3 - 1.97816765\alpha^2 - 3.22558688\alpha - 1.29759793,$$
$$p_3(\alpha) \approx 3.43467644\alpha^3 - 0.51633002\alpha^2 - 6.541427002\alpha - 17.7666993.$$

As the polynomial $p_4$ has only 2 real roots, the considered system has only two real solutions

$$(z_{1a}^*, z_{2a}^*) \approx (-14.693346428044632240, -0.85248929481003427092),$$
$$(z_{1b}^*, z_{2b}^*) \approx (-13.701549420630826548, 1.6907716352120896856).$$

As a function of $\alpha$, we can find explicitly the values $\det \mathrm{J}(\mathcal{L}_1, \mathcal{L}_2)_{(z_1^*, z_2^*)}$ and $\mathcal{L}_3(z_1^*, z_2^*)$. They are also polynomials of degree 3 in $\alpha$ with rational coefficients that approximately write

$$\det \mathrm{J}(\mathcal{L}_1, \mathcal{L}_2)_{(z_1^*, z_2^*)} \approx 611.007749\alpha^3 + 530.704211\alpha^2 - 846.328099\alpha - 1100.76580,$$
$$\mathcal{L}_3(z_1^*, z_2^*) = 1.53111109\alpha^3 + 0.80160181\alpha^2 - 3.66450101\alpha - 3.44764247.$$

It can be seen that all the polynomials of degree 3, $p_3$, $\det \mathrm{J}(\mathcal{L}_1, \mathcal{L}_2)_{(z_1^*, z_2^*)}$, $\mathcal{L}_3(z_1^*, z_2^*)$ have no common zeros with $p_4$ because their respective resultants, with respect to $\alpha$, are non zero rational numbers. This proves the transversality and that the last Lyapunov constant is non vanishing.

In Figure 6.1, we draw the level curves of $\mathcal{L}_k$, for $k = 1, 2, 3$, in a neighborhood of the intersection points. The transversality is clear. $\square$

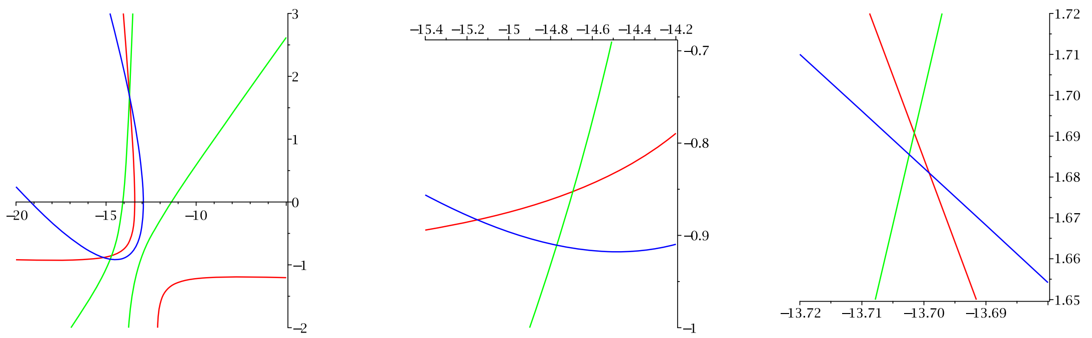

FIGURE 6.1. The level curves $\mathcal{L}_1 = 0$, $\mathcal{L}_2 = 0$, and $\mathcal{L}_3 = 0$ in red, green, and blue, respectively

## 1.7. Final comments

Taking a look of all the considered systems it is clear that we need new good examples to get higher lower bounds for the local cyclicity. The main difficulty is to know how to get them to ensure that only with developments of order 1 it is enough to get the value conjectured by Giné, $n^2 + 3n - 7$. In the language of Zoladek, see [**63**], this is equivalent to find systems with maximal codimension. It is also clear that with this mechanism we will never provide upper bounds. In the next chapter we will see that this conjecture fails, at least for degree 3 polynomial vector fields.

We notice that the importance of Christopher work in [**17**] is that he pointed out that the computation of the Lyapunov constants near a fixed center can be done without knowing their explicit expressions. This fact has been crucial to perform all the computations made in this chapter and allow us to go further in determining the best lower bounds for $M(n)$ for lower degrees $n$. In particular, to design our parallelization algorithm.

For studying this local problem, the parallelization mechanism has been really a good tool. It has two computational advantages, the first is the decreasing of the total computation time, the second is the decreasing of memory necessities. Both because partial computations require less time and less memory. Among these advantages, the difficulties now do not depend on the computation mechanisms. They are the size of the objects of higher developments, the knowledge of the local intersection of the varieties, and the high number of variables.

Finally, the numerical computations are also not easy. Because as the degrees are quite high, to have small approximation errors we need to work with very high precision.

### 1.8. Appendix

Next two technical results will help us to find upper and lower bounds for a polynomial of $n$ variables in a $n$-dimensional cube. The proofs of them can be found in [**22**].

Lemma 1.23 ([**22**]). *Consider $h>0$, $p>0$, $q$ real numbers such that $p \in [\underline{p}, \overline{p}]$, with $\underline{p}\overline{p}>0$, and $q \in [\underline{q}, \overline{q}]$, with $\underline{q}\overline{q}>0$.*

*i) Then, $\sigma^l(q,p) \le qp \le \sigma^r(q,p)$, where*

$$\sigma^l(q,p) = \begin{cases} q\underline{p} & , \text{if } q>0, \\ q\overline{p} & , \text{if } q<0, \end{cases}$$
$$\sigma^r(q,p) = \begin{cases} q\overline{p} & , \text{if } q>0, \\ q\underline{p} & , \text{if } q<0. \end{cases}$$

*ii) If $u_j \in [-h,h]$, for $j=1,\dots,n$ and denoting $u^i = u_1^{i_1}\dots u_n^{i_n}$ for $i=(i_1,\dots,i_n)\neq 0$, we have $\mathcal{X}^l(q,u^i) \le qu^i \le \mathcal{X}^l(q,u^i)$, where*

$$\mathcal{X}^l(q,u^i) = \begin{cases} 0, & \text{if } q>0 \text{ and } i_k \text{ even for all } k=1,\dots,n, \\ -\overline{q}h^{i_1+\cdots+i_n}, & \text{if } q>0 \text{ and } i_k \text{ odd for some } k=1,\dots,n, \\ \underline{q}h^{i_1+\cdots+i_n}, & \text{if } q<0. \end{cases}$$

*and*

$$\mathcal{X}^r(q,u^i) = \begin{cases} -\overline{q}h^{i_1+\cdots+i_n}, & \text{if } q>0 \text{ and } i_k \text{ even for all } k=1,\dots,n,, \\ 0, & \text{if } q<0 \text{ and } i_k \text{ odd for some } k=1,\dots,n, \\ \underline{q}h^{i_1+\cdots+i_n}, & \text{if } q<0. \end{cases}$$

*Furthermore, $\mathcal{X}^l(q,1)=\underline{q}$ and $\mathcal{X}^r(q,1)=\overline{q}$.*

Lemma 1.24 ([**22**]). *Let $h>0$ and $p_j$ be a positive non rational numbers such that $p_j \in [\underline{p_j}, \overline{p_j}]$ with $\underline{p_j}, \overline{p_j}$ rational numbers satisfying $\underline{p_j}, \overline{p_j}>0$, for $j=1,\dots,m$. Consider the polynomial*

$$\mathcal{U}(u_1,\dots,u_n) = \sum_{i_1+\cdots+i_n=0}^{M} \left( \sum_{j=1}^{m} U_{j,i}p_j \right) u^i$$

*with $u^i = u_1^{i_i}\dots u_n^{i_n}$, $i=(i_i,\dots,i_n)$ and $U_{j,i}$ rational numbers. Then*

$$U_i^l \le \sum_{j=1}^{M} U_{j,i}p_j \le U_i^r,$$

*with $U_i^l = \sum_{j=1}^{m} U_{j,i}\sigma^l(U_{j,i},p_j)$ and $U_i^r = \sum_{j=1}^{m} U_{j,i}\sigma^r(U_{j,i},p_j)$. Moreover, if $u_j \in [-h,h]$, for $j=1,\dots,n$ and $U_i^l > U_i^r$ then*

$$\underline{\mathcal{U}} = \sum_{i_1+\cdots+i_n=0}^{M} \mathcal{X}^l(U_i^l,u^i) \le \mathcal{U}(u_1,\dots,u_n) \le \sum_{i_1+\cdots+i_n=0}^{M} \mathcal{X}^t(U_i^t,u^i) = \overline{\mathcal{U}}$$

CHAPTER 2

# Lower bounds for the local cyclicity for families of centers

In this chapter we are interested on how the local cyclicity of a family of centers depends on the parameters. In fact that, that the genericity is broken in some special values. This fact, was pointed out in [**62**], to prove that there exist a family of cubic centers, labeled by $CD_{31}^{12}$ in [**65**], with more local cyclicity than expected. In this family there is a special center such that at least twelve limit cycles of small amplitude bifurcate from the origin when we perturb it in the cubic polynomial general class. The original proof has some mistakes that we correct here. We take the advantage of better understanding of the bifurcation phenomenon to show two new cubic systems exhibiting 11 limit cycles. Finally, we study the local cyclicity of holomorfic quartic centers, proving that 21 limit cycles of small amplitude bifurcate from the origin, when we perturb in the class of quartic polynomial vector fields.

## 2.1. Introduction

The study of limit cycles began at the end of the 19th century with Poincaré. Years later, in 1900, Hilbert presents a list with problems unsolved. From the original 23 problems of the Hilbert's list, the 16th still open. The second part of this problem consist in determining the maximal number (named $H(n)$) of limit cycles, and their relative positions, of a planar polynomial systems of degree $n$. However, there are also other versions of 16th Hilbert's problem. Arnold in [**4**] proposed a version focused on studying the number of limit cycles bifurcating from the period annulus of Hamiltonians systems. In this chapter, we are interested in provide the number $M(n)$ of small amplitude limit cycles bifurcating from an elementary center or an elementary focus, in special for the cases $3 \leq n \leq 4$. The main idea is to study the local cyclicity of families of centers depending on a finite number of parameters.

As we have detailed in the previous chapter, for $n = 2$, Bautin proved in [**6**] that $M(n) = 3$. In [**64, 66**] Zoladek shown that $M(3) \geq 11$. Christopher, in [**17**], gave a simple proof of Zoladek's result perturbing another cubic center with a rational first integral, using only the linear parts of the Lyapunov constants. The interest of this result is that, as we have shown in Chapter 1, we can compute the linear Lyapunov constants development near a center without having the complete expressions. Basically the used technique consists in to choose a point on the center variety and at this point consider the linear term of the Lyapunov constants, if the point is chosen on a component of the center variety of codimension $k$, then the first $k$ linear terms of the Lyapunov constants are independent. This is a direct application of the Implicit Function Theorem to prove that $M(n) \geq k$. Usually we use this technique to provide lower bounds for the local cyclicity problem in the class of polynomial vector fields of degree $n$. In [**32, 33**], Giné presents a conjecture that the number $M(n)$ is bounded below by $n^2+3n-7$. From the study of previous chapter we have that $M(2) = 3$, $M(3) \geq 11$, $M(4) \geq 20$, $M(5) \geq 33$, $M(7) \geq 61$, $M(8) \geq 76$, and $M(9) \geq 88$.

In [**62**], Pei Yu and Yun Tian point out that the one parameter family of centers labeled by $CD_{31}^{12}$ in [**65**] is quite special because it can exhibit one more limit cycle than expected in Giné's conjecture. This family has the next rational first integral

$$H(x,y) = \frac{(xy^2+x+1)^5}{x^3(xy^5+5xy^3/2+5y^3/2+15xy/8+15y/4+a)^2} \tag{27}$$

and it have, following Zoladek computations, codimension 12. The original proof has some mistakes that we correct here, proving effectively that there exist some special values of the parameters of $CD_{31}^{12}$ such that 12 limit cycles of small amplitude bifurcate from the origin when we perturb in the class of complete cubic polynomial vector fields. This family was also studied by Christopher in [**17**] and it was the first clear proof about the existence of at least 11 limit cycles of small amplitude bifurcating from an equilibrium in polynomial vector fields of degree 3.

The main result of this chapter is the following.

THEOREM 2.1. *The number of limit cycles of small amplitude bifurcating from an equilibrium of monodromic type in the classes of polynomial vector fields of degrees* 3 *and* 4 *are* $M(3) \geq 12$ *and* $M(4) \geq 21$*, respectively.*

The proof of the above result is based on an extension of Theorems 1.2 and 1.3 when the considered center has some parameters. This is Theorem 2.2, proved in Section 2.2. Of course, the parallelization algorithm introduced in Chapter 1 is crucial to get all the needed computations. In Section 2.3 we do the proof of the statement of Theorem 2.1 corresponding to degree 3 vector fields. Moreover, we study also the bifurcation diagrams of limit cycles of small amplitude bifurcating from two families of centers. The first is 1-parametric and it is the rational reversible center family labeled by $CR_{12}^{17}$ in [**65**]. The second is a 2-parameter holomorphic cubic center family. In Section 2.4 we study the bifurcation diagram for a 2-parameter center family of degree 4 that allow us to prove the statement of Theorem 2.1 corresponding to degree 4 vector fields. Finally, we also study partially the bifurcation diagram for a 4-parameter quartic holomorphic family of centers.

## 2.2. Local cyclicity depending on parameters

This section is devoted to extend Theorems 1.2 and 1.3 to families of centers that depend on some parameters. Let $(\dot{x}, \dot{y}) = (P_c(x, y, \mu), Q_c(x, y, \mu))$ be a family of polynomial centers of degree $n$ depending on a parameter $\mu \in \mathbb{R}^\ell$, having a center equilibrium point at the origin. We consider the perturbed polynomial system

$$\begin{cases} \dot{x} &= P_c(x, y, \mu) + \alpha y + P(x, y, \lambda), \\ \dot{y} &= Q_c(x, y, \mu) + \alpha x + Q(x, y, \lambda), \end{cases} \tag{28}$$

with $P, Q$ polynomials of degree $n$ having no constant nor linear terms. More concretely,

$$P(x, y, \lambda) = \sum_{k+l}^{n} = a_{k,l=2} x^k y^l, \quad Q(x, y, \lambda) = \sum_{k+l=2}^{n} = b_{k,l} x^k y^l,$$

with $\lambda = (a_{20}, a_{11}, a_{02}, \ldots, b_{20}, b_{11}, b_{02}) \in \mathbb{R}^{n^2+3n-4}$. As in the previous chapter, the trace parameter $\alpha$ sometimes is also denoted by $L_0$.

THEOREM 2.2. *When* $a = 0$*, we denote by* $L_j^{(1)}(\lambda, b)$ *the first order development, with respect to* $\lambda \in \mathbb{R}^k$*, of the* $j-$*Lyapunov constant of system* (28)*. We assume* *that, after a change of variables in the parameter space if necessary, we can write*

$$L_j = \begin{cases} \lambda_j + O_2(\lambda), \ \textit{for } j = 1, \ldots, k-1, \\ \sum\limits_{l=1}^{k-1} g_{j,l}(\mu)\lambda_l + f_{j-k}(\mu)\lambda_k + O_2(\lambda), \ \textit{for } j = k, \ldots, k+\ell. \end{cases} \tag{29}$$

*Where with $O_2(\lambda)$ we denote all the monomials of degree higher or equal than 2 in $\lambda$ with coefficients analytic functions in $\mu$. If there exist a point $\mu^*$ such that $f_0(\mu^*) = \cdots = f_{\ell-1}(\mu^*) = 0$, $f_\ell(\mu^*) \neq 0$, and Jacobian matrix of $(f_0, \ldots, f_{\ell-1})$ with respect to $\mu$ has rank $\ell$ at $\mu^*$, then system (28) has $k+\ell$ hyperbolic limit cycles of small amplitude bifurcating from the origin.*

Remark 2.3. *We remark the importance, in the above result, of the number of components in parameters $\lambda$ and $\mu$. Because, if there are more parameters than $k$ in $\lambda$, in $O_2$ can appear monomials of degree 2 that can affect the monomials of degree 1 and the result could be not valid.*

Proof of Theorem (2.2). We assume first that the trace parameter $\alpha$ is zero. Then, using Proposition 1.4 we can remove the sums in (29) and consider a simpler list

$$L_j = \begin{cases} \lambda_j + O_2(\lambda), \ \text{for } j = 1, \ldots, k-1, \\ f_{j-k}(\mu)\lambda_k + O_2(\lambda), \ \text{for } j = k, \ldots, k+\ell. \end{cases}$$

With the Implicit Function Theorem in the first $k-1$ components and writing $\lambda_k = u_k$ the above expression writes as

$$L_j = \begin{cases} u_j, \ \text{for } j = 1, \ldots, k-1, \\ f_{j-k}(\mu)u_k + O_2(u), \ \text{for } j = k, \ldots, k+\ell. \end{cases} \tag{30}$$

From the hypothesis on the functions $f_j$ at $\mu = \mu^*$, using again the Implicit Function Theorem, we can write, close to $\mu = \mu^*$, $f_{j-k}(\mu) = v_{j-k} + O_2(v)$, with $v_{j-k} = \mu_{j-k} - \mu^*_{j-k}$, for $j = k, \ldots, k+\ell-1$, and $v = (v_0, \ldots, v_{\ell-1})$.

Now, we consider the change of variables, like a partial blowup, $u_j = zw_j$ for $j = 1, \ldots, k-1$, $u_k = z$, and $v_{j-k} = w_j$ for $j = k, \ldots, k+\ell-1$. Then (30) write as

$$L_j = \begin{cases} zw_j, \ \text{for } j = 1, \ldots, k-1, \\ z(w_j + A_j z + O_2(z, w_1, \ldots, w_{k+\ell-1})), \ \text{for } j = k, \ldots, k+\ell-1, \end{cases} \tag{31}$$

for some real numbers $A_j$. They, as the higher order terms in $u$, come from the terms $O_2(u)$ and the terms $O_2(v)$, after the change to $(z, w_1, \ldots, w_{k+\ell-1})$ coordinates. Moreover, the last Lyapunov constant writes as

$$L_{k+\ell} = z(B + O_1(z, w_1, \ldots, w_{k+\ell-1})).$$

Finally, in (31) we can use again the Implicit Function Theorem to write $z_j = w_j$ for $j = 1, \ldots, k-1$ and $z_j = w_j + A_j z + O_2(z, w_1, \ldots, w_{k+\ell-1})$, for $j = k, \ldots, k+\ell-1$.

