A FAMILY OF PERIODIC ORBITS FOR THE EXTENDED HAMILTONIAN SYSTEM OF THE VAN DER POL OSCILLATOR

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Abstract. In this paper we show a new way of using the averaging theory for studying families of periodic orbits of a Hamiltonian system. We do this study computing a new family of periodic orbits of the extension of the Van der Pol oscillator to a Hamiltonian system of two degrees of freedom.

1. Introduction and statement of the main results

The classical Van der Pol oscillator is modeled by the second-order differential equation

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0, \tag{1}$$

where $x = x(t)$ is the position coordinate at time t, μ is a parameter, and the dot denotes derivative with respect to the time t, see [9, 10, 11].

Initially the differential equation (1) allowed to Van der Pol to explain the stable oscillations observed in electrical circuits employing vacuum tubes. Later on this differential equation has been used for explaining different phenomena in biology, physics, ... Thus in biology was utilised as a model for studying the action potentials of neurons, see for instance [3, 4, 7]. While in physics equation (1) was also used in phonation to model the right and left vocal fold oscillators (see [6]), and in seismology to model two plates in a geological fault (see [1]), ...

More recently in 2015, see equations (9) of [8], the Van der Pol oscillator was written in the Hamiltonian formalism by extending it to a four-dimensional autonomous differential as follows

$$\begin{aligned} &\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0, \\ &\ddot{y} + \mu(1 - x^2)\dot{y} + y = 0. \end{aligned} \tag{2}$$

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Observe that the motion of the Van der Pol oscillator (1) is not affected because the evolution of the variable $x(t)$ is independent of the variable $y(t)$.

Defining $p_y = \dot{x}$ and $p_x = \dot{y} + \mu(1-x^2)y$ from (2) we obtain that this differential system of two differential equations of second order admits the following Hamiltonian formalism

$$
\begin{aligned}
\dot{x} &= p_y,\\
\dot{y} &= p_x - \mu(1-x^2)y,\\
\dot{p_x} &= -y - \mu 2xyp_y,\\
\dot{p_y} &= -x + \mu(1-x^2)p_y,
\end{aligned} \tag{3}
$$

with the Hamiltonian

$$
H = p_x p_y + xy - \mu(1-x^2)yp_y. \tag{4}
$$

Of course p_x and p_y are the conjugate momenta of the variables x and y, respectively. In [2] it is shown that the Hamiltonian system (3) establishes a connection between the phase of the limit cycle of the Van der Pol system with the Hannay angle of the Hamiltonian system.

In this paper our objective is to study analytically the periodic orbits of the Hamiltonian system (3) using the averaging theory. In general the periodic orbits of the Hamiltonian systems are studied numerically because its analytical study is in general difficult and sometimes impossible. Here the technique used with the averaging theory can be extended to many other Hamiltonian systems.

Our main result is the following.

Theorem 1. *For μ sufficiently small the following statements hold for the Hamiltonian system* (3).

(a) *For each $h \in \mathbb{R}$ there exists a periodic orbit*

$$
\gamma_h(t) = \big(x(t;\mu,h), y(t;\mu,h), p_x(t;\mu,h), p_y(t;\mu,h)\big)
$$

such that when the parameter $\mu \to 0$ satisfies that $\gamma_h(t)$ tend to the periodic orbit

$$
\left(2\cos\left(\frac{\sqrt{4-\mu^2}}{2}t\right), \frac{|h|}{2}\cos\left(\varphi^* + \frac{\sqrt{4-\mu^2}}{2}t\right),\right.
$$
$$
\left. -\frac{|h|}{2}\sin\left(\varphi^* + \frac{\sqrt{4-\mu^2}}{2}t\right), -2\sin\left(\frac{\sqrt{4-\mu^2}}{2}t\right)\right)
$$

of the Hamiltonian system (3) *with $\mu = 0$.*

(b) *The periodic orbit $\gamma_h(t)$ live on the energy level $H = h$, and when h varies in $\mathbb{R}$ we get a family of periodic orbits.*

Theorem 1 is proved in the section 3.

In section 2 we present the algorithm that we follow for studying the families of periodic orbits of the Hamiltonian systems.

