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Flow curvature manifold and energy of generalized Liénard systems

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ABSTRACT

In his famous book entitled *Theory of Oscillations*, Nicolas Minorsky wrote: "*each time the system absorbs energy the curvature of its trajectory decreases* and *vice versa*". By using the *Flow Curvature Method*, we establish that, in the ε-vicinity of the *slow invariant manifold* of generalized Liénard systems, the *curvature of trajectory curve* increases while the *energy* of such systems decreases. Hence, we prove Minorsky's statement for the generalized Liénard systems. These results are then illustrated with the classical Van der Pol and generalized Liénard *singularly perturbed systems*.

1. Introduction

At the end of the 1930s, a general equation of *self-sustained oscillations* (1) was stated by the French engineer Alfred Liénard [25]. It encompassed the prototypical equation of the Dutch physicist Balthasar Van der Pol [32] modeling the so-called *relaxation oscillations*[2]

$$\frac{d^2x}{dt^2} + \omega f(x)\frac{dx}{dt} + \omega^2 x = 0. \tag{1}$$

Less than fifteen years later, a more general form was provided by the American mathematicians Norman Levinson and his former student Oliver K. Smith [23]:

$$\frac{d^2x}{dt^2} + \mu f(x)\frac{dx}{dt} + g(x) = 0. \tag{2}$$

At that time, the classical geometric theory of differential equations developed originally by Andronov [1], Tikhonov [31] and Levinson [24] stated that such *singularly perturbed systems* possess *invariant manifolds* on which trajectories evolve slowly, and towards which nearby orbits contract exponentially in time (either forward or backward) in the normal directions. So, these manifolds have been called asymptotically stable (or unstable) *slow invariant manifolds*. Their *local invariance* has then been stated according to Fenichel [4,5,6,7] theory[3] for the *persistence of normally hyperbolic invariant manifolds*.

During the last century, various methods have been developed to approximate the *slow invariant manifold* equation in the form of an asymptotic expansion in power of ε. The seminal works of Wasow [33], Cole [3], O'Malley [27,28] and Fenichel [4,5,6,7] to name but a few, gave rise to the so-called *Geometric Singular Perturbation Method*. Fifteen years ago, a new approach of n-dimensional singularly perturbed dynamical systems of ordinary differential equations with two time scales, called *Flow Curvature Method* has been developed [10]. This method gives an implicit non intrinsic equation, because it depends on the euclidean metric. A 'kinetic energy metric' has been introduced in [19] for chemical kinetic systems and an extremum principle for computing *slow invariant manifolds* has been formulated [20,21] which can be viewed as minimum curvature geodesics. In [14] a curvature-based differential geometry formulation for the *slow manifold* problem has been used for the purpose of a coordinate-independent formulation of the invariance equation. In his famous book entitled *Theory of Oscillations*, the Russian mathematician Nicolas Minorsky [29] wrote:

> "*each time the system absorbs energy the curvature of its trajectory decreases* and *vice versa* when the energy is supplied by the system (e. g. braking) the curvature increases."

Thus, according to Minorsky, *energy* and *curvature of the trajectory* are linked by a relationship that he unfortunately didn't give. So, the aim of this work is to prove this statement in the ε-vicinity of the *slow*

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[2] For more details see J.-M. Ginoux [12].

[3] The theory of invariant manifolds for an ordinary differential equation is based on the work of Hirsch et al. [15].

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