We notice that $z$ is small enough and we have, near the origin of the parameter space, a curve (parametrized) by $z$ of weak-foci of order $k+\ell$ that unfolds exactly, using the Weierstrass Preparation Theorem, $k+\ell-1$ hyperbolic limit cycles of small amplitude bifurcating from the equilibrium point located at the origin. The last limit cycle appears using the trace parameter $\alpha$ in a classical Hopf bifurcation as we have explained in Chapter 1. □

Christopher in [**17**] comments the generic unfolding of $k$ limit cycles in families of polynomial vector fields when we consider centers on a component of the center variety of codimension $k$. This is the aim of Theorems 1.2 and 1.3. The above result shows that on some special points on such component the cyclicity can increase. This is the mechanism that we have used in the following sections to improve the known lower bounds for the local cyclicity $M(n)$ for some low values of $n$. In particular for $n=3$ and $n=4$. We think that the Gine's conjecture in [**32, 33**] about the lower bound for $M(n)=n^2+3n-7$ can be thought in the sense of generic centers. We remark that, for providing higher values for $M(n)$ for higher degree $n$, we need to know better center families. Because the known families or have low codimension of they have too many parameters and the computational difficulties, as we will see in the following examples, increase so fast.

The fact that the cyclicity of Hamiltonian families depends on the parameters was previously studied by Han and Yu in [**38**]. Here we extend this result for other type of center families.

## 2.3. Bifurcation diagrams for local cyclicity in families of cubic centers

In this section we use Theorem 2.1 to study the bifurcation diagram for some families of cubic centers, lying in components of the center variety of codimension 11, 10, and 9. The first, in Proposition 2.4, is the family labeled $CD_{31}^{(12)}$ that has generically cyclicity 11 and was studied previously by Christopher in [**17**], for only one parameter value $a=2$ in (27), and by Yu and Tian in [**62**]. This proposition proves partially the main Theorem 2.1. The family labeled as $CR_{17}^{(12)}$ in [**65**] is studied in Proposition 2.5. We have studied the local cyclicity for some values of the parameter in the family up high order and we have found only 10 limit cycles. But using Theorem 2.2 we can get an extra limit cycle. Up to our knowledge this is the first time that the cyclicity of this family has been studied. The last cubic family has 2 free parameters, see Proposition 2.7, and we show that generically the origin has cyclicity at least 9 and that there are curves with cyclicity at least 10 and some special points with cyclicity at least 11. According Gasull, Jarque, and Garijo in [**29**], any holomorphic center is also a Darboux center. Liang and Torregrosa in [**45**] show that, for some values of the cubic family the cyclicity is as least 9. Here we explain that the cyclicity will increase depending on the specific center that we select. Up to our knowledge the studies of the bifurcation diagrams are new for these families.

PROPOSITION 2.4. *Consider system* (28) *with* $n = 3$ *and the unperturbed center*

$$\left\{\begin{array}{rcl} \dot{x} & = & -10(256a^3xy + 384a^3y - 96a^2x^2 - 384a^2y^2 - 16a^2x - 600axy \\ & & -480ay + 225x^2 + 900y^2 - 225x)(32a^2x + 48a^2 - 75x + 150), \\ \dot{y} & = & 16384a^5xy^2 + 24576a^5y^2 - 61440a^4y^3 + 16384a^5x + 56320a^4xy \\ & & -76800a^3xy^2 - 7680a^4y - 384000a^3y^2 + 288000a^2y^3 - 32000a^3x \\ & & -96000a^2xy + 90000axy^2 - 132000a^2y + 765000ay^2 - 337500y^3 \\ & & +168750ax - 84375xy - 337500y, \end{array}\right. \tag{32}$$

*with* $a$ *such that* $(32a^2 - 75)(16384a^6 - 14400a^4 + 165000a^2 + 84375) > 0$. *Then, there exist only six parameter values* $a^*$ *such that 12 limit cycles of small amplitude bifurcate from the origin. They are approximately* $\pm 2.019925086$, $\pm 7.444369217$, *and* $\pm 15.62631048$. *For almost all other values of* $a$ *only* 11 *limit cycles bifurcate from the origin.*

PROOF. The system corresponding to the rational first integral (27) has a center at the point $(x, y) = (6(8a^2 + 25)/(32a^2 - 75), 70a/(32a^2 - 75))$. Then, translating it to the origin we get system (32).

Let us consider (28) with $b_{30} = 0, b_{12} = 0$, and $b_{03} = 0$. After computing the first 12 Lyapunov constants up to order 1, we have that, generically for every $a$, $L_1^{(1)}, \ldots, L_{10}^{(1)}$ are linearly independents with respect to the parameters

$$a_{02}, a_{03}, a_{11}, a_{12}, a_{20}, a_{21}, a_{30}, b_{02}, b_{11}, b_{20}.$$

Then, we can write, after a linear change of parameters, $L_k = u_k + O_2(u)$, for $k = 1, \ldots, 10$, where $u_{11} = b_{21}$ and $O_2(u)$ denotes the monomials in $u$ of degree higher than 2 with coefficients rational functions in the parameter $a$. Moreover, we have that $L_j$ write as (29) with

$$L_{10}^{(1)} = \sum_{l=1}^{10} g_{10,l}(a)u_l + g(a)f_0(a)u_{11}, \quad L_{11}^{(1)} = \sum_{l=1}^{10} g_{11,l}(a)u_l + g(a)f_1(a)u_{11},$$

where $f_0$ and $f_1$ polynomials of degree 26 and 39 in $a^2$, respectively, $g$ is a rational function without common factors with $f_0$ nor $f_1$. Additionally, the numerator and denominator of $g$ are polynomials of degrees 69 and 90 in $a^2$ and $g_{10,l}$ and $g_{11,l}$ are also rational functions. All the involved polynomials are polynomials with rational coefficients.

The proof follows applying Theorem 2.2. To do that, we need to check that $f_0$ has real simple zeros and that the resultant of $f_0$ and $f_1$ with respect to $a$ is a non zero rational number. So, there should be at least a special value $a = a^*$ such that $f_0(a^*) = 0$, $f_0'(a^*) \neq 0$ and $f_1(a^*) \neq 0$. Finally, it can be checked that there only six possible values for $a^*$. The numerical approximation values for $a^*$ are the ones given in the statement. □

In the proof of the existence of the extra limit cycle done in [**62**] the computations of $L_k^{(1)}$ are the same that we obtain. As we have understood, the mistake is that his proof is not based directly in a result like Theorem 2.2 which we have perfectly identified the perturbation parameters and we have restricted the perturbation in order to apply it. Their proof is based in the fact that $L_{11}^{(1)}$ vanishes and $L_{12}^{(1)}$ not. This is not enough because the terms of order 2 of $L_{11}$ can appear and the weak-focus order does not increase. In fact, if we only consider $f_0(a^*) = 0$ then $L_{11}^{(2)} = u_{11}^2\, g_1(a)/g_2(a)$, with $g_1$ and $g_2$ polynomials of degree 66 and 103 in $a^2$. Moreover, in [**62**] the control of the number of essential parameters as we have commented in Remark 2.3 is not clear.

Proposition 2.5. *Consider the system*

$$\left\{\begin{array}{rcl} \dot{x} & = & (x-ay+a+2)(2\eta-3\eta_y+3x^2+6x+6)-3\eta\eta_y \\ & & -9x^2\eta_y+9(2ax^2+(2a-1)x+2a), \\ \dot{y} & = & 3(y(x-ay+a+2)(-3x+y+2)+3(x^2+x-2)), \end{array}\right. \tag{33}$$

*with* $\eta = xy - ay^2 + 2x + 2(1+a)y + 1 - a$, $-1/6 < a < 0$, $1/3 < a < 1$, *or* $1 < a$. *Then, it has a center at*

$$(x^*, y^*) = \left(\frac{3(a-1)}{6a+1}, -\frac{3a^2-4a+1}{a(6a+1)}\right),$$

*and the next properties hold.*

*(i) If* $g(a) \neq 0$ *and* $f_0(a) \neq 0$ *the local cyclicity, perturbing with polynomials of degree* 3, *is at least* 10.

*(ii) If* $g(a) \neq 0$, $f_0(a) = 0$ *the local cyclicity, perturbing with polynomials of degree* 3, *is at least* 11. *Moreover,* $f_0$ *has only* 4 *simple roots in the considered intervals. The numerical approximation are* $\{-0.12245, 0.39672, 0.61983, 2.70517\}$.

*The expressions of polynomials* $f_0$ *and* $g$ *are*

$$\begin{aligned} f_0(a) = {} & 115567116089031205 20a^{26} - 82791934329314091672a^{25} \\ & + 22819540504618684701 0a^{24} + 90491533122780174 24a^{23} \\ & - 157081144205847844346 4a^{22} + 33591807504814739820 39a^{21} \\ & - 315147810716332642769 4a^{20} - 32595532439923382979 6a^{19} \\ & + 142113712204693890075 06a^{18} - 386703672836697106216 11a^{17} \\ & + 568689349826650362654 06a^{16} - 543771793261786440069 63a^{15} \\ & + 308039087840735069073 36a^{14} - 901927704569663238347 7a^{13} \\ & - 66492299673756816877 8a^{12} + 29638923904721400008 13a^{11} \\ & - 176229630977894669307 6a^{10} + 40834318924969633194 3a^{9} \end{aligned}$$

$$
\begin{aligned}
&- 5342376894194351959 2a^8 + 3688723106531530364 7a^7 \\
&- 1326383678363391115 2a^6 + 1484165815203151098a^5 \\
&+ 85191877643707008a^4 - 114163404746428485a^3 \\
&+ 1130289090405930a^2 + 1973552231555520a + 103574370739840, \\
g(a) = {} & 4413012875799720164280 0a^{31} - 2525013156212545596840 00a^{30} \\
&+ 5679972508489162450201 80a^{29} - 8137938285118733498371 80a^{28} \\
&+ 2399279362949988891138 690a^{27} - 2777203364308983128745 270a^{26} \\
&- 1117982977709921462960 8785a^{25} + 5110034312827876920102 3051a^{24} \\
&- 9672273456885616905558 9531a^{23} + 1010724142371470731557 82098a^{22} \\
&- 8191116789244198181292 3273a^{21} + 9154373799722590388166 5763a^{20} \\
&- 1234642089357580685865 25599a^{19} + 1353853355799434724068 67144a^{18} \\
&- 1074703166613425094760 35270a^{17} + 5932298567720321123817 6126a^{16} \\
&- 2246844350391022929360 3606a^{15} + 6323085724047239916867 708a^{14} \\
&- 1656039645590378761238 526a^{13} + 3513462751671847804347 30a^{12} \\
&+ 1240755469220636887172 4a^{11} - 2921779219862791558927 8a^{10} \\
&+ 3200041670276393240067a^9 + 933095466480821343399a^8 \\
&- 81964651107172872879a^7 - 23554321764806596878a^6 \\
&- 5264497532389501 89a^5 + 210455326225541295a^4 \\
&+ 20323154636412705a^3 + 375301845557100a^2 \\
&- 28137453964620a - 1083684121520.
\end{aligned}
$$

REMARK 2.6. *We notice that, although the family* (33) *is considered of codimension* 12 *by Zoladek in* [**65**], *we have not found more than* 10 *limit cycles of small amplitude as it is stated in the above result for* $a = 2$ *and computing up to order* 10. *We think that the same will happen for other values of* $a$ *except the ones in Proposition 2.5 such that vanishes* $f_0$.

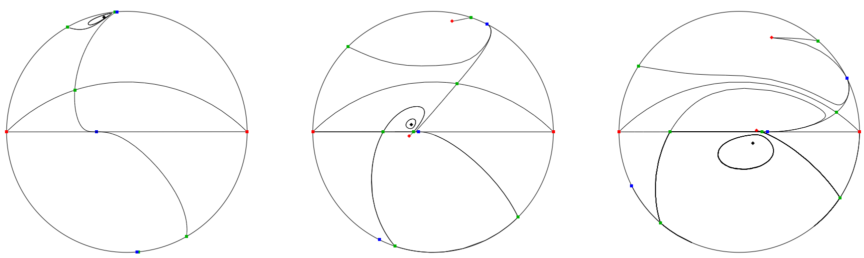

FIGURE 3.1. Phaseportraits in the Poincaré disk of the center (33) for $a = -1/12$, $a = 1/2$, and $a = 2$

Proof of Proposition 2.5. Doing a translation in order that the center $(x^*, y^*)$ of system (33) moves to the origin, we get

$$\left\{\begin{array}{rcl} \dot{x} & = & -(6a+1)(648a^7y^3 - 1944a^7y^2 - 2430a^6x^2y - 1944a^6xy^2 + 216a^6y^3 \\ & & +1458a^7y + 729a^6x^2 - 2916a^6xy - 3564a^6y^2 + 972a^5x^3 - 1701a^5x^2y \\ & & -1296a^5xy^2 + 18a^5y^3 - 972a^6y + 2187a^5x^2 + 486a^5xy - 1404a^5y^2 \\ & & +1134a^4x^3 - 216a^4x^2y - 270a^4xy^2 - 486a^5y + 1053a^4x^2 + 1782a^4xy \\ & & -144a^4y^2 + 486a^3x^3 + 72a^3x^2y - 18a^3xy^2 + 54a^4y - 162a^3x^2 \\ & & +594a^3xy + 90a^2x^3 + 18a^2x^2y - 36a^3y - 189a^2x^2 + 54a^2xy \\ & & +6x^3a + ax^2y - 18a^2y - 33ax^2 - x^2), \\ \dot{y} & = & 3(3a+1)^4(6a^2y - 3a^2 + ay + 4a - 1)(6a^2y - 9a^2 + ay + 3a - 1)x. \end{array}\right.$$

Then we can consider equation (28). The proof that this family has a center follows from a rational symmetry and it can be found in [**63, 65**].

Next step is the computation of $L_k(1)$, for $k = 1, \ldots, 9$ and we consider them as linear functions depending on $a_{02}, a_{03}, a_{11}, a_{12}, a_{20}, a_{21}, b_{02}, b_{03}, b_{20}$. Hence, we write, after a linear change of coordinates adding $b_{21} = u_{10}$,

$$L_j = \ u_j + O_2(u), \text{ for } j = 1, \ldots, 9.$$

The other parameter values in (28) have been taken as zero. In $O_2$ appear some denominators in $a$ which are non zero under the hypotheses of the statement. In particular the condition $g(a) \neq 0$ appear solving the above linear change. It can be seen also in the following expressions of the next two Lyapunov constants. Their simplified expressions, using Proposition 1.4, are, except non zero multiplicative constants,

$$\begin{aligned} L_{10} &= \frac{(3a+1)^{13}(6a+1)^{18}}{a^3(9a^2-3a+1)^{10}(a-1)^9}\frac{f_0(a)}{g(a)}u_{10} + O_2(u), \\ L_{11} &= \frac{(3a+1)^{13}(6a+1)^{19}}{a^4(3a-1)(9a^2-3a+1)^{12}(a-1)^{11}}\frac{f_1(a)}{g(a)}u_{10} + O_2(u), \end{aligned}$$

where $f_0$ and $g$ are defined in the statement and $f_1$ is

$$\begin{aligned} f_1(a) = {} & 7245364776085722378808880a^{32} + 640589325778944777413782 80a^{31} \\ & - 61014448185975758622340155 6a^{30} + 2360973008978454210093841374a^{29} \\ & - 3106072481972105279560942206a^{28} - 7847548346783924455871215944a^{27} \\ & + 3735046528119834043057366 6575a^{26} - 6591294991279570315334914 1583a^{25} \\ & + 5521383491291137923493255 8885a^{24} + 6562489081413077400265003 1070a^{23} \\ & - 3860975104162813854835688 57175a^{22} + 8522590404897635458648691 24460a^{21} \\ & - 1193508332460445900562643 584016a^{20} + 1139964285706711135528711 455009a^{19} \\ & - 730726233625740844361877322266a^{18} + 280817510225041089315898703766a^{17} \\ & - 214872020845367124995261 19540a^{16} - 543042191500608606082521 08112a^{15} \end{aligned}$$

$$
\begin{aligned}
&+ 4289458988037004468371728967 6a^{14} - 1609808118602118645935944517 4a^{13}\\
&+ 2841857092329161976333442044a^{12} - 401707003814285433422087250a^{11}\\
&+ 2785195208840768927049212 01a^{10} - 899226261654887424089680 47a^{9}\\
&- 2194343731942112410768 17a^{8} + 5240045076877491398959122a^{7}\\
&- 1141062554605305892208985a^{6} - 151240365953281702155 96a^{5}\\
&+ 4292814380007330375309 0a^{4} - 982397541469926950 55a^{3}\\
&- 576005750186099035950a^{2} - 28747061161858522560a + 21647043484626560.
\end{aligned}
$$

Clearly $9a^2 - 3a + 1$ is non vanishing and with the restriction on $a$ given in the statement all the rational functions are well defined.