Since the differential system formed by the equations $\dot{x}$ and $\dot{p_y}$ is the first order differential system associated to the Van der Pol oscillator (1), and this differential system does not depend on the variables y and p_y, it follows immediately from Theorem 1 the following corollary.

Corollary 2. *For μ sufficiently small and for each $h \in \mathbb{R}$ the Van der Pol oscillator has a periodic orbit*

$$\gamma_h(t) = \big(x(t;\mu,h), p_y(t;\mu,h)\big)$$

such that when the parameter $\mu \to 0$ satisfies that $\gamma_h(t)$ tend to the periodic orbit

$$\left(2\cos\left(\frac{\sqrt{4-\mu^2}}{2}t\right), -2\sin\left(\frac{\sqrt{4-\mu^2}}{2}t\right)\right),$$

solution of the Van der Pol oscillator with $\mu = 0$.

Of course, for a fixed value of μ sufficiently small the periodic orbit $\gamma_h(t)$ of Corollary 2 is the classical limit cycle of the Van der Pol oscillator, because it is known that such a limit cycle is unique for a given value of μ.

2. The algorithm

The algorithm that we will follow for studying the families of periodic orbits of a Hamiltonian system has the following steps.

1. We select an equilibrium point of the Hamiltonian system having at least one pair of complex conjugate eigenvalues and translate it to the origin of coordinates.

2. We do a change of variables which writes the matrix of the linear part of the Hamiltonian system at the origin of coordinates in its real Jordan normal form. Then, in general the obtained differential system have lost its Hamiltonian structure.

3. The coordinates of the differential system whose linear part at the origin of coordinates produce complex eigenvalues are changed to polar coordinates. More precisely, if the real Jordan normal form of the linear part at the origin of coordinates has the block

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix},$$

associated to the coordinates x and y with α non identically zero. Then we do the change of variables $(x,y) \to (r,\theta)$, where $x = r\cos\theta$ and $y = r\sin\theta$. We shall take α as the small parameter μ necessary for applying the averaging theory, see the appendix.

But if there are more than one block associated to complex eigenvalues, for all the other blocks different from the first we do the following change of variables. Take one of these blocks, for instance

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

associated to the coordinates u and v and do the change of variables $(u,v) \to (R,\varphi)$, where $x = R\cos(\varphi + k\theta)$, $y = R\sin(\varphi + k\theta)$, and the constant k is chosen in such a way that the differential system in the new coordinates must satisfy $\dot{\varphi} = O(\alpha)$, with α sufficiently small. This step will become more clear when we apply the algorithm to the Hamiltonian system (3) in the next section.

4. Now we take the angle variable θ as the new independent variable.

5. Since generically the periodic orbits in the Hamiltonian systems appear in families of periodic orbits parametrized by h, being h the different values of the energy levels $H = h$ of the Hamiltonian H, and the averaging theory only detects isolated orbits, we must apply the averaging theory in the energy levels $H = h$, eliminating one variable using the relation $H = h$. If the Hamiltonian H is the unique independent first integral of the Hamiltonian system the differential system obtained on the energy level $H = h$ is into the normal form (14) of the appendix, ready for applying to it the averaging theory for studying its periodic orbits.

6. In case that the Hamiltonian system has additional independent first integrals F_k, also independent with the Hamiltonian H of the Hamiltonian system, we need to fix them, i.e. $F_k = f_k$ and work on these invariant subspaces in order that we can apply the averaging theory and to study the periodic orbits of the Hamiltonian system.

3. Proof of Theorem 1

Now we apply the algorithm described in section 2 to the Hamiltonian system (3) for proving Theorem 1. We will indicate all the steps of the algortihm using its numbers.

1. It is easy to verify that the Hamiltonian system (3) has a unique equilibrium point localized at the origin of coordinates, and that its four eigenvalues are $(\pm\mu \pm \sqrt{\mu^2-4})/2$.

