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Strongly formal Weierstrass non-integrability for polynomial differential systems in $\mathbb{C}^2$

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Abstract. Recently a criterion has been given for determining the weakly formal Weierstrass non-integrability of polynomial differential systems in $\mathbb{C}^2$. Here we extend this criterion for determining the strongly formal Weierstrass non-integrability which includes the weakly formal Weierstrass non-integrability of polynomial differential systems in $\mathbb{C}^2$. The criterion is based on the solutions of the form $y = f(x)$ with $f(x) \in \mathbb{C}[[x]]$ of the differential system whose integrability we are studying. The results are applied to a differential system that contains the famous force-free Duffing and the Duffing–Van der Pol oscillators.

Keywords: Liouville integrability, Weierstrass integrability, polynomial differential systems.

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1 Introduction and statement of the main result

One of the main problems in the qualitative theory of differential systems is the integrability problem. For differential systems in $\mathbb{C}^2$ this problem consists in to determine if the system has or not an explicit first integral. When this first integral can be expressed as quadratures of elementary functions we have the so-called Liouville integrability, which is the most studied, see for instance [16, 30, 31] and references therein. The Liouville integrability is based on the cofactors of the invariant algebraic curves and the exponential factors (see definitions below). Some generalizations of the Liouville integrability theory defining the generalized cofactors have been obtained, see [7, 8, 10, 11, 19, 20, 30, 31].

Some differential systems have an explicit first integral that cannot be expressed as quadratures of elementary functions. Hence these systems are not Liouville integrable. Sometimes these first integrals can be expressed in terms of special functions, as for instance functions that are solutions of second order linear differential equations (in [11, 19, 29] several examples are given). To determine when a differential system is not Liouville integrable is an open problem, see [25]. A partial answer to this question has been recently given in [23].

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