Statement (i) follows from Theorem 1.2. Statement (ii) follows as the proof of Proposition 2.4. That is, computing the resultant of $f_0$ and $f_1$ and the discriminant of $f_0$ with respect to $a$, and checking that $f_0$ have real zeros, which will be simple. From Theorem 2.2 we know that for the values of $a$ such that $f_0$ vanishes we have 11 limit cycles bifurcating from the origin. $\square$

The next result provides a complete bifurcation diagram for all holomorphic cubic centers having the coefficient of $z^2$ non vanishing. In this case it is not restrictive, rescaling if necessary, to assume that it is 1. In complex coordinates they write as

$$\dot{z} = iz + z^2 + (a + ib)z^3. \tag{34}$$

Proposition 2.7. *Consider system* (28) *with $n = 3$ and the unperturbed center*

$$
\begin{cases}
\dot{x} &= ax^3 - 3axy^2 - 3bx^2y + by^3 + x^2 - y^2 - y,\\
\dot{y} &= 3ax^2y - ay^3 + bx^3 - 3bxy^2 + 2xy + x,
\end{cases}
$$

*for every value of the parameters $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ and the polynomials*

$$
\begin{aligned}
f_0(a, b) &= 8a^6 + 24a^4b^2 + 24a^2b^4 + 8b^6 + 282a^4b + 564a^2b^3 + 282b^5 - 37569a^4\\
&\quad - 45954a^2b^2 - 8385b^4 - 91924a^2b - 162484b^3 - 646020a^2 - 37860b^2,\\
f_1(a, b) &= 2448a^6b + 7344a^4b^3 + 7344a^2b^5 + 2448b^7 + 3208a^6 + 95916a^4b^2\\
&\quad + 182208a^2b^4 + 89500b^6 - 12055032a^4b - 15179760a^2b^3 - 3124728b^5\\
&\quad - 19489169a^4 - 64437898a^2b^2 - 665400089b^4 - 285166044a^2b\\
&\quad - 926888444b^3 - 310735620a^2 - 182106 60b^2,\\
f_2(a, b) &= 145864a^8 - 3776a^6b^2 - 886512a^4b^4 - 1178240a^2b^6 - 441368b^8\\
&\quad + 3892522a^6b - 9022362a^4b^3 - 29722290a^2b^5 - 16807406b^7\\
&\quad - 708522105a^6 + 1379959497a^4b^2 + 2743262973a^2b^4 + 6547813 71b^6\\
&\quad + 8068743920a^4b + 18906063664a^2b^3 + 160167059 84b^5
\end{aligned}
$$

$$
\begin{aligned}
&- 5202830396a^4 + 86382442952a^2b^2 + 486067338828b^4 \\
&+ 185131413648a^2b + 337911194128b^3 + 93466173600a^2 + 5477584800b^2, \\
g(a,b) &= 27936a^6 + 83808a^4b^2 + 83808a^2b^4 + 27936b^6 - 162180a^4b \\
&- 324360a^2b^3 - 162180b^5 - 199825a^4 - 227714a^2b^2 - 27889b^4 \\
&- 347172a^2b - 23172b^3 + 30636a^2 - 5364b^2.
\end{aligned}
$$

*Then,*

*(i) if* $f_0(a,b)g(a,b) \neq 0$ *there are* 9 *limit cycles of small amplitude bifurcating from the origin;*

*(ii) if* $f_0(a,b) = 0$ *and* $f_1(a,b)g(a,b) \neq 0$ *there are* 10 *limit cycles of small amplitude bifurcating from the origin;*

*(iii) if* $f_0(a,b) = f_1(a,b) = 0$ *and* $f_2(a,b)g(a,b) \neq 0$ *there are* 11 *limit cycles of small amplitude bifurcating from the origin.*

*Moreover, there exist only two transversal intersection points of the curves* $f_0(a,b) = 0$ *and* $f_1(a,b) = 0$ *which are* $(a^*, b^*) \approx (\pm 69.66852455, -6.617950485)$.

The above result provides the bifurcation diagram for the local cyclicity of the 2-parameter holomorphic family (34). The curves $f_0$, $f_1$ and $f_2$ are drawn in Figure 3.2 in red, green and blue, respectively. Generically, the local cyclicity is 9. On the red curve, generically, the cyclicity is 10 and in the intersection point of the curves red and green the cyclicity is 11.

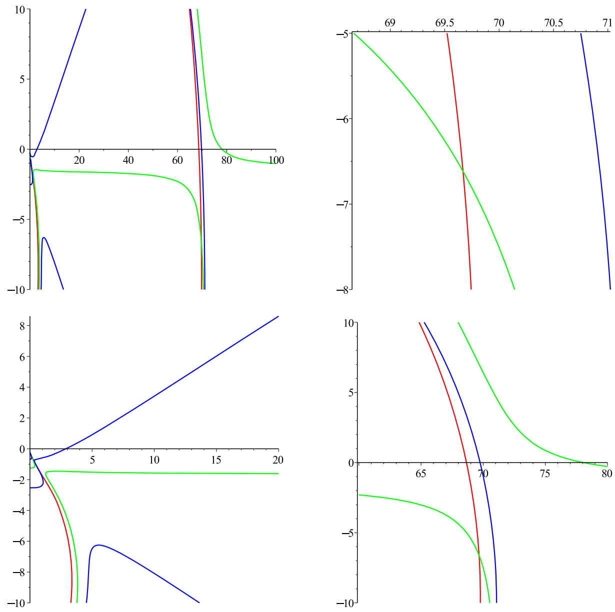

FIGURE 3.2. The curves, with some zooms, $f_0(a,b) = 0$, $f_1(a,b) = 0$, and $f_2(a,b) = 0$ given in Proposition 2.7 in red, green, blue, respectively

Proof of Proposition 2.7. After a change of sign if necessary we can restrict our analysis to $a>0$. For every $a, b$ different from $(0,0)$ and taking zero the parameters $b_{20}$, $b_{11}$, $b_{30}$, $b_{12}$, we compute, with the parallelized algorithm described in Chapter 1, the linear terms of the first 11 Lyapunov constants, with respect to the essential parameters $a_{20}, a_{11}, a_{02}, b_{02}, a_{30}, a_{21}, a_{12}, a_{03}, b_{21}$. If $g(a,b)\neq 0$ then, up to a linear change of parameters, we can write $L_j^{(1)}=u_j$, for $j=1,\ldots,8$, and

$$
\begin{aligned}
L_9^{(1)} &= \frac{(81a^2+(9b+2)^2)(a^2+b^2)^3}{g(a,b)} f_0(a,b)u_9,\\
L_{10}^{(1)} &= \frac{(81a^2+(9b+2)^2)(a^2+b^2)^3}{g(a,b)} f_1(a,b)u_9,\\
L_{11}^{(1)} &= \frac{(81a^2+(9b+2)^2)(a^2+b^2)^3}{g(a,b)} f_2(a,b)u_9.
\end{aligned}
$$

To simplify we have divided, if necessary, by non zero multiplicative constants.

Computing the resultants of the pairs $(f_0, f_1)$ and $(f_0, f_2)$ with respect to $a$, we get

$$
\begin{aligned}
&b^6(4b-9)^2(9b-59)^2(512192700b^4+13330993797b^3+610349822291b^2\\
&\quad -330283585509b-10270019239)^2,\\
&b^6(4b-9)^2(9b-59)^2(570698912585670507000b^7+22990976281237387495014b^6\\
&\quad +36881578284839814317085b^5-44788809152838367037764940b^4\\
&\quad -9505227203153802766492979b^3+38476609139880937030659126b^2\\
&\quad +13351954188119085151405788b+26961888682015305774809 60)^2.
\end{aligned}
$$

Removing the common factors, the above two polynomials in $b$ of degrees 4 and 7 have no common roots, becuase it resultant, with respect to $b$ is non vanishing. Then any intersection point of the curves $f_0=0$ and $f_1=0$ is not in the curve $f_2=0$. Then straighforward computations shows that the curves $\{f_0=0, f_1=0\}$ have only one real intersection point $(a^*, b^*)\approx(69.66852455, -6.617950485)$. Moreover, it is a transversal intersection and $f_2(a^*,b^*)\neq 0$.

The proof follows using Theorem 2.2 in each item in the statement. $\square$

## 2.4. Bifurcation diagrams for local cyclicity in families of quartic centers

This section is devoted to prove the second part of the statement of our main result, Theorem 2.1. It follows from the next proposition. We provide the bifurcation diagram of local cyclicity of the cubic center given by Bondar and Sadovski in [**7**] adding a straight line of equilibria. This problem can be studied to get 19 limit cycles. Here show a curious fact, the cyclicity depends on the selected straight line. We work with two parameters $(a,b)$, showing the existence of a curve with cyclicity at least 20 and a point with at least 21. Our server needs around one day to get the expressions of all necessary Lyapunov constants. Moreover the size of each text file containing them has size higher than 170 MB. We use a Computer Assisted Proof using the Poincaré Miranda Theorem, Theorem 1.5, the Gershgorin Theorem, Theorem 1.6 and technical Lemmas 1.23 and 1.24.

Finally, we do a partial study of the bifurcation diagram of the local cyclicity of the holomorphic center of degree $n = 4$, depending on 4 parameters. We prove the existence of a holomorphic center with 20 limit cycles of small amplitude bifurcating from the origin. We have strong numerical evidences that there are values of the parameters such that 21 limit cycles bifurcate from the origin, but the calculus are hard and an analytical proof is impossible to be done. For the moment, we only present the analytical proof for 20 limit cycles.

PROPOSITION 2.8. *Consider equation* (28) *for* $n = 4$ *with the unperturbed system*

$$\begin{cases} \dot{x} &= -y(1183x^2-68x+1)(1-ax-by), \\ \dot{y} &= (672x^3+1484x^2y-945xy^2-84y^3-58x^2-44xy+30y^2+x)(1-ax-by). \end{cases} \tag{35}$$

*Then, there exist a pair* $(a^*, b^*) \approx (-0.8159251773700849, 0.55062996428210239)$ *such that, for* $\lambda$ *small enough,* (28) *bifurcate 21 limit cycles of small amplitude from origin.*

PROOF. System (35), without the straight line of equilibria, has a center at the origin because it has a rational first integral, see (21) or [**7**]. We restrict our study to $b_{11} = 0, b_{21} = 0, b_{30} = 0, b_{31} = 0, b_{40} = 0$ in (28). After a linear change of coordinates we move from $a_{02}, a_{03}, a_{04}, a_{11}, a_{12}, a_{13}, a_{20}, a_{21}, a_{22}, a_{30}, a_{31}, a_{40}$, $b_{02}, b_{03}, b_{04}, b_{11}, b_{12}, b_{13}$ to $u_1, \ldots, u_{18}$ and write $L_k^{(1)} = u_k$ for $k = 1, \ldots, 18$. As we have done in the previous proofs, writing $b_{20} = u_{19}$, and removing the common factors, which are rational functions in $(a, b)$, in the linear development of the next Lyapunov constants we can write

$$L_{19}^{(1)} = f_0(a,b)u_{19}, \quad L_{20}^{(1)} = f_1(a,b)u_{19}, \quad L_{21}^{(1)} = f_2(a,b)u_{19}. \tag{36}$$

The coefficients $f_0, f_1, f_2$ are polynomials with rational coefficient of degrees 180, 182, and 184, respectively. The total number of monomials are, respectively, 16329, 16694, and 17063. We have not added here the expressions because of their size.

Numerically we can find the solution $(a^*, b^*)$ in the statement of the algebraic system $\{f_0 = 0, f_1 = 0\}$. Moreover, the intersection is transversal because the determinant of the Jacobian matrix at the intersection point is $-8.7569521108153076 \cdot 10^{570}$. At this point we have $f_2(a^*, b^*) = -1.7191356490086216 \cdot 10^{290}$.

To get an analytic proof we will use a Computer Assisted Proof with the help of Lemmas 1.23 and 1.24. We will use Theorem 1.5 for the existence of the intersection point of $f_0$ and $f_1$ and Theorem 1.6 to prove the transversality. The technical lemmas also are used to check that at point $f_2$ is non vanishing. We fix a square $\mathcal{Q} = [-h, h]^2$ with $h = 10^{-12}$ and we do a rational affine change of coordinates such that a good rational approximation of $(a^*, b^*)$ be inside $\mathcal{Q}$. This affine change of variables is chosen such that the Taylor series of degree 1 of $f_0$ and $f_1$ at $(a^*, b^*)$ will be the new coordinates. Then

$$\begin{aligned}
\tilde{f}_0(S_0^-) &\subset [-1.31146 \times 10^{-12}, -8.44847 \times 10^{-13}],\\
\tilde{f}_0(S_0^+) &\subset [1.15471 \times 10^{-12}, 6.89142 \times 10^{-13}],\\
\tilde{f}_1(S_0^-) &\subset [-1.15545 \times 10^{-12}, -6.90604 \times 10^{-13}],\\
\tilde{f}_1(S_0^+) &\subset [1.30878 \times 10^{-12}, 8.44982 \times 10^{-13}],\\
\tilde{f}_2(\mathcal{Q}) &\subset [0.9035737600, 1.096426240],
\end{aligned}$$

and the have proved the existence of $(a^*, b^*)$ such that $f_2$ is non vanishing. In the computations we have worked with rational numbers with numerators and denominators of around 15000 digits. To simplify the computations we have worked with the functions $\tilde{f}_j(a, b) = f_j(a, b)/f_j(0, 0)$.

The last part is to check the transversality. Instead of compute the determinant of the Jacobian matrix of $(f_0, f_1)$ with respect to $(a, b)$, we use the technical lemmas to get that the elements in the Jacobian matrix for the transformed variables are, varying in $\mathcal{Q}$, $A_{11}, A_{22} \subset (0.84568065, 1.15431935)$ and r $A_{12} \subset (-0.15535611, 0.15535611)$ and $A_{21} \subset (-0.15498852, 0.15498852)$. Then with Theorem 1.6, the both eigenvalues are positive and belong in $(0.74, 1.25)$. Therefore, the determinant is different from zero. □

Remark 2.9. *We remark the computational difficulties of the numeric in the above result. We should work with very high precision. In fact, working with 1000 digits the evaluations of $f_0$ and $f_1$ at $(a^*, b^*)$ are $-2.19920305995245 \cdot 10^{-397}$ and $3.595005930091451 \cdot 10^{-390}$, respectively. Moreover, the necessary affine change of variables has need more than one computation day. Finally, the curves in Figure 4.3 has been drawn computing the points one by one working with very high precision and then using linear interpolation. In fact, the first time that we got $(a^*, b^*)$ was from the intersection of this linear interpolation curves.*

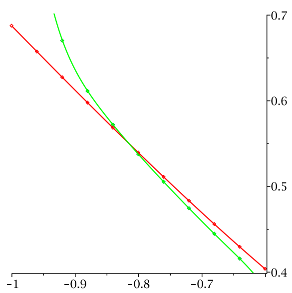


Figure 4.3. Drawing the zero level sets of $f_0$ and $f_1$ in (36) in red and green, respectively

Proposition 2.10. *Consider equation* (28) *for $n = 4$ with the unperturbed system written in complex coordinates, $z = x + iy$, as*

$$\dot{z} = iz + z^2 + (a_1 + ia_2)z^3 + (a_3 + ia_4)z^4. \tag{37}$$

*If $a_1 = 1$ and $a_3 = 3$, there exist two algebraic curves $f_0(a_2, a_4)$ and $f_1(a_2, a_4)$ such that, generically on $f_0(a_2, a_4)$, there are small parameters $\lambda$ for which* (28) *has has at least* 19 *limit cycles of small amplitude bifurcating from the origin. Moreover, there are at least three transversal intersection points,*

$$(a_2^*, a_4^*) \in \{(-6.788836, 2.856062), (-4.387174, 4.549274), (-4.619905, -4.565876)\},$$

*of $f_0$ and $f_1$ for which* (28) *has at least* 20 *limit cycles of small amplitude bifurcating from the origin.*

Proof. In cartesian coordinates, taking $a_1 = 1$ and $a_3 = 3$, system (2.10) writes as

$$\begin{cases} \dot{x} &= -4a_4x^3y + 4a_4xy^3 - 3a_22x^2y + a_2y^3 + 3x^4 - 18x^2y^2 + 3y^4 + x^3 \\ & \quad -3xy^2 + x^2 - y^2 - y, \\ \dot{y} &= a_4x^4 - 6a_4x^2y^2 + a_4y^4 + a_2x^3 - 3a_2xy^2 + 12x^3y - 12xy^3 + 3x^2y \\ & \quad -y^3 + 2xy + x. \end{cases}$$

We will restrict our analysis to $b_{11} = 0, b_{20} = 0, b_{21} = 0, b_{30} = 0, b_{31} = 0, b_{40} = 0$. The Lyapunov constants up to order 1, with the algorithm explained in Chapter 1 and similarly as the proof of Proposition 2.8, can be computed and written as $L_k^{(1)} = u_k$, for $k = 1, \ldots, 17$. Here we have done a linear change of coordinates in the parameter space changing the linear independent parameters

$$a_{12}, a_{02}, a_{03}, a_{04}, a_{11}, a_{13}, a_{20}, a_{21}, a_{22}, a_{30}, a_{31}, a_{40}, b_{02}, b_{03}, b_{04}, b_{12}, b_{13},$$

by $u_1, \ldots, u_{17}$. Changing the last one $b_{22}$ to $u_{18}$ we have, as in the previous proofs and except a multiplicative rational function in $a_2, a_4$ as a common factor,

$$L_{18}^{(1)} = f_0(a_2, a_4)u_{18}, \quad L_{19}^{(1)} = f_1(a_2, a_4)u_{18}, \quad L_{20}^{(1)} = f_2(a_2, a_4)u_{18}. \tag{38}$$

The proof follows similarly as the proof of Theorem 2.7 to get the transversal intersection points in the statement. Computing the necessary resultants with respect to $a_2$ and $a_4$ to apply Theorem 2.2.