2. Now we write the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -\mu & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & \mu \end{pmatrix}$$

of linear part of the Hamiltonian system (3) into its real Jordan normal form

$$\begin{pmatrix} \dfrac{\mu}{2} & -\dfrac{1}{2}\sqrt{4-\mu^2} & 0 & 0 \\ \dfrac{\sqrt{4-\mu^2}}{2} & \dfrac{\mu}{2} & 0 & 0 \\ 0 & 0 & -\dfrac{\mu}{2} & -\dfrac{1}{2}\sqrt{4-\mu^2} \\ 0 & 0 & \dfrac{\sqrt{4-\mu^2}}{2} & -\dfrac{\mu}{2} \end{pmatrix},$$

doing the change of variables $(x, y, p_x, p_y) \to (X, Y, Z, V)$, where

$$(x, y, p_x, p_y) = \left(X, Z, \frac{1}{2}\left(\mu Z - \sqrt{4-\mu^2}V\right), \frac{1}{2}\left(\mu X - \sqrt{4-\mu^2}Y\right)\right).$$

Then the Hamiltonian system (3) becomes

$$\begin{aligned} \dot{X} &= \frac{\mu}{2}X - \frac{\sqrt{4-\mu^2}}{2}Y, \\ \dot{Y} &= \frac{\sqrt{4-\mu^2}}{2}X + \frac{\mu}{2}Y + \frac{\mu^2}{\sqrt{4-\mu^2}}X^3 - \mu X^2 Y, \\ \dot{Z} &= -\frac{\sqrt{4-\mu^2}}{2}V - \frac{\mu}{2}Z + \mu X^2 Z, \\ \dot{V} &= -\frac{\mu}{2}V + \frac{\sqrt{4-\mu^2}}{2}Z + \frac{3\mu^2}{\sqrt{4-\mu^2}}X^2 Z - 2\mu XYZ, \end{aligned} \tag{5}$$

and the Hamiltonian (4) becomes the first integral

$$
\begin{aligned}
H = \ & \frac{1}{4}\left(\mu Z - \sqrt{4-\mu^2}V\right)\left(\mu X - \sqrt{4-\mu^2}Y\right) - \\
& \frac{1}{2}\mu\left(1-X^2\right)Z\left(\mu X - \sqrt{4-\mu^2}Y\right) + XZ.
\end{aligned} \tag{6}
$$

of the differential system (5).

3. We do the change of variables $(X,Y,Z,V) \to (r,\theta,R,\varphi)$ where

$$
(X,Y,Z,V) = \big(r\cos\theta, r\sin\theta, R\cos(\varphi+k\theta), R\sin(\varphi+k\theta)\big),
$$

and the differential system (5) writes in the new variables as

$$
\begin{aligned}
\dot{r} = \ & \mu\left(\frac{1}{2}r + \sin\theta\cos^2\theta\left(\frac{\mu\cos\theta}{\sqrt{4-\mu^2}} - \sin\theta\right)r^3\right), \\
\dot{\theta} = \ & \frac{\sqrt{4-\mu^2}}{2} + \mu\cos^3\theta\left(\frac{\mu\cos\theta}{\sqrt{4-\mu^2}} - \sin\theta\right)r^2, \\
\dot{R} = \ & \mu\left(-\frac{1}{2}R + \frac{1}{2}\cos\theta\cos(\varphi+\theta)\left(\frac{6\mu\cos\theta\sin(\varphi+\theta)}{\sqrt{4-\mu^2}} + \right.\right. \\
& \left.\left. 3\cos(\varphi+2\theta) - \cos\varphi\right)r^2R\right), \\
\dot{\varphi} = \ & \mu\cos\theta\left(\cos\theta\cos(\varphi+\theta)\left(\frac{3\mu\cos(\varphi+\theta)}{\sqrt{4-\mu^2}} - \sin(\varphi+\theta)\right)\right. \\
& \left. -\frac{\mu\cos^3\theta}{\sqrt{4-\mu^2}} - 2\sin\theta\cos^2(\varphi+\theta) + \sin\theta\cos^2\theta\right)r^2,
\end{aligned} \tag{7}
$$

where we have take $k=1$ in order that $\dot{\varphi} = O(\mu)$. The first integral H given in (6) in the variables (r,θ,R,φ) writes

$$
\begin{aligned}
H = \ & rR\cos\theta\cos(\varphi+\theta) - \frac{1}{2}\mu R\cos(\varphi+\theta)\left(1 - r^2\cos^2\theta\right) \\
& \left(\mu r\cos\theta - \sqrt{4-\mu^2}r\sin\theta\right) + \frac{\mu r\cos\theta - \sqrt{4-\mu^2}r\sin\theta}{4} \\
& \left(\mu R\cos(\varphi+\theta) - \sqrt{4-\mu^2}R\sin(\varphi+\theta)\right).
\end{aligned} \tag{8}
$$

