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Lower bounds for the local cyclicity for families of centers

Jaume Giné [a,*], Luiz F.S. Gouveia [b,c], Joan Torregrosa [b,d]

[a] Departament de Matemàtica, Universitat de Lleida, Avda. Jaume II, 69; 25001 Lleida, Catalonia, Spain
[b] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Cerdanyola del Vallès, Barcelona, Catalonia, Spain
[c] Departamento de Matemática, Universidade Estadual Paulista, 15054-000 São José do Rio Preto, Brazil
[d] Centre de Recerca Matemàtica, Campus de Bellaterra, 08193 Cerdanyola del Vallès, Barcelona, Catalonia, Spain

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Abstract

In this paper, we are interested in how the local cyclicity of a family of centers depends on the parameters. This fact was pointed out in [21], to prove that there exists a family of cubic centers, labeled by CD_{31}^{12} in [25], with more local cyclicity than expected. In this family, there is a special center such that at least twelve limit cycles of small amplitude bifurcate from the origin when we perturb it in the cubic polynomial general class. The original proof has some crucial missing points in the arguments that we correct here. We take advantage of a better understanding of the bifurcation phenomenon in nongeneric cases to show two new cubic systems exhibiting 11 limit cycles and another exhibiting 12. Finally, using the same techniques, we study the local cyclicity of holomorphic quartic centers, proving that 21 limit cycles of small amplitude bifurcate from the origin, when we perturb in the class of quartic polynomial vector fields.

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* Corresponding author.
E-mail addresses: jaume.gine@udl.cat (J. Giné), fernandosg@mat.uab.cat, fernando.gouveia@unesp.br (L.F.S. Gouveia), torre@mat.uab.cat (J. Torregrosa).

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1. Introduction

The study of limit cycles began at the end of the 19th century with Poincaré. Years later, in 1900, Hilbert presents a list of unsolved problems. From the original 23 problems of this list, the 16th is still open. The second part of this problem consists in determining a uniform bound of the maximal number of limit cycles (named $\mathcal{H}(n)$), and their relative positions, of a planar polynomial system of degree n. However, there are also weak versions of 16th Hilbert's problem. Arnol'd in [1] proposed a version focused on studying the number of limit cycles bifurcating from the period annulus of Hamiltonian systems. In this paper, we are interested in providing the maximal number $M(n)$ of small amplitude limit cycles bifurcating from an elementary center or an elementary focus, in special for degrees 3 and 4. The main idea is to study the local cyclicity of families of centers depending on a finite number of parameters.

As it is well known, for $n = 2$, Bautin proved in [2] that $M(2) = 3$. The case $n = 3$ but without quadratic terms (homogeneous cubic perturbation) was studied in [3,19] and solved in [23], then $M_h(3) = 5$. In [24,26] Żołądek shown that $M(3) \geq 11$. Christopher, in [5], gave a simple proof of Żołądek's result perturbing another cubic center with a rational first integral, using only the linear parts of the Lyapunov constants. The interest of this result is that we can compute these linear parts [5], in a parallelized way [13,17], near a center without having the complete expressions of the Lyapunov constants. Basically the used technique consists in to choose a point on the center variety and at this point consider the linear term of the Lyapunov constants, if the point is chosen on a component of the center variety of codimension k, then the first k linear terms of the Lyapunov constants are independent. This is a direct application of the Implicit Function Theorem to prove that $M(n) \geq k$. Usually, we use this technique to provide lower bounds for the local cyclicity problem in the class of polynomial vector fields of degree n. In [10,11], Giné presents a conjecture that the number $M(n)$ is bounded below by $n^2 + 3n - 7$ and studies the cyclicity of different families of centers presented in [9]. In [14,13] new lower bounds for $M(n)$ and n small have been obtained. The new values are $M(4) \geq 20$, $M(5) \geq 33$, $M(6) \geq 44$, $M(7) \geq 61$, $M(8) \geq 76$, and $M(9) \geq 88$.