In Figure 4.4, we have drawn the algebraic curves $f_k(a_2, a_4) = 0$ for $k = 18, 19, 20$ in red, blue, and green, respectively. Notice, that in each picture it is clear the existence of a transversal intersection of $f_0 = 0$ and $f_1 = 0$ where $f_3$ is non vanishing. □

Remark 2.11. *Taking $a_1 = 1$ in* (37) *we can compute the corresponding algebraic functions $f_0, f_1, f_2, f_3$. They have around $10^5$ monomials and degrees $100, 101, 102, 103$,*

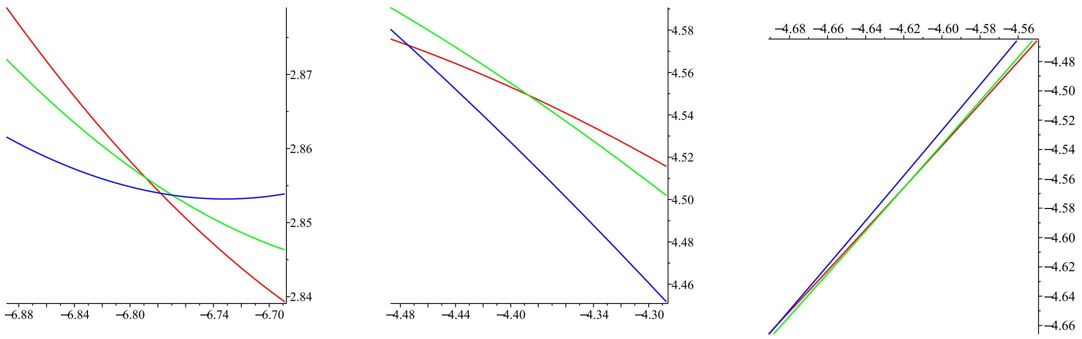


FIGURE 4.4. Drawing the zero level sets of $f_0$, $f_1$, and $f_2$ in (38) in red, green, and blue, respectively

*respectively. Then, we can solve numerically with high precision the first three obtaining*

$$a^* = (a_2^*, a_3^*, a_4^*) \approx (0.26423354653702, 2.06583351382191, 2.26983478766641).$$

*The evaluation at this point gets*

$$f_0(a^*) \approx 4.35 \cdot 10^{-281}, \quad f_1(a^*) \approx 3.2 \cdot 10^{-275},$$
$$f_2(a^*) \approx 3.67 \cdot 10^{-272}, \quad f_3(a^*) \approx 1.091295989718 \cdot 10^{126},$$

*and the determinant of the Jacobian matrix of* $(f_0, f_1, f_2)$ *with respect to* $a$ *at the intersection point* $a^*$ *is* $-3.82703230760 \cdot 10^{363}$. *This gives a numerical evidence that the holomorphic family of degree* 4 *exhibits also* 21 *limit cycles of small amplitude bifurcating from the origin.*

## 2.5. Final comments

The computations in this chapter are quite high although basically we have worked only with developments of order 1 in the Lyapunov constants. This is because the existence of parameters in the unperturbed centers makes the things more complicated. Before the simplifications, the polynomials appearing as coefficients of the perturbation parameters are of very high degree and with rational coefficients with high number of digits. In fact this is why we have only considered vector fields of degrees $n = 3$ and $n = 4$.

In [**45**] the holomorphic centers are considered and it is proved that for low degree $4 \leq n \leq 13$ the cyclicity of the center is at least $n^2 + n - 2$ and for $n = 3$ it is at least 9. The results of this chapter provides higher values of the cyclicity but only for $n = 3$ and $n = 4$. Obtaining as new lower bounds 11 and 21, respectively, even though this last value is not analytic. We have also worked with other holomorphic centers, $n = 5, 6, 7$ but only with one parameter. In all cases we have found at least one extra limit cycle than the ones obtained in [**45**]. But as the obtained lower bounds for $M(n)$ are worse than other obtained in other chapters of this work we have not added here.

In all the proofs it is very important to restrict our studies to exactly the number of parameters $k$ and $\ell$ in Theorem 2.2. Then we will have always only lower bound for the cyclicity in families. This restriction ensures that the higher order terms do no affect in the expressions in the first order developments.

CHAPTER 3

# Local cyclicity in lower degree piecewise polynomial vector fields

In this chapter, we are interested in crossing limit cycles in piecewise polynomial vector fields defined in two zones separated by a straight line. In particular, in isolated periodic orbits of small amplitude. They are all surrounding one equilibrium point or an sliding segment. We provide lower bounds for the local cyclicity for piecewise polynomial systems with degree 2, 3, and 4. More concretely, $M_p^c(2) \geq 13$, $M_p^c(3) \geq 26$, and $M_p^c(4) \geq 40$. Clearly, all of them are in only one nest. The computations use a parallelization algorithm.

## 3.1. Introduction

The study of piecewise linear systems was started by Andronov, see [**3**]. It has been widely studied in the last years, since many problems of engineering, physics, and biology can be modeled by such systems, see [**1, 23**]. One of the most studied problem is given by a straight line separating two half-planes and as in the case of the classical qualitative theory of polynomial systems, the study of limit cycles have received a special attention, see for example [**21, 39, 48**]. In particular, it can be seen as an extension of the 16th-Hilbert problem for piecewise polynomial systems. More details of this problem in analytic vector fields can be seen in [**41**].

In this chapter, we are interested in the study of isolated periodic orbits, the so-called limit cycles, bifurcating from the origin, for piecewise differential equations of the form

$$\begin{cases} (x', y') = (P^+(x,y,\lambda), Q^+(x,y,\lambda)), \text{ when } y \geq 0, \\ (x', y') = (P^-(x,y,\lambda), Q^-(x,y,\lambda)), \text{ when } y < 0, \end{cases} \tag{39}$$

where $P^{\pm}(x,y,\lambda)$ and $Q^{\pm}(x,y,\lambda)$ are polynomials. The straight line $\Sigma = \{y = 0\}$ divides the plane in two half-planes $\Sigma^{\pm} = \{(x,y) : \pm y > 0\}$ and the trajectories on $\Sigma$ are defined following the Filippov convention, see [**26**]. We will consider only limit cycles of crossing type, that is, when both vector fields point out in the same direction in the intersection points with the discontinuity line $\Sigma$.

For polynomial vector fields of degree $n$, we denote by $M(n)$ the maximum number of limit cycles bifurcating from the origin and by $H(n)$ the maximum number of limit cycles. Clearly $M(n) \leq H(n)$. For piecewise systems, we call $M_p^c(n)$ the maximum number of limit cycles bifurcating from a monodromic singular point and $H_p^c(n)$ the maximum number of limit cycles in the piecewise case. Clearly $M_p^c(n) \leq H_p^c(n)$. It is well-know that linear systems have no limit cycles, so $H(1) = M(1) = 0$. This is not the case for piecewise linear systems defined in two zones separated by a straight line. Huan and Yang in [**40**] firstly showed a numerical evidence that $H_p^c(1) \geq 3$. In [**48**] Llibre and Ponce provide an analytical proof of this fact. One year later, using the averaging technique this lower bound was reobtained by Buzzi et al, [**10**]. Recently, also the same number was obtained in [**28**] by Freie et al. The three limit cycles in [**28**] are explained studying the full return map, two appear near the origin and the other one far from it. In fact, these two limit cycles appearing from an equilibrium point provide the lower bound $M_p^c(1) \geq 2$. This value can be proved with the results in [**27**]. We will show this in the next section.

For quadratic vector fields is also well known that $H(2) \geq 4$, see [**56**]. But only 3 can bifurcate from the origin, that is $M(2) = 3$. This fact was proved by Bautin in [**6**]. For piecewise quadratic systems it is not proved yet which will be that local maximum. Moreover, there are few works providing good lower bounds. Using averaging theory of order five, and perturbing the linear center, Llibre and Tang in

[**49**] proved that $H_p^c(2) \geq 8$. Recently, da Cruz and et al. in [**22**] provide a better lower bound, $H_p^c(2) \geq 16$. These limit cycles appear using also averaging method up to order 2 and perturbing some quadratic isochronous systems. The new lower bound is quite surprising because is higher than what it can be expected a priori, that is doubling (because we have two vector fields) the value 4 obtained for usual quadratic vector fields.

The best known lower bound for the number of limit cycles in cubic and quartic systems is $H(3) \geq 13$ and $H(4) \geq 28$, see [**44**] and [**53**], respectively. But for the local cyclicity the results in the previous chapters are the best up to now, $M(3) \geq 12$ and $M(4) \geq 20$. In piecewise polynomial vector fields there are no so much results studying $H_p^c(3)$ nor the local cyclicity problem. The very recent work [**35**] provides $H_p^c(3) \geq 18$ in two nests of 9 limit cycles each.

THEOREM 3.1. *The local cyclicity for piecewise polynomial vector fields of degree* $n = 3$ *and* $n = 4$ *is* $M_p^c(3) \geq 26$ *and* $M_p^c(4) \geq 40$, *respectively. In particular,* $H_p^c(3) \geq 26$ *and* $H_p^c(4) \geq 40$.

Moreover, we provide a quadratic system exhibiting at least 13 limit cycles of small amplitude. Furthermore, the limit cycles are all of crossing type and in only one nest, surrounding the same equilibrium point. Our approach is based in the degenerate Hopf bifurcation, studying the limit cycles of small amplitude bifurcating from an equilibrium point of center-focus type. That is, through the computation of the linear parts of the Lyapunov constants but for piecewise differential systems. The main idea is based in the Implicit Function Theorem as was stated by Chicone and Jacobs in [**14**] for an equivalent problem. Our work is the piecewise extension of the one done by Christopher in [**17**]. However, as the computations are quite hard, we implement the parallelization mechanisms introduced in [**45**] and in Chapters 1 and 2.

The chapter is structured as follows, in Section 3.2 we present how to compute the Lyapunov constants and to use them for Hopf and pseudo-Hopf bifurcations. In particular, we prove the existence of generic unfolding of $2n+1$ limit cycles of small amplitude bifurcating from a weak-focus of such order. We detail also the differences in the order between weak-focus of analytic and piecewise analytic vector fields. Section 3.3 is devoted to show how the proposed technique works to find good lower bounds for the local cyclicity in quadratic, cubic and quartic piecewise vector fields, which provide the proof of the main result of this chapter. We perturb quadratic, cubic and quartic centers with piecewise systems defined in two zones separated by a straight line. We finish this chapter, in Section 3.4, with the advantages and difficulties to get further in the results for other piecewise systems.

## 3.2. Degenerated Hopf and pseudo-Hopf bifurcations

Let is introduce the concepts of sewing and sliding which are necessary for the study of dynamics of a piecewise system. Given the system in the form

$$Z(x,y)=\begin{cases} Z^+(x,y), & \text{when } f(x,y)\geq 0,\\ Z^-(x,y), & \text{when } f(x,y)<0,\end{cases}\tag{40}$$

where $f:\mathbb{R}^2\to\mathbb{R}$ is a $\mathcal{C}^1$ function such that 0 is a regular value. The discontinuity curve is given by $\Sigma=f^{-1}(0)$, and $Z^\pm=(X^\pm,Y^\pm)$. Following the notation introduced by Filippov in [**26**], in $\Sigma$ when both vector fields meet, we can have on $\Sigma$ three behaviors, that are crossing, escaping and sliding. We will denote them by $\Sigma^C$, $\Sigma^E$ and $\Sigma^S$, respectively. Given a point $p\in\Sigma$, we say that $p\in\Sigma^c$ if, and only if $Z^+f(p)\cdot Z^-f(p)>0$ where $Z^\pm f(p)=\langle\nabla f(p),Z^\pm(p)\rangle$. Consequently, we have $p\in\Sigma^E\bigcup\Sigma^S$ if, only if $Z^+f(p)\cdot Z^-f(p)<0$. Figure 2.1 illustrates how is the vector field near these three regions.

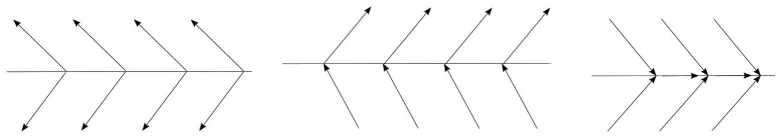

FIGURE 2.1. Escaping, sewing and sliding segments

Now we detail the algorithm that we have implemented to compute the coefficients of the return map, $\Pi(\rho)$, near the origin, when it is of monodromic type, in a piecewise vector field. As, from (40), we have two vector fields we have two half return maps, $\Pi^\pm(\rho)$, and the global one can be defined by composition, $\Pi(\rho)=\Pi^-(\Pi^+(\rho))$. But for simplicity, we will compute the difference map $\Delta(\rho)=\Pi^+(\rho)-(\Pi^-)^{-1}(\rho)$. Here, equivalently as we have introduced in previous chapters, the coefficients of this function are called also the Lyapunov constants for piecewise polynomial vector fields.

As in this chapter we are dealing with polynomial centers perturbed with different polynomial vector fields in $y>0$ than in $y<0$, we write our system in the form

$$\begin{cases} \dot{x} &= -y+P^\pm(x,y,\lambda^\pm),\\ \dot{y} &= x+Q^\pm(x,y,\lambda^\pm),\end{cases}\tag{41}$$

such that the origin is a center point. Writing (41) in polar coordinates we have

$$\begin{cases} \dfrac{dr}{d\theta}=\displaystyle\sum_{k=2}^{\infty}R_k^\pm(\theta,\lambda^+)r^k, & \theta\in[0,\pi],\\ \dfrac{dr}{d\theta}=\displaystyle\sum_{k=2}^{\infty}R_k^\pm(\theta,\lambda^-)r^k, & \theta\in[\pi,2\pi],\end{cases}\tag{42}$$

where $R_k(\theta,\lambda)$ are polynomials in the variables $\sin\theta$, $\cos\theta$. The perturbative parameters are denoted by $\lambda^{\pm}$. Writing $r^{\pm}(\theta,\rho)$ the solution of (42) such that $r^{\pm}(0,\rho)=\rho$, close to $\rho=0$ we have

$$r^{\pm}(\theta,\rho,\lambda^{\pm}) = \rho + \sum_{k=2}^{\infty} r_k^{\pm}(\theta,\lambda^{\pm})\rho^k, \tag{43}$$

with $r_k^{\pm}(0)=0$ for $k\geq 2$. As our piecewise systems are defined separated by the straight line $\{y=0\}$, the half-Poincaré maps close to the origin are given by

$$\Pi^{+}(\rho) = -\rho + \sum_{k=2}^{\infty} r_k^{+}(\pi,\lambda^{+})\rho^k,$$
$$\widetilde{\Pi}^{-}(\rho) = -\rho + \sum_{k=2}^{\infty} r_k^{-}(-\pi,\lambda^{-})\rho^k.$$

Here the map $\widetilde{\Pi}^{-}$ is the inverse of $\Pi^{-}$ because of the definition of $r^{\pm}$ and the fact that both are defined from the same initial value problem, with initial condition over $\theta=0$ but $\rho>0$. Therefore, it is better to consider, as we have mentioned above, the difference map. That writes as

$$\Delta(\rho) = \Pi^{+}(\rho) - \widetilde{\Pi}^{-}(\rho) = \sum_{k=2}^{\infty} L_k\rho^k. \tag{44}$$

Remark 3.2. *The computations of $r^{\pm}(\theta,\rho,\lambda^{\pm})$ in* (43) *are done for general perturbations computing $r(\theta,\rho,\lambda)$ and then evaluating at $\pm\pi$ changing $\lambda$ by $\lambda^{\pm}$, which decrease the computation time. For more details about Poincaré maps for piecewise systems, see* [**50**].

The coefficients $L_k$ are known as the Lyapunov constants associated to system (41). Consequently, for the perturbed system, the first nonvanishing $L_k$ provides the stability of the origin. In this case we say that the origin is a generalized weak-focus of order $k$. We have followed the classical Lyapunov algorithm scheme. For more details we refer the reader to [**2**]. As the usual Lyapunov constants, see [**18**], to solve the analytic center-focus problem for non degenerate centers, $L_k$ are polynomials in the parameters $\lambda$ with rational coefficients. Moreover, they are also defined when the previous vanish. The main difference in (44) between the analytic study, done in Chapter 1 versus the piecewise one is the fact that in the first, the Lyapunov constants with even indices are zero while in the second not.

Usually, the computation of the Lyapunov constants needs a hard effort and high memory computers. As in the analytic scenario, a parallelization algorithm can be used. We have used the procedure developed by Liang and Torregrosa in [**45**] and Theorem 1.9. Without it, the involved computations using the described algorithm and the same computers are impossible to be obtained.

For analytic vector fields with a weak-focus of order 1 at the origin only one limit cycle bifurcate from the origin using the trace parameter. This phenomenon is the classical Hopf bifurcation that we have explain in Chapter 1. See more details in [**2**]. Next result is the generalization of this property for piecewise analytic, where two crossing limit cycles appear. This result follows from the study of the return map near the origin given in [**27**].

PROPOSITION 3.3. *Consider the perturbed system*

$$\begin{cases} \dot{x} = -(1+c^2)y + \sum\limits_{k+\ell=2}^{\infty} a_{k\ell}^{+} x^k y^\ell, \\ \dot{y} = x + 2cy + \sum\limits_{k+\ell=2}^{\infty} b_{k\ell}^{+} x^k y^\ell, \end{cases} \qquad \begin{cases} \dot{x} = -y + \sum\limits_{k+\ell=2}^{\infty} a_{k\ell}^{-} x^k y^\ell, \\ \dot{y} = d + x + \sum\limits_{k+\ell=2}^{\infty} b_{k\ell}^{-} x^k y^\ell, \end{cases} \tag{45}$$

*for $y \geq 0$ and $y < 0$, respectively. If $a_{11}^{+} - a_{11}^{-} + 2(b_{02}^{+} - b_{02}^{-}) + b_{20}^{+} - b_{20}^{-} \neq 0$ then there exist $c$ and $d$ small enough such that two crossing limit cycles bifurcate from the origin.*

PROOF. When $c = d = 0$ we can use (44) to compute

$$L_2 = \frac{2}{3}\left(a_{11}^{+} - a_{11}^{-} + 2(b_{02}^{+} - b_{02}^{-}) + b_{20}^{+} - b_{20}^{-}\right).$$

From the condition given in the statement, the origin is stable or unstable because

$$\Delta(x) = \Pi(x) - x = L_2 x^2 + \cdots .$$

So, for $c, d$ small enough, computing the return map as in the proof of Proposition 7.3 of [**27**] we can write, for $x \geq 0$,

$$\Delta(x) = \Pi(x) - x = \Delta_0(c,d) + \Delta_1(c,d)x + \Delta_2(c,d)x^2 + \cdots ,$$

where $\Delta_0(c,d) = d$, $\Delta_1(c,d) = e^{\pi c} - 1$, and $\Delta_2(0,0) = L_2$. As $c$ and $d$ are arbitrary parameters, two crossing limit cycles bifurcate from the origin. □

An alternative way to get the same bifurcation to obtain a first crossing limit cycle assuming that there is no sliding segment, $\Delta(0) = 0$, computing the $\Delta_1$ and $\Delta_2$ and checking that they have opposite sign. This is the strategy followed in [**20**] where the return map of a focus-focus point is studied without the existence of an sliding segment. The second limit cycle can be obtained adding an sliding segment with an adequate stability. This is done in [**22**]. This second mechanism is known as pseudo-Hopf bifurcation, see also [**12**].