4. We take now the variable θ as the new independent variable and the differential system (7) becomes

(9)

$$
\begin{aligned}
r' &= \frac{\mu}{2}\left(r - 2r^3 \sin^2\theta \cos^2\theta\right) + O(\mu^2),\\
R' &= -\frac{\mu}{2}\left(1 + r^2 \cos\theta \cos(\varphi+\theta)(\cos\varphi - 3\cos(\varphi+2\theta))\right) R + O(\mu^2),\\
\varphi' &= \frac{\mu r^2}{4} \cos\theta(\sin(2\varphi+\theta) - 3\sin(2\varphi+3\theta) - 3\sin\theta + \sin(3\theta)) + O(\mu^2),
\end{aligned}
$$

where the prime denotes derivative with respect to the variable θ.

5. From (8) and $H = h$ we obtain that

$$
\varphi = \arccos\left(\frac{h}{rR}\right) + O(\mu). \tag{10}
$$

Substituting this φ given in (10) we obtain the following expression for the differential system (9) restricted to the energy level $H = h$

(11)

$$
\begin{aligned}
r' =& \ \mu\left(\frac{r}{2} - r^3 \sin^2\theta \cos^2\theta\right) + O(\mu^2)\\
=& \ \mu F_{11}(\theta, r, R) + O(\mu^2),\\
R' =& \ \frac{\mu}{2}\left(-hr\cos\theta\cos\left(\arccos\left(\frac{h}{rR}\right)+\theta\right) - R\right) +\\
& 3r^2 R\cos\theta\cos\left(\arccos\left(\frac{h}{rR}\right)+\theta\right)\cos\left(\arccos\left(\frac{h}{rR}\right)+2\theta\right)\\
& +O(\mu^2)\\
=& \ \mu F_{12}(\theta, r, R) + O(\mu^2).
\end{aligned}
$$

The differential system (11) is already written in the normal form (14) of the appendix for applying the averaging theory. Here using the notation of the appendix we have that $F = (F_{11}, F_{12})$, $x = (r, R)$ and $T = 2\pi$ then we compute the averaged function $f(x) = (f_{11}(r, R), f_{12}(r, R))$ defined in (16) of the appendix, and we get

$$
\begin{aligned}
f_{11}(r, R) &= \frac{1}{2}\left(r - \frac{r^3}{4}\right),\\
f_{12}(r, R) &= \frac{1}{8}\left(3r^2 - 4\right) R - \frac{h^2}{4R}.
\end{aligned} \tag{12}
$$

The real solutions of the system $f_{11}(r, R) = 0$ and $f_{12}(r, R) = 0$ with r and R positive are $r = 2$ and $R = |h|/2$. Moreover, on this solution the Jacobian (17) for our differential system (12) is

(13)
$$\det\left(\begin{array}{cc} \dfrac{1}{8}\left(4-3r^2\right) & 0 \\ \dfrac{3rR}{4} & \dfrac{1}{8}\left(\dfrac{2h^2}{R^2}+3r^2-4\right) \end{array}\right)\Bigg|_{(r,R)=(2,|h|/2)} = -8\pi^2 \neq 0.$$

From Theorem 3 of the appendix we get for μ sufficiently small that for each $h \in \mathbb{R}$ we have the periodic solution

$$\gamma_h^1(\theta;\mu,h) = \big(r(\theta;\mu,h), R(\theta;\mu,h)\big)$$

of the differential system (12) such that $\gamma_h^1(\theta;\mu,h) \to (2,|h|/2)$, when $\mu \to 0$.

The periodic orbit $\gamma_h^1(\theta;\mu,h)$ provides the following periodic orbit of the differential system (9)

$$\gamma_h^2(\theta;\mu,h) = \big(r(\theta;\mu,h), R(\theta;\mu,h), \varphi(\theta;\mu,h)\big),$$

such that $\gamma_h^2(\theta;\mu,h) \to (2,|h|/2,\varphi^*)$, when $\mu \to 0$, where $\varphi^* = 0$ if $h > 0$ and $\varphi^* = \pi$ if $h < 0$.