In [21], Yu and Tian point out that the 1-parameter family of centers labeled by CD_{31}^{12} in [25] is quite special because it can exhibit one more limit cycle than expected in Giné's conjecture. This family has the next rational first integral

$$H(x, y) = \frac{(xy^2 + x + 1)^5}{x^3(xy^5 + 5xy^3/2 + 5y^3/2 + 15xy/8 + 15y/4 + a)^2} \tag{1}$$

and it has, following Żołądek computations, codimension 12. The original proof has some crucial missing points in the arguments that we correct here, proving effectively that there exist some special values of the parameter a in (1) such that 12 limit cycles of small amplitude bifurcate from the origin when we perturb in the class of complete cubic polynomial vector fields. This family was also studied by Christopher in [5] and it was the first clear proof about the existence of at least 11 limit cycles of small amplitude bifurcating from an equilibrium in polynomial vector fields of degree three.

The main result of this paper is the following.

Theorem 1.1. *The number of limit cycles of small amplitude bifurcating from an equilibrium of monodromic type in the classes of polynomial vector fields of degrees* 3 *and* 4 *is* $M(3) \geq 12$ *and* $M(4) \geq 21$*, respectively.*

After the above result, clearly, the commented general lower bound for $M(n)$ should be updated to be $n^2 + 3n - 6$. We remark that the total number of parameters for polynomial vector fields of degree n is $n^2 + 3n + 2$. Then, the new conjecture removes 8 to this total number of parameters. Six corresponding to an affine change of variables that writes the linear part in its normal form, one corresponding to a rotation and another to a rescaling. The previous conjecture took into account that the number of limit cycles in a center component does not change. But this is only generically. In this work, we will provide examples where this property fails. Hence we establish the following conjecture.

Conjecture 1.2. *The number of limit cycles of small amplitude bifurcating from an equilibrium of monodromic type in the class of polynomial vector fields of degree n is* $M(n) = n^2 + 3n - 6$.

The proof of the above theorem is based on an extension of Christopher results ([5]) for linear and higher-order studies when the considered center has parameters. This new result, Theorem 3.1, is proved in Section 3. For completeness we also include here the Christopher results, see Theorems 2.2 and 2.4 in the next section, where we have also added a detailed description of how they should be used together. We remark that the parallelization algorithm introduced in [13,17] is crucial to get the results because facilitates all the needed computations. In Section 4 we do the proof of the statement of Theorem 1.1 corresponding to degree 3 vector fields. Moreover, we study also the bifurcation diagrams of limit cycles of small amplitude bifurcating from three families of centers. The first is 1-parametric and it is the rational reversible center family labeled by CR_{12}^{17} in [25]. The second is a 2-parameter holomorphic cubic center family. The third is labeled by $CD_{10}^{(11)}$ in [25], it has also 2 parameters and no maximal codimension 12 but, from it, also 12 limit cycles bifurcate from the center inside the cubic polynomial class. In Section 5 we study the bifurcation diagram for a 2-parameter center family of degree 4 that allows us to prove the statement of Theorem 1.1 corresponding to vector fields of degree 4. Finally, we also study partially the bifurcation diagram for a 4-parameter quartic holomorphic family of centers.

We have used a cluster of computers with 128 processors simultaneously with 725 GB of total ram memory. All the computations have been made with `Maple` [18].