We notice that in (41) the we have not considered the perturbation monomials corresponding to parameters $c$ and $d$ in (45). This is because as we are interested only in lower bounds for the number of limit cycles of small-amplitude, we consider the limit cycles bifurcating under the hypotheses $c = 0$ and $d = 0$ and then we add two extra limit cycles. This is possible because all the periodic orbits obtained with the first mechanism are hyperbolic and also the next two.

The next two propositions show how these two mechanisms can be used to study the local cyclicity of a quadratic centers, perturbing with some quadratic monomials. In the first we consider the complete Lyapunov constants, while in the second only developments of some order.

PROPOSITION 3.4. *Consider the piecewise quadratic perturbed system*

$$\begin{cases} \dot{x} &= -y - c^2y + x^2 - y^2 + a_1x^2 + a_2xy, \\ \dot{y} &= x + 2cy + 2xy, \end{cases} \quad \text{for } y \geq 0,$$
$$\begin{cases} \dot{x} &= -y + x^2 - y^2 + a_3xy, \\ \dot{y} &= d + x + 2xy + a_4x^2, \end{cases} \quad \text{for } y \leq 0. \tag{46}$$

*Then, there exist parameters $a_1, a_2, a_3, a_4, c,$ and $d$ such that 6 limit cycles of small amplitude bifurcate from the origin.*

PROOF. We prove that 4 limit cycles bifurcate from the origin when $c = d = 0$. Then, Proposition 3.3 applies and the statement follows.

Straightforward computations, using the algorithm described at the beginning of this section, show that the first Lyapunov constants, see (44), write as

$$\begin{aligned}
L_2 &= \frac{2}{3}(a_2 - a_3 - a_4), \\
L_3 &= \frac{1}{8}\pi(a_2a_1 - 2a_2 + 6a_3), \\
L_4 &= \frac{2}{45}(7a_1^2a_2 - 6a_2^3 + 15a_3a_2^2 - 9a_2a_3^2 + 24a_1a_2 + 33a_2 - 33a_3), \\
L_5 &= -\frac{\pi}{4704}(960a_2^3 - 1294a_2^2a_3 - 590a_2a_3^2 + 924a_3^3 - 6611a_2 + 7635a_3), \\
L_6 &= \frac{2}{4862025}(45360a_2^5 - 442890a_2^4a_3 + 1135764a_2^3a_3^2 - 1173690a_2^2a_3^3 \\
&\quad + 435456a_2a_3^4 - 506352a_1a_2^3 + 1430520a_1a_2^2a_3 - 858312a_1a_2a_3^2 + 3222681a_1^2a_3 \\
&\quad - 2203848a_2^3 + 10309338a_3a_2^2 - 11563623a_2a_3^2 + 3458133a_3^3 + 1913056a_1a_2 \\
&\quad + 7902048a_3a_1 + 3015060a_2 - 3015060a_3).
\end{aligned}$$

Then solving the system $\{L_2 = 0, L_3 = 0, L_4 = 0, L_5 = 0\}$ we have two solutions. The first is very simple, $a_2 = a_3 = a_4 = 0$, and the second, $a^* = (a_1^*, a_2^*, a_3^*, a_4^*)$, depending on $\beta$, any real solution of the polynomial

$$\beta^8 - \frac{3571}{60}\beta^6 + \frac{138277}{120}\beta^4 - \frac{389439}{50}\beta^2 + \frac{353322}{25}. \tag{47}$$

More concretely,

$$
\begin{aligned}
a_1^* &= \frac{-144600\beta^6 + 5085310\beta^4 - 41118785\beta^2 + 68287464}{3384216}, \\
a_2^* &= \beta, \\
a_3^* &= \beta\frac{(144600\beta^6 - 5085310\beta^4 + 41118785\beta^2 - 61519032)}{20305296}, \\
a_4^* &= -\frac{\beta\,(144600\beta^6 - 5085310\beta^4 + 41118785\beta^2 - 81824328)}{20305296}.
\end{aligned}
$$

It is easy to check that the above polynomial has exactly 8 simple roots. Approximatelly they are $\pm 4.942476362, \pm 4.861760293, \pm 2.934038109, \pm 1.686209152$. At this point, we have that

$$
\begin{aligned}
L_6(a^*) = \frac{-\beta}{1891844428320}&(1460409164760\beta^6 - 52388583528350\beta^4 \\
&+ 444066605940121\beta^2 - 873428786956332)
\end{aligned}
$$

and the Jacobian of $(L_2, L_3, L_4, L_5)$ with respect to $a$ at $a^*$ is

$$
\begin{aligned}
&\frac{\pi^2\beta}{865431385211184723093763915776 0}(-312101550963588949029174435840 0\beta^6 \\
&\quad + 11203571677465708836528983758848 0\beta^4 \\
&\quad - 950914725725481509117124444211200\beta^2 \\
&\quad + 1869664212507384372502857918726144).
\end{aligned}
$$

The statement follows from the fact that the above two polynomials have no common roots with (47). Because their respective resultants with respect to $\beta$ are non vanishing. ☐

Proposition 3.5. *Consider the piecewise quadratic perturbed system* (46). *Then, the local cyclicity, considering $a_1, a_2, a_3, a_4, c,$ and $d$ small enough parameters, is at least* 4 *using developments up to order* 6.

Proof. We prove that 2 limit cycles bifurcate from the origin when $c = d = 0$. Then, as in the previous result, Proposition 3.3 applies and the statement follows.

As in the proof of Proposition 3.4 we compute the linear developments of the first Lyapunov constants, see (44), and we get $L_k^{(1)} = u_k$ for $k = 2, \ldots, 4$ and $L_k^{(1)} = 0$ for $k = 5, 6$. Then, the result follows for order 1. Computing the higher developments, up to order 6, and using the simplification mechanism described in Section 1.2, we get that $L_5^{(j)} = 0$ and $L_6^{(j)} = 0$ for $j = 2, \ldots, 6$. Then, the result follows for higher orders. ☐

We notice that in the above proof we have not computed more Lyapunov constants nor higher order developments because our interest here is not to study the local cyclicity of such fixed quadratic system that it has nothing special. In fact, the results of following sections improve the local cyclicity because we consider general quadratic perturbations.

THEOREM 3.6. *Consider the class of piecewise analytic systems* (40) *without sliding segment and such that both* $Z^{\pm}$ *have equilibria at the origin. If* $Z$ *has a weak-focus of order* $2n+1$ *at the origin then, the local cyclicity is at most* $2n$. *Moreover, there are analytic perturbations inside the same class* (40) *without constant terms, such that* $2n$ *hyperbolic limit cycles of small amplitude bifurcate from the origin.*

REMARK 3.7. *We remark that the above result also includes the case when the unperturbed system is analytic. We notice that the piecewise perturbation exhibits* $2n$ *limit cycles of small amplitude instead of the analytic perturbation that only* $n$ *bifurcate from the origin. See* [**54**].

The next corollary is a direct application of the above result together with the psedo-Hopf bifurcation provided in Theorem 3.3.

COROLLARY 3.8. *In Theorem 3.6, considering that the perturbation can have constants terms, i.e. adding an sliding segment, at least one more limit cycle bifurcates from the origin. That is, generically* $2n+1$ *limit cycles of small amplitude bifurcate from a weak-focus of order* $2n+1$ *in piecewise analytic vector fields defined in two zones separated by a straight line.*

PROOF OF THEOREM 3.6. The proof follows with the ideas in [**2, 54**]. From the mechanism described in the beginning of the section, if we have $Z$ a vector field having a weak-focus of order $2n+1$, the difference map writes as $\Delta(x)=\Pi(x)-x=L_{2n+1}x^{2n+1}+\cdots$, with $L_{2n+1}\neq 0$. Considering a general perturbation of (40) we have that

$$\Delta(x)=f_1(\lambda)x+f_2(\lambda)x^2+\cdots+f_{2n}(\lambda)x^{2n}+L_{2n+1}x^{2n+1}+\cdots,$$

and $\Delta(0)=0$ because we have no sliding segment by hypothesis. Using the Weierstrass Preparation Theorem, because $\left.\frac{\partial^{2n+1}\Delta}{\partial^{2n+1}x}\right|_{(0,0)}\neq 0$, there exist analytic functions $\tilde{f}_k$ and $F$ such that

$$\Delta(x)=(\tilde{f}_1(\lambda)x+\tilde{f}_2(\lambda)x^2+\cdots+\tilde{f}_{2n}(\lambda)x^{2n}+\tilde{f}_{2n+1}(\lambda)x^{2n+1})F(x,\lambda),$$

where $F(0,0)\neq 0$ and $f_{2n+1}(0)=L_{2n+1}$. Then, clearly, the function $\Delta$ can have at most $2n$ solutions as the first statement ensures.

Let us consider now the perturbed vector field $Z_\lambda=Z+\tilde{Z}$ with

$$\tilde{Z}(\rho,\theta,\lambda)=\begin{cases}(\dot\rho,\dot\theta)=(\lambda_1\rho+\lambda_2\rho^2+\ldots+\lambda_{2n}\rho^{2n},1), & \text{when } 0\leq\theta\leq\pi,\\ (\dot\rho,\dot\theta)=(0,1), \text{when} -\pi<\theta\leq 0,\end{cases}$$

and it corresponding difference map $\Delta_{Z_\lambda}(\rho)$. Observe that $\Delta_{Z_0}(\rho)=\Delta_Z(\rho)=L_{2n+1}\rho^{2n+1}+\cdots$. From the mechanism described in the beginning of the section, the difference map is computed from $Z^{\pm}_{\lambda}$. But here only $\Pi^{+}_{\lambda}$ is necessary to be computed, because $\Pi^{-}_{\lambda} = \Pi^{-}_{0}$, the vector field in $y < 0$ remains unchanged. Taking $\lambda_j = 0$, for $j = 1, \ldots, 2n-1$ and $\lambda_{2n}$ small enough such that $\lambda_{2n} L_{2n+1} < 0$, we can compute $\Pi^{+}_{\lambda}(\rho)$ from the solution given by $r_{\lambda}(\theta, \rho) = \rho + \sum_{i=2}^{\infty} r^{+}_{i}(\theta, \lambda)\rho^{i}$ and evaluating at $\theta = \pi$. Then as above the first non vanishing coefficient of the difference map is now $\lambda_{2n}$ and it exists a crossing limit cycle because the stability of the origin has changed. The next crossing limit cycles appear similarly using, in an ordered way, the parameters $\lambda_j$ alternating sign, $\lambda_j \lambda_{j+1} < 0$, but in such a way $|L_{2n+1}| \gg |\lambda_{2n}| \gg |\lambda_{2n-1}| \gg \cdots \gg |\lambda_2| \gg |\lambda_1|$. With this mechanism, the bifurcation of each limit cycle is controlled by each $\lambda_j$. Then the second part of the statement follows. □

We notice that the above proof is the same than in the analytic unfolding of a weak-focus but without checking the symmetry property that vanish all the terms corresponding to even exponents of the initial condition in the difference map.

As we have shown in the above result and we have commented in Remark 3.7 in piecewise vector fields all the coefficients in the return map appear after a generic perturbation. Then the order of a weak-focus in analytic and piecewise analytic means different but they have a relation. Because when we say that an analytic vector field we have a weak-focus of order $n$ the difference map starts with the monomial $x^{2n+1}$, while, in this case, for piecewise analytic vector fields we say that the weak-focus has order $2n+1$.

As we have shown in the proof of Proposition 3.4, computing the first order terms of the Lyapunov constants we can use the Implicit Function Theorem study the bifurcation of hyperbolic limit cycles of small amplitude from the origin. Then we can generalize Theorems 1.2 and 1.3 for piecewise polynomial vector fields. The proofs follows directly from the proofs of both results. Using exactly the same ideas. That is, the Implicit Function Theorem directly or using previously an specific blowup. Of course, here we need to use also Proposition 3.3. First studying the hyperbolic limit cycles bifurcating from the origin with $c = d = 0$ in (45) and then adding 2 extra hyperbolic limit cycles.

Theorem 3.9. *Consider the perturbed system of the form* (39),

$$\begin{cases} (\dot{x}, \dot{y}) = \left( P_c(x,y) + \sum\limits_{k+\ell=0}^{n} a^{+}_{k\ell} x^k y^\ell, Q_c(x,y) + \sum\limits_{k+\ell=0}^{n} b^{+}_{k\ell} x^k y^\ell \right) & \text{for } y \geq 0, \\ (\dot{x}, \dot{y}) = \left( P_c(x,y) + \sum\limits_{k+\ell=0}^{n} a^{-}_{k\ell} x^k y^\ell, Q_c(x,y) + \sum\limits_{k+\ell=0}^{n} b^{-}_{k\ell} x^k y^\ell \right) & \text{for } y < 0, \end{cases} \tag{48}$$

*where the polynomial vector field of degree $n$, $(\dot{x}, \dot{y}) = (P_c(x,y), Q_c(x,y))$ has a center at the origin. Then if the first $k$ Lyapunov constants, $L_1, \ldots, L_k$, have independent linear parts then the cyclicity of the origin of* (48) *is at least $k$.*

THEOREM 3.10. *Consider the perturbed system of the form* (48)*. Then, assuming that, after a change of variables if necessary,* $L_1 = \cdots = L_k = 0$ *and the next Lyapunov constants* $L_i = h_i(u) + O_{m+1}(u)$, $i = k+1, \ldots, k+l$, *where* $h_i$ *are homogeneous polynomials of degree* $m \geq 2$ *and* $u = (u_{k+1}, \ldots, u_{k+l})$*. If there exists a line* $\ell$*, in the parameter space, such that* $h_i(\ell) = 0$, $i = k+1, \ldots, k+l-1$*, the hypersurfaces* $h_i = 0$ *intersect transversally along* $\ell$ *for* $i = k+1, \ldots, k+l-1$*, and* $h_{k+l}(\ell) \neq 0$*, then there are perturbations of the center which can produce* $k+l$ *limit cycles.*

### 3.3. Lower bounds for the local cyclicity in piecewise systems

In this section we illustrate how the degenerated Hopf bifurcation together with Proposition 3.3 provide a good mechanism to obtain new lower bounds for the limit cycles of small amplitude bifurcating from the origin. That is, to get lower bounds for the local cyclicity $M_p(n)$. We present the results for $n = 2, 3$, and 4.

PROPOSITION 3.11. *Consider the perturbed system of the form* (48) *with* $n = 2$ *and* $P_c(x, y) = -y + 18x^2 + 8xy - 8y^2$ *and* $Q_c(x, y) = x + 4x^2 + 14xy - 4y^2$*. Then, there exist small enough values of the parameters* $a^{\pm}_{k\ell}, b^{\pm}_{k\ell}$ *such that* (48) *has at least* 13 *hyperbolic crossing limit cycles of small amplitude bifurcating from the origin.*

PROOF. The origin of the unperturbed system (48) is a Darboux center with the rational first integral, well defined at the origin,

$$H(x, y) = \frac{(80x^3 - 480x^2y + 960xy^2 - 640y^3 + 120xy - 240y^2 - 30y - 1)^2}{(20x^2 - 80xy + 80y^2 + 20y + 1)^3}.$$

First we consider that the constant and linear perturbation monomials are zero in (48). By using the algorithm described in Section 3.2 the first two terms, the corresponding to constant and linear one, are zero and the linear part, $L_k^{(1)}$, of the first Lyapunov constants are

$$\begin{aligned}
L_2^{(1)} =& \frac{2}{3}((a^+_{11} - a^-_{11}) + 2(b^+_{02} - b^-_{02}) + (b^+_{20} - b^-_{20})),\\
L_3^{(1)} =& \frac{5\pi}{4}(-4a^-_{11} + 6a^+_{11} - 13b^-_{02} - 10b^-_{20} + 7b^+_{02}),\\
L_4^{(1)} =& \frac{8}{15}(100(a^+_{02} - a^-_{02}) + 54(a^+_{20} - a^-_{20}) - 250(b^+_{02} - b^-_{02})\\
&+ 2(b^+_{11} - b^-_{11}) - 261(b^+_{20} - b^+_{20})),\\
L_5^{(1)} =& \frac{125\pi}{6}(110a^+_{02} - 90a^-_{02} + 60a^+_{20} - 48a^-_{20} - 250(b^+_{02} - b^-_{02})\\
&- 4b^-_{11} - 261(b^+_{20} - b^-_{20})),\\
L_6^{(1)} =& \frac{32}{21}(78120(a^+_{02} - a^-_{02}) + 42540(a^+_{20} - a^-_{20}) - 203043(b^+_{20} - b^-_{20})\\
&+ 194470(b^+_{02} - b^-_{02})),
\end{aligned}$$

$$
\begin{aligned}
L_7^{(1)} =&\frac{625\pi}{2127}(3675a_{02}^+ - 152565a_{02}^- - 85080a_{20}^- - 173200b_{02}^+ + 215740b_{02}^- \\
&- 139233b_{20}^+ + 266853b_{20}^-), \\
L_8^{(1)} =&\frac{2560}{402003}(1942830(a_{02}^+ - a_{02}^-) - 21484712(b_{02}^+ - b_{02}^-) \\
&+ 1142229(b_{20}^+ - b_{20}^-)),
\end{aligned}
$$

which are linearly independent. That is $L_k^{(1)} = u_k + \cdots$ , for $k = 2, \ldots, 8$. Then, using the Implicit Function Theorem, there exists an analytic change of coordinates in the parameter space such that $L_k = v_k$, for $k = 2, \ldots, 8$. Computing the Lyapunov constants up to order 2 and eliminating the linear parts, using $v_k = 0$ for $k = 2, \ldots, 8$, we obtain that the Lyapunov constants of order 2, $L_k^{(2)}$ for $k = 9, \ldots, 13$, are homogeneous polynomials of degree 2 on the essential parameters $u_9, \ldots, u_{13}$. The statement follows using Theorem 3.10 proving that the varieties $L_k^{(2)}$, for $k = 9, \ldots, 13$, intersect transversally along a straight line in the parameters space and then Proposition 3.3. After the 11 limit cycles have bifurcated from the origin, the value corresponding to $L_2 = L_2^{(1)}$ is non zero. Then we can apply Proposition 3.3 to get two more limit cycles.