The periodic orbit $\gamma_h^2(\theta;\mu,h)$ provides the following periodic orbit of the differential system (7)

$$\gamma_h^3(t;\mu,h) = \big(r(t;\mu,h), \theta(t;\mu,h), R(t;\mu,h), \varphi(t;\mu,h)\big),$$

such that $\gamma_h^3(t;\mu,h) \to \left(2, \sqrt{4-\mu^2}\,t/2, |h|/2, \varphi^*\right)$, when $\mu \to 0$.

The periodic orbit $\gamma_h^3(t;\mu,h)$ provides the following periodic orbit of the differential system (5)

$$\gamma_h^4(t;\mu,h) = \big(X(t;\mu,h), Y(t;\mu,h), Z(t;\mu,h), V(t;\mu,h)\big),$$

such that

$$\begin{aligned}\gamma_h^4(t;\mu,h) \to \quad & \left(2\cos\left(\frac{\sqrt{4-\mu^2}}{2}t\right), 2\sin\left(\frac{\sqrt{4-\mu^2}}{2}t\right),\right. \\ & \left.\frac{|h|}{2}\cos\left(\varphi^* + \frac{\sqrt{4-\mu^2}}{2}t\right), \frac{|h|}{2}\sin\left(\varphi^* + \frac{\sqrt{4-\mu^2}}{2}t\right)\right),\end{aligned}$$

when $\mu \to 0$.

The periodic orbit $\gamma_h^4(t;\mu,h)$ provides the following periodic orbit of the differential system (3)

$$\gamma_h(t;\mu,h) = \big(x(t;\mu,h), y(t;\mu,h), p_x(t;\mu,h), p_y(t;\mu,h)\big),$$

such that

$$\gamma_h(t;\mu,h) \to \left(2\cos\left(\frac{\sqrt{4-\mu^2}}{2}t\right), \frac{|h|}{2}\cos\left(\varphi^* + \frac{\sqrt{4-\mu^2}}{2}t\right),\right.$$
$$\left.-\frac{|h|}{2}\sin\left(\varphi^* + \frac{\sqrt{4-\mu^2}}{2}t\right), -2\sin\left(\frac{\sqrt{4-\mu^2}}{2}t\right)\right),$$

when $\mu \to 0$. This completes the proof of Theorem 1.

Appendix

Here we summarize the basic results from averaging theory for computing periodic orbits that we shall use for proving Theorem 1, for a proof see Theorem 11.5 of [12], or Theorem 1.2.1 of [5].

First we consider the initial value problem

$$\dot{x} = \mu F(\theta, x) + \mu^2 R(\theta, x, \mu), \ x(0) = x_0, \tag{14}$$

with $x \in U$ where U is an open subset of $\mathbb{R}^n$, and $\theta \geq 0$. We assume that $F(\theta, \mathrm{x})$ and $R(\theta, x, \mu)$ are periodic functions in θ of period T. Of course the dot in (14) denotes derivative with respect to the variable θ.

Second we consider in the open set U the following initial value problem for the averaged differential system

$$\dot{y} = \mu f(y), \ y(0) = x_0, \tag{15}$$

where $f(x)$ is the averaged function of the function $F(\theta, x)$, i.e.

$$f(x) = \frac{1}{T}\int_0^T F(\theta, x)d\theta. \tag{16}$$

Theorem 3. *For the two initial value problems* (14) *and* (15) *we assume:*

- (i) *the function F is C^2 and bounded by a constant which does not depend on μ in $[0, \infty) \times U$ for all $\mu \in (0, \mu_0]$.*
- (ii) *the functions F and R are periodic in θ of period T, and T is independent of μ.*
- (iii) *the solution $y(\theta)$ of* (15) *belongs to U for all $\theta \in [0, 1/\mu]$.*

Then for each equilibrium point p of the averaged system (15) *satisfying*

$$\det\left(\frac{\partial f}{\partial y}\right)\Bigg|_{y=p} \neq 0, \tag{17}$$

there is a periodic solution $x(\theta, \mu)$ *of period* T *for the system* (14) *verifying* $x(\theta, \mu) \to p$ *as* $\mu \to 0$.

Declaration of Interest

The results of this paper are new and original. They do not have any conflict of interest.

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