2. Lyapunov constants and parallelization

Let us consider the system

$$\begin{cases} \dot{x} &= -y + P_n(x, y), \\ \dot{y} &= x + Q_n(x, y), \end{cases}$$

with P_n and Q_n polynomials of degree n in variables x, y. Writing the system in complex coordinates, we have

$$R(z, \bar{z}) = \mathrm{i}\, z + R_n(z, \bar{z}), \tag{2}$$

where $R_n(z, \bar{z})$ are polynomials of degree n in variables $(z, \bar{z})$. We seek for a first integral in the form $H(z, \bar{z})$ in a neighborhood of the origin such as

$$\mathcal{X}(H) = \sum_{k=0}^{\infty} v_k (z\bar{z})^{k+1},$$

where $\mathcal{X}$ is the vector field associated to (2) and v_k the *k-Lyapunov constant*. Clearly, the origin will be a center if, and only if $v_n(2\pi) = 0$ for all n. If $v_n(2\pi) \neq 0$, for some n, so we have a focus of order n.

In most cases, the process of calculating Lyapunov constants is very hard, being impossible to calculate them manually. Therefore, the use of an algebraic manipulator system is necessary. Moreover, the Parallelization process will be so important in our work, as it offers us a reduction in the time of very large computations. The first result about Parallelization is given by Liang and Torregrosa in [17] and using it we can obtain the linear part of the Lyapunov constants of the more easy way.

Theorem 2.1 *([17]). Let $p(z, \bar{z})$ and $Q_j(z, \bar{z})$, $j = 1, \ldots, s$ be polynomials with monomials of degree higher or equal than two such that the origin of $\dot{z} = \mathrm{i}\, z + p(z, \bar{z})$ is a center. If $L_{k,j}^{(1)}$ denotes the linear part, with respect to $\lambda_j \in \mathbb{R}$, of the k-Lyapunov constant of equation*

$$\dot{z} = \mathrm{i}\, z + p(z, \bar{z}) + \lambda_j Q_j(z, \bar{z}), \quad j = 1, \ldots, s,$$

then the linear part of the k-Lyapunov constants, with respect to $\Lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N$, of equation

$$\dot{z} = \mathrm{i}\, z + p(z, \bar{z}) + \sum_{j=1}^{N} \lambda_j Q_j(z, \bar{z}),$$

is $L_k^{(1)} = \sum_{j=1}^{N} L_{k,j}^{(1)}$.

The next theorem due to Christopher, details how we can use the first-order Taylor approximation of the Lyapunov constants to obtain lower bounds on the number of limit cycles near elementary center equilibrium points.

Theorem 2.2 *([5]). Suppose that s is a point on the center variety and that the first k Lyapunov constants, $L_1, \ldots, L_k$, have independent linear parts (with respect to the expansion of L_i about s), then s lies on a component of the center variety of codimension at least k and there are bifurcations which produce k limit cycles locally from the center corresponding to the parameter value s. If, furthermore, we know that s lies on a component of the center variety of codimension k, then s is smooth point of the variety, and the cyclicity of the center for the parameter value s is exactly k. In the latter case, k is also the cyclicity of a generic point on this component of the center variety.*

We notice that to perform higher-order parallelization, that is, to calculate high-order of Lyapunov constants we need to decompose the global problem in simpler problems having exactly ℓ parameters or monomials. However, as many parameters appear in some of the simple perturbation problems we need to correct the obtained coefficients of the developments of order ℓ.

Theorem 2.3 *([13]). Let $p(z,\bar z)$ and $Q_j(z,\bar z)$, $j=1,\dots,N$ be polynomials with monomials of degree higher or equal than two such that the origin of $\dot z = \mathrm{i}\, z + p(z,\bar z)$ is a center. For $\ell \le N$, we denote by $L_k^{(\ell)}$ the Taylor approximation of k-Lyapunov constant up to degree ℓ of equation*

$$\dot z = \mathrm{i}\, z + p(z,\bar z) + \sum_{j=1}^{N} \lambda_j Q_j(z,\bar z), \tag{3}$$