Straightforward computations show that, writing $u_k = z v_k$, for $k = 9, \ldots, 12$ and $u_{13} = z$ we have $L_k^{(2)} = z^2 \mathcal{L}_k(v_9, \ldots, v_{12})$, for $k = 9, \ldots, 13$. Then, there exists a solution of $\{\mathcal{L}_9 = \mathcal{L}_{10} = \mathcal{L}_{11} = \mathcal{L}_{12} = 0\}$ such that $v_9^* = \alpha$, $v_{10}^* = p_4(\alpha)/q_3(\alpha)$, $v_{11}^* = p_2(\alpha)/q_1(\alpha)$, $v_{12}^* = \hat{p}_1(\alpha)/\hat{q}_1(\alpha)$, where $p_j, q_j, \hat{p}_j, \hat{q}_j$ are polynomials of degree $j$ with coefficients polynomials in $\pi$ with rational coefficients, and $\alpha$ is a real root of a polynomial $\phi$ of degree 2 with coefficients polynomials of degree 2 in $\pi$ with rational coefficients. Moreover, it can be checked that $\varphi$ has only two real solutions, which are simple. Finally, $\mathcal{L}_{13}$ the determinant of the Jacobian of $(\mathcal{L}_9, \mathcal{L}_{10}, \mathcal{L}_{11}, \mathcal{L}_{12})$ with respect to $(v_9, v_{10}, v_{11}, v_{12})$ evaluated at $(v_9^*, v_{10}^*, v_{11}^*, v_{12}^*)$ are also rational functions of $\alpha$ and $\pi$. As the resultant, with respect to $\alpha$, of both numerators and denominators with $\phi$ are nonvanishing the transversality and the existence of an analytic curve of weak-foci of order 13 is proved. The transversality prove also the complete unfolding, as it is said in the statement. □

We notice that, in the above system, we have 12 parameters in the homogeneous quadratic parts, then with the applied technique we can bifurcate 11 limit cycles. Additionally, we have two extra with the trace parameter and the constant parameter. We think that we have obtained the maximum number with this way to bifurcate limit cycles.

PROPOSITION 3.12. *Consider the perturbed system of the form* (48) *with* $n = 3$ *and* $P_c(x, y) = -y(1 - 68x + 1183x^2)$ *and* $Q_c(x, y) = x - 58x^2 - 44xy + 30y^2 + 672x^3 + 1484x^2y - 945xy^2 - 84y^3$. *Then, there exist small enough values of the* *parameters $a^{\pm}_{k\ell}, b^{\pm}_{k\ell}$ such that* (48) *has at least* 26 *hyperbolic crossing limit cycles of small amplitude bifurcating from the origin.*

Proof. The origin of the unperturbed system (48) is a Darboux center with the rational first integral

$$H(x,y)=\frac{(42x-7y-1)^3h(x,y)}{(448x^2+336xy+63y^2-44x-12y+1)^3(1183x^2-68x+1)},$$

with $h(x,y)=10752x^3+29568x^2y+17640xy^2+3024y^3-1600x^2-2760xy-576y^2+74x+57y-1$. Note that it is well defined at the origin. Straightforward computations show that $L_2^{(1)},\ldots,L_{24}^{(1)}$, are linearly independent. Then, using the Implicit Function Theorem, there are small enough values of the parameters in (48) such that $L_k=v_k$, for $k=2,\ldots,24$.

We do not show here the complete expressions because of their size, but only the first three.

$$\begin{aligned}
L_2^{(1)} &= -\frac{2}{3}((a_{11}^+-a_{11}^-)+2(b_{02}^+-b_{02}^-)+(b_{20}^+-b_{20}^-)),\\
L_3^{(1)} &= -\frac{\pi}{8}(+128(a_{02}^++a_{02}^-)-44(a_{11}^+-a_{11}^-)+a_{12}^++a_{12}^-\\
&\quad+184(a_{20}^++a_{20}^-)+3(a_{30}^++a_{30}^-)-44b_{02}^++132b_{02}^-+3(b_{03}^++b_{03}^-)\\
&\quad+28(b_{11}^++b_{11}^-)+88b_{20}^-+b_{21}^++b_{21}^-),\\
L_4^{(1)} &= \frac{2}{45}(+4856(a_{02}^+-a_{02}^-)-768(a_{03}^+-a_{03}^-)+88(a_{12}^+-a_{12}^-)-10796(a_{20}^+-a_{20}^-)\\
&\quad-372(a_{21}^+-a_{21}^-)-132(a_{30}^+-a_{30}^-)+17408(b_{02}^+-b_{02}^-)-3920(b+_{11}-b_{11}^-)\\
&\quad+12(b_{12}^+-b_{12}^-)+1007(b_{20}^+-b_{20}^-)-176(b_{21}^+-b_{21}^-)-102(b_{30}^+-b_{30}^-)).
\end{aligned}$$

The next step is the computation of the developments of order 2 and simplify using the first $L_2(2),\ldots,L_{24}^{(2)}$, to write $L_{25}^{(2)}$ and $L_{26}^{(2)}$ as homogeneous polynomials of degree 2 of only two essential parameters $v_{25}$ and $v_{26}$. As in Proposition 3.11, we can do a blowup $v_{25}=zz_{25}$ and $v_{26}=z$, and write $L_k^{(2)}=z^2\mathcal{L}_k(z_{25})$, for $k=25,26$ with $\mathcal{L}_k$ polynomials of degree 2 in $z_{25}$ with coefficients polynomials of degree 2 in $\pi$ with rational coefficients. The proof follows using Theorem 3.10 because $\mathcal{L}_{25}$ has simple real zeros where $\mathcal{L}_{26}$ is non vanishing. $\square$

We notice that the cubic center that we have used to perturb in the last result is of Darboux type and it provides 11 limit cycles in the non-piecewise scenario. In fact, they also appear only computing the linear parts of the Lyapunov constants and also considering the trace as another perturbation parameter. See [**7**].

Proposition 3.13. *Consider the perturbed system of the form* (48) *with $n=4$ and $P_c(x,y)=x^4-6x^2y^2+y^4+x^3-3xy^2+x^2-y^2-y$ and $Q_c(x,y)=4x^3y-4xy^3+3x^2y-y^3+2xy+x$. Then, there exist small enough values of the parameters $a^{\pm}_{k\ell}, b^{\pm}_{k\ell}$*

*such that* (48) *has at least* 36 *hyperbolic crossing limit cycles of small amplitude bifurcating from the origin.*

Proof. The unperturbed system $(\dot{x}, \dot{y}) = (P_c(x, y), Q_c(x, y))$, in complex coordinates $z = x + iy$ writes as $\dot{z} = iz + z^2 + z^3 + z^4$. Which is an holomorphic system, consequently it has a center at the origin. The proof follows as the proof of Proposition 3.11 but only computing first order Lyapunov constants. Because $L_2^{(1)}, \ldots, L_{36}^{(1)}$ are linearly independent.

Here we show only the expressions of the first three Lyapunov constants, because of their size,

$$
\begin{aligned}
L_2^{(1)} &= \frac{2}{3}(a_{11}^+ - a_{11}^- + b_{20}^+ - b_{20}^-),\\
L_3^{(1)} &= \frac{1}{8}\pi(a_{02}^+ + a_{02}^- + a_{12}^+ + a_{12}^- + 3a_{30}^+ + 3a_{30}^- + 3b_{03}^+ + 3b_{03}^-\\
&\quad - 4b_{20}^+ - 4b_{20}^- + b_{21}^+ + b_{21}^-),\\
L_4^{(1)} &= 8b_{20}^+ - 8b_{20}^- + 4a_{02}^+ - 4a_{02}^- - 6b_{21}^+ + 6b_{21}^- + 4b_{22}^+ - 4b_{22}^-\\
&\quad - 2a_{11}^+ + 2a_{11}^- + 4a_{12}^+ - 4a_{12}^- + 4a_{13}^+ - 4a_{13}^- - 16a_{20}^+ + 16a_{20}^-\\
&\quad + 6a_{30}^+ - 6a_{30}^- + 6a_{31}^+ - 6a_{31}^- - 4b_{03}^+ + 4b_{03}^- + 16b_{04}^+ - 16b_{04}^-\\
&\quad - 14b_{11}^+ + 14b_{11}^-.
\end{aligned}
$$

□

Proposition 3.14. *Consider the perturbed system of the form* (48) *with* $n = 4$ *and* $P_c(x, y) = (1 - x - y)y(1183x^2 - 68x + 1)$, *and* $Q_c(x, y) = (1 - x - y)(-672x^3 - 1484x^2y + 945xy^2 + 84y^3 + 58x^2 + 44xy - 30y^2 - x)$ *Then, there exist small enough values of the parameters* $a_{k\ell}^{\pm}, b_{k\ell}^{\pm}$ *such that* (48) *has at least* 40 *hyperbolic crossing limit cycles of small amplitude bifurcating from the origin.*

Proof. The unperturbed system in the statement is the same cubic system given in (3.12) but multiplied by a line of equilibrium points. Then we have a center at the origin. Straightforward computations show that the linear terms of the first Lyapunov constants, $L_2^{(1)}, \ldots, L_{39}^{(1)}$, are linearly independent, then the statement follows as the previous results using Theorem 3.9.

Here we show only the expressions of the first three Lyapunov constants, because of their size,

$$
\begin{aligned}
L_2^{(1)} &= \frac{2}{3}(a_{11}^+ - a_{11}^- + 2b_{02}^+ - 2b_{02}^- + b_{20}^+ - b_{20}^-),\\
L_3^{(1)} &= \frac{1}{9}(560(-b_{20}^- + b_{20}^+ + (a_{11}^+ - a_{11}^-) + 1120(-b_{02}^- + b_{02}^+)),
\end{aligned}
$$

$$\begin{aligned}L_4^{(1)} = -\frac{2}{15}(&-4b_{20}^+ + 4b_{20}^- - 2a_{02}^+ + 2a_{02}^- + 3b_{21}^+ - 3b_{21}^- - 2b_{22}^+ + 2b_{22}^- + a_{11}^+ - a_{11}^- \\ &- 2a_{12}^+ + 2a_{12}^- - 2a_{13}^+ + 2a_{13}^- + 8a_{20}^+ - 8a_{20}^- - 3a_{30}^+ + 3a_{30}^- - 3a_{31}^+ + 3a_{31}^- \\ &+ 2b_{03}^+ - 2b_{03}^- - 8b_{04}^+ + 8b_{04}^- + 7b_{11}^+ - 7b_{11}^-.\end{aligned}$$

□

### 3.4. Computational difficulties

The main computational difficulties found in this chapter have been related with the mechanism to get the Lyapunov constants. Because at the system is defined in two parts we have only the Lyapunov mechanism. That is, the computation of the solution in polar coordinates. This method needs a higher computational effort because of the trigonometrical expressions appearing in the recursive primitives. Among we have only computed, for degrees $n = 3$ and $n = 4$, only order 1 or 2, the computation time is very high. For example, in the proof of the last result, the total computation time has been around 4 days, working with 5 servers. Because of the memory and the parallelization of the linear terms, we have done the computations with only 4 monomials in each server. The advantage of the parallelization here is clear. To go further in higher degree we need a new mechanism for the computation of the return map.

CHAPTER 4

# Local cyclicity using the first Melnikov function

In [**15**], Chicone and Jacobs prove the equivalence between the computation of the developments of the Lyapunov constants of order 1 and the developments of the first Melnikov function in the study of limit cycles of small amplitude bifurcating from a quadratic center. This equivalence is used here for other polynomial vector fields of degree $n$. We use this extended result to show that $M(6) \geq 44$. Finally, we use also this equivalence to get limit cycles in piecewise polynomial vector fields obtaining $M_p^c(5) \geq 59$.

## 4.1. Introduction

We consider equations containing a privileged small parameter $\varepsilon$. Lagrange, in his study about the three-body problem, formulated the idea of averaging. During many years, this method was used in many fields without people bothering about proofs validity. In 1928, Fatou in [**25**] gave the first analytic proof. Nowadays, the averaging theory is also used for proving the existence of limit cycles of planar differential systems which are perturbations of a period annulus. Another mechanism used for the same purpose is the method of Melnikov. Moreover, this method is an excellent tool for studying global bifurcations that occur near homoclinic or heteroclinic loops on near one-parameter families of periodic orbits.

Let us consider the Hamiltonian system in the form

$$\begin{cases} \dot{x} = -H_y + \varepsilon P(x, y, \varepsilon, \lambda), \\ \dot{y} = \quad H_x + \varepsilon Q(x, y, \varepsilon, \lambda). \end{cases} \tag{49}$$

Then, the first Melnikov function writes as

$$\mathcal{M}(h) = \int_{\Gamma_h} Q(x, y, 0, \lambda)dx - P(x, y, 0, \lambda)dy, \tag{50}$$

where $\Gamma_h = \{H(x, y) = h\}$ are closed ovals. The Melnikov theory for first order analysis is based in the Implicit Function Theorem. In fact, the simple zeros of $\mathcal{M}(h)$ correspond to limit cycles of (49). That is, if $h^*$ is satisfies $\mathcal{M}(h^*) = 0$ and $\mathcal{M}'(h^*) \neq 0$ then there exists an hyperbolic limit cycle of (49) that goes to $\Gamma_h$ when $h$ goes to $h^*$.

This result can be generalized also to other centers, not necessarily Hamiltonian. If the system

$$\begin{cases} \dot{x} = P_c(x, y) + \varepsilon P(x, y, \varepsilon, \lambda) = -\dfrac{H_y}{V(x)} + \varepsilon P(x, y, \varepsilon, \lambda), \\ \dot{y} = Q_c(x, y) + \varepsilon Q(x, y, \varepsilon, \lambda) = \dfrac{H_x}{V(x)} + \varepsilon Q(x, y, \varepsilon, \lambda), \end{cases}$$

has an inverse integrating factor, $V(x, y)$, the generalized first Melnikov function (50) is given by

$$\mathcal{M}(h) = \int_{\Gamma_h} \frac{Q(x, y, 0, \lambda)dx - P(x, y, 0, \lambda)dy}{V(x, y)}. \tag{51}$$

We have only considered the Melnikov function, (50), for the study of perturbation of periodic orbits near planar autonomous differential systems. In this case, there are some works explaining the equivalence between Melnikov studies and the averaging theory, see [**8, 37**]. For more details on the Melkinov and the Averaging theories we refer the reader to [**9, 38, 55, 59**].

The averaging theory can be used also in piecewise vector fields, see [**46, 47**]. Consider the piecewise vector field defined in two zones separated by a straight line,

$$Z_\varepsilon = \begin{cases} Z^+(x,y) + \varepsilon(P_1^+(x,y,\varepsilon,\lambda), Q_1^+(x,y,\varepsilon,\lambda)), \text{ when } y \geq 0, \\ Z^-(x,y) + \varepsilon(P_1^-(x,y,\varepsilon,\lambda), Q_1^-(x,y,\varepsilon,\lambda)), \text{ when } y < 0. \end{cases} \tag{52}$$

Changing to polar coordinates, (52) becomes the piecewise differential equation

$$\frac{d\rho}{d\theta} = \varepsilon F_1(\rho,\theta,\lambda) + O(\varepsilon^2),$$

where

$$F_1(\rho,\theta,\lambda) = \begin{cases} F_1^+(\rho,\theta,\lambda), \text{ when } 0 < \theta < \pi, \\ F_1^-(\rho,\theta,\lambda), \text{ when } -\pi < \theta < 0. \end{cases}$$

with $F_1^\pm : (0,\rho^*) \times [-\pi,+\pi] \to \mathbb{R}$ are $2\pi$ periodic analytical functions. We notice that the period annulus is well defined for $\rho \in (0,\rho^*)$. Then, the first order averaging function for piecewise system writes as

$$\mathcal{F}_1(\rho) = \int_0^\pi (F_1^+(\rho,\theta,\lambda) + F_1^-(\rho,\theta,\lambda))d\theta. \tag{53}$$

From the mechanism described, the functions (50), (51), and (53) provides the number of limit cycles bifurcating from the period annulus up to first order perturbation. In the previous chapters we have seen that a good and simple tool for studying the number of limit cycles of small amplitude bifurcating from the origin is the first order development of the Lyapunov constants. Then, a natural question arises. *Are there any relation between both mechanisms?* Chicone and Jacobs in [**15**] provide a positive answer for families of quadratic centers. But, their proof applies also to any polynomial vector field. Hence, we can say that the next result comes from their original work. These ideas appear also in the works of Han and Yu, see [**38, 61**].

THEOREM 4.1 ([**15**]). *Let $(\dot{x},\dot{y}) = (P_c(x,y), Q_c(x,y))$ be a polynomial vector field of degree $n$, with a non degenerated center at the origin. Consider the perturbed systems, in the class of polynomial vector fields of degree $n$,*

$$\begin{cases} \dot{x} = P_c(x,y) + \displaystyle\sum_{k+l=2}^{n} a_{kl}x^k y^l, \\ \dot{y} = Q_c(x,y) + \displaystyle\sum_{k+l=2}^{n} b_{kl}x^k y^l, \end{cases} \tag{54}$$

*and*

$$\begin{cases} \dot{x} = P_c(x,y) + \varepsilon \displaystyle\sum_{k+l=2}^{n} a_{kl}x^k y^l, \\ \dot{y} = Q_c(x,y) + \varepsilon \displaystyle\sum_{k+l=2}^{n} b_{kl}x^k y^l. \end{cases} \tag{55}$$

*If we denote by* $L_k^{(1)}$ *the linear terms of the Lyapunov constants of* (54) *then, for* $\rho$ *small, the first Melnikov function of* (55) *is*

$$\mathcal{M}(\rho) = \sum_{k=1}^{N} L_k^{(1)} \left(1 + \sum_{j=1}^{\infty} \alpha_{kj0}\rho^j\right) \rho^{2k+1}, \tag{56}$$

*with the Bautin ideal* $\langle L_1, \ldots, L_N, \ldots\rangle = \langle L_1, \ldots, L_N\rangle$.

We can use the above result to obtain new lower bounds for $M(n)$. A direct consequence if the next corollary.

