A NOTE ON A CONJECTURE OF POINCARÉ

GERARD GÓMEZ and JAUME LLIBRE

Secció de Matemàtiques, Facultat de Ciències, Universitat Autònoma de Barcelona, Spain

(Received 5 December, 1979; Accepted 7 April, 1980)

Abstract. We prove the following weakened version of Poincaré's conjecture on the density of periodic orbits of the restricted three-body problem: The measure of Lebesgue of the set of bounded orbits which are not contained in the closure of the set of periodic orbits goes to zero when the mass parameter does.

1. Introduction

Let m_1, m_2 be the masses of the primaries normalized in such a way that $m_1 = 1 - \mu$, $m_2 = \mu$, $\mu \in [0, 1]$. Units of length and time are chosen in order to have one unit of distance between the primaries and a mean motion equal to one.

For the position of the infinitesimal body m_3 we use both the sideral system of coordinates (X, Y) and the synodical one (x, y). In the last system the two primaries are fixed at $(\mu, 0)$ and $(\mu - 1, 0)$, respectively. The equations of motion are (see [8]):

$$\ddot{x} - 2\dot{y} = \frac{\partial \Omega}{\partial x}, \qquad \ddot{y} + 2\dot{x} = \frac{\partial \Omega}{\partial y}, \tag{1}$$

where

$$\Omega(x, y) = \tfrac{1}{2}(x^2 + y^2) + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} + \tfrac{1}{2}\mu(1 - \mu),$$

and

$$r_1^2 = (x - \mu)^2 + y^2, \qquad r_2^2 = (x + 1 - \mu)^2 + y^2.$$

System (1) has the Jacobian integral

$$C = 2\Omega(x, y) - (\dot{x}^2 + \dot{y}^2).$$

Poincaré (1892) conjectured (see [6] p. 82) the following: "If a particular solution of the restricted problem is given, one can always find a periodic solution (with a period which might be very long) such that the difference between these two solutions is as small as desired for any given length of time." Obviously the boundness of the particular solution must be assumed.

Schwarzschild's version of Poincaré's conjecture is (see [7]): "In an arbitrarily small neighborhood of any point in the phase space there is a point representing a periodic orbit (if the first point corresponds to a bounded orbit)".

The main result of this note is the following:

THEOREM. *For any fixed value of the Jacobian constant and for any $\varepsilon > 0$, there exists a $\mu_0 > 0$ such that if the mass parameter $\mu \in [0, \mu_0]$, then the set of bounded orbits which are not contained in the closure of the set of periodic orbits has measure of Lebesgue smaller than ε.*

Celestial Mechanics **24** (1981) 335–343. 0008–8714/0244–0335. $01.35.