with $\Lambda = (\lambda_1,\dots,\lambda_N) \in \mathbb{R}^N$. Let S be the set of all combinations of the components of Λ taken ℓ at a time. That is, $S = \{(\lambda_1,\dots,\lambda_\ell),(\lambda_2,\dots,\lambda_{\ell+1}),\dots,(\lambda_{\ell-N},\dots,\lambda_N)\}$ and having $\binom{N}{\ell}$ elements. For each element $S_j \in S$, we denote by $\sigma(j,\zeta)$ the subscript of the parameters in S_j at position ζ, i.e. $S_j = (\lambda_{\sigma(j,1)},\dots,\lambda_{\sigma(j,\ell)})$, and we denote by $L_{k,j}^{(\ell)}$ the Taylor approximation up to degree ℓ with respect to Λ of the k-Lyapunov constant of equation

$$\dot z = \mathrm{i}\, z + p(z,\bar z) + \sum_{l=1}^{\ell} \lambda_{\sigma(j,l)} Q_{\sigma(j,l)}(z,\bar z). \tag{4}$$

Then

$$L_k^{(\ell)} = \sum_{l=1}^{N} \hat L_{k,j}^{(\ell)},$$

where $\hat L_{k,j}^{(\ell)} = \sum_p \dfrac{\mu_{k,j,p}}{\binom{N-s(p)}{\ell-s(p)}} \Lambda_j^p$, for $\Lambda_j^p = \lambda_{\sigma(j,1)}^{p_1}\lambda_{\sigma(j,2)}^{p_2}\cdots\lambda_{\sigma(j,\ell)}^{p_\ell}$ and $p=(p_1,\dots,p_\ell)$ writing $L_{k,j}^{(\ell)} = \sum_p \mu_{k,j,p}\Lambda_j^p$ with $s(p) = \sum_{l=1}^{\ell} \operatorname{sgn}(p_l)$ where $\operatorname{sgn}(x) = \begin{cases} 1, & \text{if } x>0,\\ 0, & \text{if } x=0.\end{cases}$

The next result, also due to Christopher in [5], is an extension of Theorem 2.2 that shows how sometimes we can obtain more limit cycles using high-order Taylor developments of the Lyapunov constants.

Theorem 2.4 *([5]). Suppose that we are in a point s where Theorem 2.2 applies. After a change of variables if necessary, we can assume that $L_0 = L_1 = \cdots = L_k = 0$ and the next Lyapunov constants $L_i = h_i(u) + O_{m+1}(u)$, for $i = k+1,\dots,k+l$, where h_i are homogeneous polynomials of degree $m \ge 2$ and $u = (u_{k+1},\dots,u_{k+l})$. If there exists a line $\boldsymbol{\ell}$, in the parameter space, such that $h_i(\boldsymbol{\ell}) = 0$, $i = k+1,\dots,k+l-1$, the hypersurfaces $h_i = 0$ intersect transversally along ℓ for $i = k+1,\dots,k+l-1$, and $h_{k+l}(\boldsymbol{\ell}) \neq 0$, then there are perturbations of the center which can produce $k+l$ limit cycles.*

We remark that from the proof of the above theorem it is clear that there exists a perturbation that produces the total number of limit cycles.

As we will see in the proofs of the results in the next sections, sometimes the application of Theorem 2.4 is not so simple. Because it depends on finding explicitly the transversal intersection line $\boldsymbol{\ell}$. Although a new proof of this result, using blow-up, has been given in [13], here we also reproduce the main idea to clarify the difficulties on the application. We start computing the Taylor approximation up to degree m of the $k+l$ Lyapunov constants and writing them in the form that Theorem 2.2 applies, i.e. $L_i = u_i + O_2(u)$, for $i = 1, \ldots, k$. The next step is the simplification of the next l Lyapunov constants assuming that the principal part of them is defined by the homogeneous polynomials h_i, writing $L_i = h_i(u) + O_{m+1}(u)$, for $i = k+1, \ldots, k+l$. The Implicit Function Theorem allows us to write $L_i = v_i$, for $i = 1, \ldots, k$, and we can restrict the analysis to $v_i = 0$, for $i = 1, \ldots, k$. It is important that, with this assumption, the polynomials h_i remain unchanged. So, after reordering parameters if necessary, we can use a blow-up change $u_{k+j} = \xi\zeta_j$, for $j = 1, \ldots, l-1$, and $u_{k+l} = \xi$. Then $h_{k+j}(u) = \xi^m g_j(\zeta)$ with $\zeta \in \mathbb{R}^{l-1}$. The transversal line $\boldsymbol{\ell}$ in the statement comes from the existence of a transversal intersection point ζ^* of the polynomials g_j of degree m. So Theorem 2.4 applies, taking $\boldsymbol{\ell}$ as the straight line $u = \xi\zeta^*$, if $g_j(\zeta^*) = 0$ for $j = 1, \ldots, l-1$, $g_l(\zeta^*) \neq 0$ and the determinant of the Jacobian matrix of $(g_1, \ldots, g_{l-1})$ with respect to ζ is nonzero at ζ^*.