Corollary 4.2. *Let* $A_m$ *be the matrix corresponding to* $[L_1^{(1)}, \ldots, L_m^{(1)}]$ *with respect to the parameters* $[a_{20}, a_{11}, a_{02}, \ldots, b_{20}, b_{11}, b_{02}, \ldots] \in \mathbb{R}^{n^2+3n-4}$, *where each* $L_k^{(1)}$ *is the linear* $k$*-Lyapunov constants of system* (54). *Then if* $\operatorname{rank} A_m = \ell$ *then, for* $\varepsilon$ *small enough, system* (55) *has* $\ell - 1$ *hyperbolic limit cycles of small amplitude bifurcating from the origin. Additionally, adding the trace parameter, there are polynomial perturbations of the corresponding system* $(\dot{x}, \dot{y}) = (P_c(x, y), Q_c(x, y))$ *exhibiting* $\ell$ *hyperbolic limit cycles of small amplitude.*

In previous chapters we have studied the return map of a differential equation near a non-degenerated monodromic point located at the origin. It is clear that Theorem 1.2 applies when $L_k^{(1)} = u_k$ for $k = 1, \ldots, N$, proving the existence of $N$ limit cycles of small amplitude bifurcating from the origin. When the next linear parts are linearly dependent we can use Theorem 1.3 to obtain more limit cycles. The advantage of Theorem 4.1 is that we can compute easily the expressions of $L_j^{(1)}$ than the coefficients of the series expansion of $\mathcal{M}$.

The next two applications of Corollary 4.2 comes from the study of a big collection of centers and their perturbations.

Theorem 4.3. *The number of limit cycles of small amplitude bifurcating from a singular monodromic point for vector fields of degree six is at least* $M(6) \geq 44$.

Theorem 4.4. *The number of crossing limit cycles of small amplitude bifurcating from a singular monodromic point for piecewise vector fields of degree five is at least* $M_p^c(5) \geq 59$.

This chapter is structured as follows. In Section 4.2, for completeness, we proof Theorem 4.3 recovering the original proof for quadratic vector fields in [**15**]. We also prove Corollary 4.2 as a natural application to get lower bounds for the number of limit cycles of small amplitude bifurcating from a center equilibrium point. We also provide the equivalent results for piecewise polynomial vector fields. In Section 4.3 we show, in a simple example of a polynomial vector field of degree 6, that the maximal computed rank of the linear developments do not coincide with the subindex of the corresponding Lyapunov constant. Then, Theorem 1.2 does not apply but using

Theorem 4.1 we can provide a higher lower bound for the number of limit cycles of small amplitude. In Section 4.4 we prove Theorem 4.3 and we study the local cyclicity problem for other vector fields of degrees 7, 8, and 10 where Theorem 4.1 applies, proving that $M(7) \geq 60$, $M(8) \geq 70$, $M(10) \geq 97$. These values are lower than the ones obtained in Chapter 1. Finally, Section 4.5 is devoted to apply Theorem 4.1 to piecewise polynomial vector fields of degrees 3 and 5, proving Theorem 4.4. We also provide a new proof of $M_p^c(3) \geq 26$, using only order 1 developments instead of the proof in Chapter 3 that uses order 2.

## 4.2. The proof of Chicone–Jacobs' result

The proof for perturbing quadratic centers can be found in [**15**]. You can see it partially also in [**52**]. Here we reproduce it but for a polynomial vector field of degree $n$, proving Theorem 4.1.

Proof. Theorem 4.1 Consider the perturbed parameters in (54) as

$$[a_{20}, a_{11}, a_{02}, \ldots, b_{20}, b_{11}, b_{02}, \ldots] = [\lambda_1, \ldots, \lambda_m] \in \mathbb{R}^m, \text{ with } m = n^2 + 3n - 4.$$

Writing them as series expansion in terms of a privileged parameter $\varepsilon$,

$$\lambda_l(\varepsilon) = \sum_{j=1}^{\infty} \lambda_{jl} \varepsilon^j,$$

we have that the displacement function writes as

$$d(\rho, \lambda) = \sum_{k=1}^{N} L_k(\lambda) \rho^{2k+1} \left( 1 + \sum_{j=1}^{\infty} \alpha_{kj}(\lambda) \rho^j \right),$$

with $\alpha_{kj}$ polynomials vanishing at zero in the variables $\lambda$. We have now that

$$d(\rho, \varepsilon) = \sum_{k=1}^{\infty} d_k(\rho) \varepsilon^k = \sum_{k=1}^{\infty} \frac{1}{k!} \left( \left. \frac{\partial^k d(\rho, \varepsilon)}{\partial \varepsilon^k} \right|_{\varepsilon=0} \right) \varepsilon^k.$$

We notice that the series representation of the displacement function is only local, but by Global Bifurcation Lemma, see [**15**], implies that the coefficients

$$d_k(\rho) = \frac{1}{k!} \left. \frac{\partial^k d(\rho, \varepsilon)}{\partial \varepsilon^k} \right|_{\varepsilon=0},$$

are defined and analytic on the full $\rho$-domain corresponding to the portion of the $x-$axis cut by the periodic trajectory surrounding the center at the origin of the system. We observe that the idea of Melnikov theory is determine $d_k(\varepsilon)$ assuming that $d_j(\varepsilon) = 0$ for all $j < k$.

Writing

$$\lambda_i(\varepsilon) = \sum_{j=0}^{\infty} \lambda_{ij} \varepsilon^j,$$

in power series expansions we have that for each $k$

$$L_k(\lambda(\varepsilon)) = \sum_{j=1}^{\infty} L_k^{(j)}(\lambda(\varepsilon))\varepsilon^j.$$

and

$$\alpha_{ki}(\lambda(\varepsilon)) = \sum_{j=1}^{\infty} \alpha_{kij}\varepsilon^j.$$

Rearranging the series for $\varepsilon$ and $\rho$ small enough it follows that

$$d(\rho,\varepsilon) = \sum_{k=1}^{N}\sum_{j=1}^{\infty} L_k^{(j)}\varepsilon^j \left(1 + \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \alpha_{kij}\rho^j\varepsilon^i\right)\rho^{2k+1}.$$

Hence, choosing the coefficient of $\varepsilon$ in the equation above, we have the expression, for $\rho$ small, of the first Melnikov function

$$\mathcal{M}(\rho) = d_1(\rho) = \sum_{k=1}^{N} L_k^{(1)}\left(1 + \sum_{j=1}^{\infty}\alpha_{kj0}\rho^j\right)\rho^{2k+1}.$$

□

PROOF. Proof of Corollary 4.2 We study first the case when the trace parameter is zero. Then we can compute the linear terms of the Lyapunov constants and Theorem 4.1 gets the expression of the first Melnikov function (56). Then it is clear that if the rank is $\ell$, we have $\ell-1$ simple zeros near $\rho = 0$ then our perturbed system exhibits $\ell - 1$ limit cycles of small amplitude. Finally, adding the trace parameter, we get the total $\ell$ as it is stated in the statement. □

We notice that the proof of Theorem 4.1 can be easily generalized for piecewise vector fields considering only that in the above the exponents of $\rho$ are all the natural numbers, not only the odd ones. It is necessary to use also that the return map is analytic in $\rho$ if we take a system without sliding segment. In this case, the constant term in (53) is zero.

The piecewise version of perturbed systems (54) and (55) are

$$\begin{cases} \dot{x} = P_c(x,y) + \displaystyle\sum_{k+l=2}^{n} a_{kl}^{\pm}x^k y^l, \\ \dot{y} = Q_c(x,y) + \displaystyle\sum_{k+l=2}^{n} b_{kl}^{\pm}x^k y^l, \end{cases} \tag{57}$$

and

$$\begin{cases} \dot{x} = P_c(x,y) + \varepsilon\displaystyle\sum_{k+l=2}^{n} a_{kl}^{\pm}x^k y^l, \\ \dot{y} = Q_c(x,y) + \varepsilon\displaystyle\sum_{k+l=2}^{n} b_{kl}^{\pm}x^k y^l. \end{cases} \tag{58}$$

They are defined for the parameters $a^+, b^+$ and $a^-, b^-$, in the regions $y \geq 0$ and $y < 0$, respectively. Here also system $(\dot{x}, \dot{y}) = (P_c(x, y), Q_c(x, y))$ has a non-degenerated center at the origin.

Next corollary is a direct consequence of the proof of Theorem 4.1 because the return map is also analytic and all the steps are equal except that the series in $\rho$ have all natural powers because here we have not the property of symmetry that vanish all the even terms in the developments. See Chapter 3.

Corollary 4.5. *If we denote by $L_k^{(1)}$ the linear terms of the Lyapunov constants of a polynomial piecewise vector field* (57) *near a monodromic singularity, then the corresponding first Melnikov function of* (58) *writes as*

$$\mathcal{F}_1(\rho) = \sum_{k=2}^{N} L_k^{(1)} \left( 1 + \sum_{j=1}^{\infty} \alpha_{kj0} \rho^j \right) \rho^k,$$

*where the Bautin ideal $\langle L_2, \ldots, L_N, \ldots \rangle = \langle L_2, \ldots, L_N \rangle$.*

Finally, as for analytic perturbations the rank of the matrix provides the lower bound for the number of limit cycles. Here we add two extra monomials in the Taylor series because we have the trace parameters and also the sliding parameter. Then we have exactly one limit cycle more than the number of linearly independent parameters. See Proposition 3.3.

Corollary 4.6. *Let $A_m$ be the matrix corresponding to $[L_2^{(1)}, \ldots, L_m^{(1)}]$ with respect to the parameters $[a_{20}, a_{11}, a_{02}, \ldots, b_{20}, b_{11}, b_{02}, \ldots] \in \mathbb{R}^{2n^2+6n-8}$, where each $L_k^{(1)}$ is the linear $k$-Lyapunov constants of system* (54)*. Then if* $\operatorname{rank} A_m = \ell$ *then, for $\varepsilon$ small enough, system* (55) *has $\ell + 1$ limit cycles of small amplitude bifurcating from the origin.*

## 4.3. A first but not trivial example

In this section we show, in a simpler example studied also in [**42**], how the study of the linear developments of the Lyapunov constants can be used to provide limit cycles but with the first Melnikov function. From the proof we can see that most probably the number of limit cycles obtained will be the maximum using only first order analysis.

Proposition 4.7. *Let us consider the system*

$$\begin{cases} \dot{x} = -y + x^5 y, \\ \dot{y} = x + x^4 y^2. \end{cases} \tag{59}$$

*Then, there exist polynomial perturbations of degree* 6 *such that from the origin of the center* (59) *bifurcate* 16 *limit cycles of small amplitude.*

Proof. System (59) has a center at the origin because it is time-reversible. It is invariant by the change $(x,y,t)\to(x,-y,-t)$. But it has also the rational first integral

$$H(x,y)=\frac{(x^2+y^2)^5}{(1-x^5)^2},$$

with the corresponding inverse integrating factor

$$V(x,y)=\frac{x^{10}-2x^5+1}{(x^2+y^2)^{3/2}}.$$

Then the proof follows using Corollary 4.2. We compute $L_k^{(1)}$ for $k=1,\ldots,40$ with the algorithm described in Chapter 1. Let $A_n$ be the matrix corresponding to $[L_1^{(1)},\ldots,L_n^{(1)}]$ with respect to the parameters $[a_{20},a_{11},a_{02},\ldots,b_{20},b_{11},b_{02},\ldots]\in\mathbb{R}^{50}$. Then $\operatorname{rank}A_k=k$ for $k=1,\ldots,12$, but $\operatorname{rank}A_{13}=12$. Then, Theorem 1.2 only ensures the existence of 12 limit cycles. To go further with this investigation line we need to compute higher order terms of $L_k$. If we continue computing the rank, adding more linear Lyapunov constants, we have that $\operatorname{rank}A_{14}=13$, $\operatorname{rank}A_{15}=14$, $\operatorname{rank}A_{19}=15$, and $\operatorname{rank}A_k=16$, for $k=20,\ldots,40$.

The explicit expression of the necessary Lyapunov constants to get the statement are:

$$\begin{aligned}
L_1^{(1)}&=\frac{2}{3}(3a_{30}+a_{12}+b_{21}+3b_{03}),\\
L_2^{(1)}&=\frac{2}{5}(b_{41}+a_{32}+b_{23}+5b_{05}+a_{14}+5a_{50}),\\
L_3^{(1)}&=\frac{2}{35}(33a_{20}+7a_{02}-b_{11}),\\
L_4^{(1)}&=\frac{2}{315}(161a_{40}-21b_{31}-9b_{13}+29a_{22}+21a_{04}),\\
L_5^{(1)}&=\frac{2}{231}(21a_{60}-7b_{33}-5b_{15}+5a_{24}+7a_{06}+7a_{42}-21b_{51}),\\
L_6^{(1)}&=-\frac{2}{715}(2661b_{03}+873b_{21}+901a_{12}+2773a_{30}),\\
L_7^{(1)}&=-\frac{2}{10725}(42775b_{05}+8639a_{14}+8755a_{32}+8447b_{23}+8051b_{41}+44259a_{50}),\\
L_8^{(1)}&=-\frac{2}{2127125}(-265929b_{11}+1931573a_{02}+9055937a_{20}),\\
L_9^{(1)}&=\frac{2}{121246125}(-40561101b_{31}-18052569b_{13}+42963501a_{04}+320497961a_{40}\\
&\qquad+58729949a_{22}),\\
L_{10}^{(1)}&=-\frac{2}{24249225}(12806957a_{60}+4471569a_{42}+4628869a_{06}-12806957b_{51}\\
&\qquad-4471569b_{33}+3270585a_{24}-3270585b_{15}),
\end{aligned}$$

$$L_{11}^{(1)} = \frac{2}{663966875}(9725859717a_{12} + 29642231001a_{30} + 28891397801b_{03} + 9538151417b_{21}),$$

$$L_{12}^{(1)} = \frac{2}{16599171875}(770680333485b_{05} + 789987106585a_{50} + 155267074197a_{14} + 156791990497a_{32} + 147654869097b_{41} + 152709627597b_{23}),$$

$$L_{13}^{(1)} = \frac{2}{3137243484375}(158564044415887a_{20} + 33790154009423a_{02} - 46806856756639b_{11}),$$

$$L_{14}^{(1)} = \frac{2}{10108895671875}(44434290786711a_{04} + 332135401856251a_{40} + 60777619220839a_{22} - 186928552178196_{13} - 421368784695116_{31}),$$

$$L_{15}^{(1)} = \frac{2}{3418644718125}(23966540375301a_{60} + 8360451650717a_{42} + 6115824649905a_{24} - 23966540375301b_{51} - 8360451650717b_{33} - 6115824649905b_{15} + 8657859935137a_{06}),$$

$$L_{16}^{(1)} = -\frac{2}{3357597491015625}(324541281870992179 7b_{03} + 1075093752985900449b_{21} + 1088751590673845749a_{12} + 3300044169461702997a_{30}),$$

$$L_{17}^{(1)} = -\frac{2}{430461216796875}(441743420639301995b_{05} + 857037972335647 99b_{41} + 877651926734416 99b_{23} + 894357364652753 99a_{32} + 888122797251454 99a_{14} + 449630929414297095a_{50}),$$

$$L_{18}^{(1)} = -\frac{2}{56468685076171875}(6343162117243756 4093a_{20} - 18833119357064660 21b_{11} + 13503765373476308497a_{02}),$$

$$L_{19}^{(1)} = -\frac{2}{621155535837890625}(464272204405895182111a_{40} + 84747082997571253579a_{22} - 591620270508416429 71b_{31} - 261011043684573805 59b_{13} + 618678635752134289 71a_{04}),$$

$$L_{20}^{(1)} = -\frac{2}{35654327757094921875}(5933622583442761308861a_{60} + 15030422421675922027 05a_{24} - 2059684414443886787537b_{33} + 2125467391558746447037a_{06} - 1503042242167592202705b_{15} - 5933622583442761308861b_{51} + 2059684414443886787537a_{42}).$$

$\square$

### 4.4. Perturbing systems of degrees $6, 7, 8$, and $10$

In this section we use Corollary 4.2 to study lower bounds for the cyclicity in some centers of low degree, $n = 6, 7, 8$, and 10. Next result proves Theorem 4.3. From the proofs, it seems that the number of limit cycles that bifurcate from the origin will be the maximum values for order 1 studies.