It is very important to remark that the number of components in the parameters $u \in \mathbb{R}^{k+l}$ is exactly the same as the total number of used Lyapunov constants. Moreover, in the above proof all polynomials h_i have the same degree. The extension analyzed in [13] removes this special condition. Although the main difficulties are the computation of the higher-order developments and the finding of solutions of polynomial systems of high degree with many variables, we do not to forget that, a priori, the order up to we need to compute is another unknown. Finally, when this intersection point is obtained numerically, we may use a computer assisted proof to prove analytically the existence of such point. This is done using Poincaré–Miranda's Theorem together with the results of the last section. For the transversality property, we can use the Gershgorin Circles Theorem. For completeness, we add them here.

Theorem 2.5 *([16], Poincaré–Miranda). Let c be a positive real number and $S = [-c, c]^n$ a n-dimensional cube. Consider $f = (f_1, \ldots, f_n) : S \to \mathbb{R}^n$ a continuous function such $f_i(S_i^-) < 0$ and $f_i(S_i^+) > 0$ for each $i \le n$, where $S_i^{\pm} = \{(x_1, \ldots, x_n) \in S : x_i = \pm c\}$. So, there exists a point $d \in S$ such that $f(d) = 0$.*

Theorem 2.6 *([8], Gershgorin). Let $A = (a_{i,j}) \in \mathbb{C}^{n\times n}$ and α_k its eigenvalues. Consider for each $i = 1, \ldots, n$*

$$D_i = \{z \in \mathbb{C} : |z - a_{i,i}| \le r_i,$$

where $r_i = \sum_{i\neq j} |a_{i,j}|$. So, for all k, each $\alpha_k \in D_i$ for some i.

The Poincaré–Miranda's Theorem was conjectured by Poincaré in the 19th century and proved by Miranda in the last century. Note that this result is a generalization of the Bolzano's Theorem for higher dimensions. The reader can get more details on Gershgorin Circles Theorem in [12].

3. Local cyclicity depending on parameters

This section is devoted to extend Theorems 2.2 and 2.4 to families of centers that depend on some parameters. Let $(\dot{x}, \dot{y}) = (P_c(x, y, \mu), Q_c(x, y, \mu))$ be a family of polynomial centers of degree n depending on a parameter $\mu \in \mathbb{R}^{\ell}$, having a center equilibrium point at the origin. We consider the perturbed polynomial system

$$\begin{cases} \dot{x} &= P_c(x,y,\mu)+\alpha y+P(x,y,\lambda), \\ \dot{y} &= Q_c(x,y,\mu)+\alpha x+Q(x,y,\lambda), \end{cases} \tag{5}$$

with P, Q polynomials of degree n having no constant nor linear terms. More concretely,

$$P(x,y,\lambda)=\sum_{k+l=2}^{n} a_{k,l}x^k y^l, \quad Q(x,y,\lambda)=\sum_{k+l=2}^{n} b_{k,l}x^k y^l,$$

with $\lambda=(a_{20},a_{11},a_{02},\ldots,b_{20},b_{11},b_{02})\in\mathbb{R}^{n^2+3n-4}$. The trace parameter α sometimes is also denoted by L_0.