Proposition 4.8. *There are polynomial perturbations of degree* 6 *such that from the origin of the center*

$$\begin{cases} \dot{x} = -y + \dfrac{128}{15}x^6 - \dfrac{128}{15}x^5 y - \dfrac{416}{45}x^4 y^2 + \dfrac{448}{45}x^3 y^3 - \dfrac{256}{15}x^2 y^4 + \dfrac{256}{45}xy^5 + \dfrac{8}{9}y^6, \\ \dot{y} = 2x - \dfrac{896}{45}x^5 y - \dfrac{1664}{45}x^4 y^2 + \dfrac{96}{5}x^3 y^3 - \dfrac{512}{45}x^2 y^4 + \dfrac{112}{45}xy^5 + \dfrac{32}{15}y^6. \end{cases} \tag{60}$$

*bifurcate at least* 44 *limit cycles of small amplitude.*

Proof. The vector field in the statement and the proof that it has a center at the origin is given by Giné in [**32**]. The proof follows from Corollary 4.2 computing $L_k^{(1)}$, for $k = 1, \ldots, 80$, and checking that $\operatorname{rank} A_k = 44$, for $k = 52, \ldots, 80$. We notice that $\operatorname{rank} A_k = k$, for $k = 1, \ldots, 32$. Because of the size, we only show the linear developments of the first 3 Lyapunov constants.

$$\begin{aligned} L_1^{(1)} &= \frac{4}{3}(2a_{12} + b_{21}) + 4(a_{30} + b_{03}), \\ L_2^{(1)} &= \frac{4}{5}(4a_{14} + b_{41} + 2a_{32} + 2b_{23}) + 4(a_{50} + 4b_{05}), \\ L_3^{(1)} &= \frac{512}{1575}(-118b_{02} - 86a_{11} + 488a_{02} + 236a_{20} - 61b_{20} + 172b_{11}). \end{aligned}$$

□

Remark 4.9. *We notice that, with the described perturbation of degree* 6*, for* (60) *we have also computed* 44 *Lyapunov constants of order* 4 *and, using the techniques presented in Chapter 1, we can prove only the existence of* 41 *limit cycles of small amplitude. For this center, with the equivalence described we get more limit cycles.*

Proposition 4.10. *There are polynomial perturbations of degree* 7 *such that from the origin of the center*

$$\begin{cases} \dot{x} &= -\dfrac{2527}{3}x^6 y - \dfrac{2968}{3}x^5 y^2 - \dfrac{4186}{3}x^4 y^3 - \dfrac{2800}{3}x^3 y^4 - 553x^2 y^5 + 56xy^6 \\ & \quad + \dfrac{184}{3}x^3 y + \dfrac{88}{3}x^2 y^2 + 48xy^3 - y, \\ \dot{y} &= 672x^7 + 1484x^6 y + \dfrac{2219}{3}x^5 y^2 + \dfrac{5684}{3}x^4 y^3 - \dfrac{742}{3}x^3 y^4 + \dfrac{1148}{3}x^2 y^5 \\ & \quad -315xy^6 - 28y^7 - 58x^4 - 44x^3 y - \dfrac{104}{3}x^2 y^2 - \dfrac{44}{3}xy^3 + 10y^4 + x, \end{cases}$$

*bifurcate at least* 60 *limit cycles of small amplitude.*

Proof. The system in the statement has a center at the origin because it has a rational first integral of the form $H(x(x^2+y^2), y(x^2+y^2))$ where $H$ is given in (21), that is,

$$H(x,y) = \frac{(42x-7y-1)^3 f_3(x,y)}{(448x^2+336xy+63y^2-44x-12y+1)^3 (1183x^2-68x+1)}$$

with $f_3(x,y) = (10752x^3 + 29568x^2y + 17640xy^2 + 3024y^3 - 1600x^2 - 2760xy - 576y^2 + 74x + 57y - 1)$.

The proof follows from Corollary 4.2 computing $L_k^{(1)}$, for $k = 1, \ldots, 80$, and checking that $\operatorname{rank} A_{63} = 60$. We notice that $\operatorname{rank} A_k = k$, for $k = 1, \ldots, 55$. We show only the first 3 Lyapunov constants.

$$\begin{aligned}
L_1^{(1)} &= \frac{1}{3}(a_{12}+b_{21}) + a_{30} + b_{03},\\
L_2^{(1)} &= \frac{1}{45}(1664a_{02} + 352a_{11} + 9a_{14} + 3952a_{20} + 9a_{32} + 45a_{50} + 968b_{02} + 45b_{05}\\
&\quad + 1064b_{11} + 616b_{20} + 9b_{23} + 9b_{41}),\\
&\quad + a_{50} + b_{05}.\\
L_3^{(1)} &= \frac{2192}{105}(a_{04} + 176a_{13} + 15a_{16} + 1392a_{22} + 528a_{31} + 9a_{34} + 6800a_{40} + 15a_{52}\\
&\quad + 105a_{70} + 880b_{04} + 105b_{07} + 912b_{13} + 528b_{22} + 15b_{25} + 1072b_{31} + 880b_{40}\\
&\quad + 9b_{43} + 15b_{61}).
\end{aligned}$$

□

Proposition 4.11. *There are polynomial perturbations of degree* 8 *such that from the origin of the center*

$$\begin{cases}
\dot{x} &= \frac{54}{175}x^8 + \frac{18}{35}x^7y - \frac{54}{175}x^6y^2 + \frac{894}{175}x^5y^3 - 2x^4y^4\\
&\quad + \frac{66}{25}x^3y^5 - \frac{26}{35}x^2y^6 - \frac{342}{175}xy^7 + \frac{16}{25}y^8 - y,\\
\dot{y} &= -\frac{198}{175}x^7y - \frac{1254}{175}x^6y^2 - \frac{586}{175}x^5y^3 - \frac{258}{35}x^4y^4\\
&\quad - \frac{22}{5}x^3y^5 + \frac{18}{25}x^2y^6 - \frac{382}{175}xy^7 + \frac{162}{175}y^8 + x.
\end{cases}$$

*bifurcate at least* 70 *limit cycles of small amplitude.*

Proof. We consider the center with quartic homogeneous nonlinearities given in (22) written in polar coordinates, $(x,y) = (r\cos\theta, r\sin\theta)$. Then, with the change $R = r^{3/5}$ and recovering again a new cartesian coordinates we obtain the system in the statement of degree 10. Then it has also a center at the origin. The proof follows from Corollary 4.2 computing $L_k^{(1)}$, for $k = 1, \ldots, 130$, and checking that $\operatorname{rank} A_{87} = 70$. We notice that $\operatorname{rank} A_k = k$, for $k = 1, \ldots, 45$. The first 3 Lyapunov constants are

$$
\begin{aligned}
L_1^{(1)} &= \frac{1}{3}(a_{12}+b_{21})+a_{30}+b_{03},\\
L_2^{(1)} &= \frac{1}{5}(a_{14}+a_{32}+5a_{50}+5b_{05}+b_{23}+b_{41}),\\
L_3^{(1)} &= \frac{1}{35}(5a_{16}+3a_{34}+5a_{52}+35a_{70}+35b_{07}+5b_{25}+3b_{43}+5b_{61}).
\end{aligned}
$$

□

PROPOSITION 4.12. *There are polynomial perturbations of degree* 10 *such that from the origin of the center*

$$
\begin{cases}
\dot{x} &= \frac{6}{25}x^{10}+\frac{2}{5}x^9y+\frac{8}{25}x^8y^2+\frac{152}{25}x^7y^3-\frac{28}{25}x^6y^4+\frac{44}{5}x^5y^5-\frac{8}{5}x^4y^6\\
&\quad +\frac{24}{25}x^3y^7+\frac{6}{25}x^2y^8-\frac{54}{25}xy^9+\frac{16}{25}y^{10}-y,\\
\dot{y} &= -\frac{6}{5}x^9y-\frac{182}{25}x^8y^2-\frac{104}{25}x^7y^3-\frac{352}{25}x^6y^4-\frac{164}{25}x^5y^5-\frac{28}{5}x^4y^6\\
&\quad -\frac{136}{25}x^3y^7+\frac{48}{25}x^2y^8-\frac{46}{25}xy^9+\frac{18}{25}y^{10}+x.
\end{cases}
$$

*bifurcate at least* 97 *limit cycles of small amplitude.*

PROOF. The center in the statement is obtained following the same procedure than in the proof of Proposition 4.11 but with the change $R = r^{3/9}$. The proof follows from Theorem 4.1 computing $L_k^{(1)}$, for $k = 1,\dots,130$, and checking that $\operatorname{rank} A_{126} = 97$. We notice that $\operatorname{rank} A_k = k$, for $k = 1,\dots,58$. Curiously, the expressions of the first 3 Lyapunov constants coincide with the ones of the previous proposition. □

## 4.5. Perturbing piecewise systems of degrees 3 and 5

This section is devoted to prove Theorem 4.4. We also show that there are other cubic centers having cyclicity also higher or equal than 26 as it was proved in Proposition 3.12 but using order 2 developments. The next result only uses first order analysis.

The general perturbed system considered in this section is

$$
\begin{cases}
(\dot{x},\dot{y}) = \left(P_c(x,y)+\sum\limits_{k+\ell=0}^{n} a_{k\ell}^{+}x^ky^\ell,\, Q_c(x,y)+\sum\limits_{k+\ell=0}^{n} b_{k\ell}^{+}x^ky^\ell\right) & \text{for } y\ge 0,\\
(\dot{x},\dot{y}) = \left(P_c(x,y)+\sum\limits_{k+\ell=0}^{n} a_{k\ell}^{-}x^ky^\ell,\, Q_c(x,y)+\sum\limits_{k+\ell=0}^{n} b_{k\ell}^{-}x^ky^\ell\right) & \text{for } y< 0.
\end{cases}
\tag{61}
$$

PROPOSITION 4.13. *There exist polynomial piecewise perturbations of degree* $n = 3$ *as* (61) *such that* 26 *crossing limit cycles of small amplitude bifurcate from the origin of system*

$$\begin{cases} \dot{x} &= -y + \dfrac{168}{125}x^2 + \dfrac{8252}{125}xy - \dfrac{2968}{125}y^2 \\ & \quad -\dfrac{44436}{625}x^3 - \dfrac{533631}{625}x^2y + \dfrac{592508}{625}xy^2 + \dfrac{69552}{625}y^3, \\ \dot{y} &= x - \dfrac{4974}{125}x^2 + \dfrac{9164}{125}xy + \dfrac{2874}{125}y^2 \\ & \quad +\dfrac{232848}{625}x^3 - \dfrac{910392}{625}x^2y + \dfrac{385231}{625}xy^2 + \dfrac{407064}{625}y^3. \end{cases}$$

PROOF. The system in the statement is the cubic center system given in (35) but rotated with the matrix

$$\begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix}$$

The proof follows from Corollary 4.6 computing $L_k^{(1)}$, for $k = 2, \ldots, 32$, and checking that rank $A_{28} = 25$. We notice that rank $A_k = k - 1$, for $k = 2, \ldots, 24$. Using the Corollary 4.6, we obtain 26 limit cycles. Because of the size, we only show the linear developments of the first 3 Lyapunov constants.

$$\begin{aligned}
L_2^{(1)} &= \frac{2}{3}((a_{11}^+ - a_{11}^-) + (b_{20}^+ - b_{20}^-) + 2(b_{02}^+ - b_{02}^-)), \\
L_3^{(1)} &= 19000(126000\pi a_{02}^- - 25200\pi a_{11}^- + 1125\pi a_{12}^- + 163800\pi a_{20}^- + 3375\pi a_{30}^- \\
&\quad + 126000\pi a_{02}^+ - 25200\pi a_{11}^+ + 1125\pi a_{12}^+ + 163800\pi a_{20}^+ + 3375\pi a_{30}^+ \\
&\quad - 135900\pi b_{02}^- + 3375\pi b_{03}^- + 18900\pi b_{11}^- - 85500\pi b_{20}^- + 1125\pi b_{21}^- - 135900\pi b_{02}^+ \\
&\quad + 3375\pi b_{03}^+ + 18900\pi b_{11}^+ - 85500\pi b_{20}^+ + 1125\pi b_{21}^+ - 577664a_{11}^- + 577664a_{11}^+ \\
&\quad - 1155328b_{02}^- - 577664b_{20}^- + 1155328b_{02}^+ + 577664b_{20}^+), \\
L_4^{(1)} &= \frac{1}{2812500}(4738650000\pi a_{02}^- - 947730000\pi a_{11}^- + 42309375\pi a_{12}^- \\
&\quad + 6160245000\pi a_{20}^- + 126928125\pi a_{30}^- + 4738650000\pi a_{02}^+ - 947730000\pi a_{11}^+ \\
&\quad + 42309375\pi a_{12}^+ + 6160245000\pi a_{20}^+ + 126928125\pi a_{30}^+ - 5110972500\pi b_{02}^- \\
&\quad + 126928125\pi b_{03}^- + 710797500\pi b_{11}^- - 3215512500\pi b_{20}^- + 42309375\pi b_{21}^- \\
&\quad - 5110972500\pi b_{02}^+ + 126928125\pi b_{03}^+ + 710797500\pi b_{11}^+ - 3215512500\pi b_{20}^+ \\
&\quad + 42309375\pi b_{21}^+ + 1547032000a_{02}^- - 84000000a_{03}^- - 14072769976a_{11}^- \\
&\quad + 3736000a_{12}^- + 2604141664a_{20}^- - 37356000a_{21}^- + 41772000a_{30}^- \\
&\quad - 1547032000a_{02}^+ + 84000000a_{03}^+ + 14072769976a_{11}^+ - 3736000a_{12}^+ \\
&\quad - 2604141664a_{20}^+ + 37356000a_{21}^+ - 41772000a_{30}^+ - 27939203952b_{02}^-
\end{aligned}$$

$$+ 68208000b_{03}^- + 6280004832b_{11}^- + 4644000b_{12}^- - 14413096320b_{20}^-$$
$$+ 42424000b_{21}^- + 9390000b_{30}^- + 27939203952b_{02}^+ - 68208000b_{03}^+$$
$$- 6280004832b_{11}^+ - 4644000b_{12}^+ + 14413096320b_{20}^+ - 42424000b_{21}^+$$
$$- 9390000b_{30}^+).$$

□

PROPOSITION 4.14. *There exist polynomial piecewise perturbations of degree* $n = 5$ *as* (61) *such that* 59 *crossing limit cycles of small amplitude bifurcate from the origin of system*

$$\begin{cases} \dot{x} &= x^5 - 10x^3y^2 + 5xy^4 + x^4 - 6x^2y^2 + y^4 + x^3 - 3xy^2 + x^2 - y^2 - y, \\ \dot{y} &= 5x^4y - 10x^2y^3 + y^5 + 4x^3y - 4xy^3 + 3x^2y - y^3 + 2xy + x. \end{cases}$$

PROOF. The vector field in the statement is a holomorphic system of degree 5, but written in cartesian coordinates, hence it has a center at the origin. The proof follows from Corollary 4.6 computing $L_k^{(1)}$, for $k = 2, \ldots, 60$, and checking that rank $A_{60} = 58$. We notice that rank $A_k = k - 1$, for $k = 2, \ldots, 58$.. Then obtain 59 crossing limit cycles. Because of the size, we only show the linear developments of the first 3 Lyapunov constants.

$$L_2^{(1)} = -\frac{2}{3}(-(a_{11}^+ - a_{11}^-) - 2(b_{02}^+ - 2b_{02}^-) - (b_{20}^+ - b_{20}^-)),$$
$$L_3^{(1)} = \frac{1}{8}\pi((a_{12}^+ + a_{12}^-) + 3(a_{30}^+ + a_{30}^-) - (b_{02}^+ + b_{02}^-) + 3(b_{03}^+ + b_{03}^-)$$
$$- 4(b_{20}^+ + b_{20}^-) + (b_{21}^+ + b_{21}^-)),$$
$$L_4^{(1)} = \frac{2}{15}(3(b_{40}^+ - b_{40}^-) + 3b_{21}^- - 3b_{21}^+ + 4b_{20}^+ - 4b_{20}^- + 2b_{03}^- - 2b_{03}^+ - 7b_{11}^+ + 7b_{11}^-$$
$$- 3a_{30}^- + 3a_{30}^+ - 8b_{04}^- + 8b_{04}^+ - 8a_{20}^+ + 8a_{20}^- + 3a_{31}^+ - 3a_{31}^- + 2a_{13}^+ - 2a_{13}^-$$
$$- 6b_{02}^- + 6b_{02}^+ + 2a_{12}^+ - 2a_{12}^- + 2b_{22}^+ - 2b_{22}^- - a_{11}^+ + a_{11}^- - 12a_{02}^+ + 12a_{02}^-).$$

□

# Conclusions and Future Works

The parallelization tool has proved to be extremely effective. Due this tool, we were able to calculate Lyapunov constants of higher order for analytic systems of degree $n$, with $3 \leq n \leq 9$, and for piecewise systems of degrees 2, 3 and 4. We would like to emphasize the power of parallelization. For a quintic system with 36 perturbative parameters, without parallelization, our computer took about one month to calculate 33 Lyapunov constants of order two. Using the parallelization, the calculations took just over an hour. Once the constants are calculated, we developed new theorems that allow us to work with such equations to obtain more limit cycles. For analytic systems, with the theory developed in this thesis, we prove analytically that $M(3) \geq 12$, concluding that the conjecture, $M(n) \geq n^2 + 3n - 7$, given by Giné is false. We notice that the number of perturbative parameters, without the trace parameter, is $n^2 + 3n - 4$. For us only two parameters can be removed, one corresponding to a reescaling and another for a rotation. Then, we think that $M(n)$ will be $n^2 + 3n - 6$.

We have also proved that $M(4) \geq 21$, $M(5) \geq 33$, $M(6) \geq 44$, $M(7) \geq 61$, $M(8) \geq 76$ and $M(9) \geq 88$. Moreover, we show three new cubic systems exhibiting also 11 limit cycles. For piecewise systems, we show that $M_p^c(3) \geq 26$, $M_p^c(4) \geq 40$, $M_p^c(5) \geq 59$. Moreover, studying the problem of bifurcations of limit cycles, we have proved that any weak-focus of order $2n + 1$ of an analytic vector field unfolds $2n + 1$ limit cycles in the analytic piecewise class. We also extended the equivalence between linear Lyapunov constants and the first order averaging method for piecewise systems. How to relate the higher order developments in both methods for piecewise systems will be a subject for future work.

It is clear that in the study of local cyclicity, we have been able to make great progress as in the calculation of Lyapunov constants as in the treatment of them. However, we are also aware of the vast work still to be done on the subject matter. Some of the future work depends on the modernization of computers. On the other hand, there exists many results without solution. In these years we have studied the local cyclicity problem for more than one hundred of vector fields. But we could not answer which special properties have to have the polynomial systems exhibiting the highest number of linear terms of Lyapunov constants independents. We just can conclude, using the papers of Bondar and Sadovski, Giné, and Zoladek, that there exist centers with very high local cyclicity, see [**7, 32, 33, 65**]. Moreover, from the papers of Giné [**32, 33**] we can see the existence of systems having centers that the weak-foci that appear after perturbation have very high order. In fact, greater than the number of perturbative parameters. All these systems have homogeneous nonlinearities of degree $n$. From the conjectures in [**32, 33, 61**] it can be seen that the order of a weak-focus for systems with homogeneous nonlinearities is given by $n^2 - n$ when $n$ is odd and $2(n^2 - n)$ when $n$ is even. Therefore, it is better, instead of taking a system with a center with homogeneous nonlinearities of degree $n$ odd, to take an $n$ even one but multiplied by a line of equilibrium points.

In piecewise systems, to get better lower bounds for $M_p^c(n)$, it is necessary find a better algorithm to compute Lyapunov constants. We can also cite as future works, the simultaneous cyclicity studies for two or more centers. Because, there are no many works in this line because of the difficulties. Finally, the study of the relationship between Lyapunov constants for classical systems and Lyapunov constants for piecewise systems.

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