Theorem 3.1. *When $a=0$, we denote by $L_j^{(1)}(\lambda,b)$ the first-order development, with respect to $\lambda\in\mathbb{R}^k$, of the j-Lyapunov constant of system (5). We assume that, after a change of variables in the parameter space if necessary, we can write*

$$L_j=\begin{cases} \lambda_j+O_2(\lambda), \text{ for } j=1,\ldots,k-1, \\ \sum\limits_{l=1}^{k-1} g_{j,l}(\mu)\lambda_l+f_{j-k}(\mu)\lambda_k+O_2(\lambda), \text{ for } j=k,\ldots,k+\ell. \end{cases} \tag{6}$$

Where with $O_2(\lambda)$ we denote all the monomials of degree higher or equal than 2 in λ with coefficients analytic functions in μ. If there exists a point μ^* such that $f_0(\mu^*)=\cdots=f_{\ell-1}(\mu^*)=0$, $f_\ell(\mu^*)\neq 0$, and the Jacobian matrix of $(f_0,\ldots,f_{\ell-1})$ with respect to μ has rank ℓ at μ^*, then system (5) has $k+\ell$ hyperbolic limit cycles of small amplitude bifurcating from the origin.

Remark 3.2. In the above result, we remark the importance of the number of components in parameters λ and μ. Because, if there are more parameters than the relevant k in λ, in $O_2(\lambda)$ term can appear monomials of degree 2 that can affect the monomials of degree 1 and the result could be not valid.

Proof of Theorem 3.1. We assume first that the trace parameter α is zero. Then, using scheme mentioned in [13], we can remove the sums in (6) and consider a simpler list

$$L_j=\begin{cases} \lambda_j+O_2(\lambda), \text{ for } j=1,\ldots,k-1, \\ f_{j-k}(\mu)\lambda_k+O_2(\lambda), \text{ for } j=k,\ldots,k+\ell. \end{cases}$$

With the Implicit Function Theorem in the first $k-1$ components and writing $\lambda_k=u_k$ the above expression writes as

$$L_j=\begin{cases} u_j, \text{ for } j=1,\ldots,k-1, \\ f_{j-k}(\mu)u_k+O_2(u), \text{ for } j=k,\ldots,k+\ell. \end{cases} \tag{7}$$

From the hypothesis on the functions f_j at $\mu=\mu^*$, using again the Implicit Function Theorem, we can write, close to $\mu=\mu^*$, $f_{j-k}(\mu)=v_{j-k}+O_2(v)$, with $v_{j-k}=\mu_{j-k}-\mu^*_{j-k}$, for $j=k,\ldots,k+\ell-1$, and $v=(v_0,\ldots,v_{\ell-1})$.

Now, we consider the change of variables, like a partial blow-up, $u_j = \zeta w_j$ for $j = 1, \dots, k-1$, $u_k = \zeta$, and $v_{j-k} = w_j$ for $j = k, \dots, k+\ell-1$. Then (7) write as

$$L_j = \begin{cases} \zeta w_j, \text{ for } j = 1, \dots, k-1, \\ \zeta(w_j + A_j \zeta + O_2(\zeta, w_1, \dots, w_{k+\ell-1})), \text{ for } j = k, \dots, k+\ell-1, \end{cases} \tag{8}$$

for some real numbers A_j. They, as the higher-order terms in u, come from the terms $O_2(u)$ and the terms $O_2(v)$, after the change to $(\zeta, w_1, \dots, w_{k+\ell-1})$ coordinates. Moreover, the last Lyapunov constant writes as

$$L_{k+\ell} = \zeta(B + O_1(\zeta, w_1, \dots, w_{k+\ell-1})).$$

Finally, in (8) we can use again the Implicit Function Theorem to write $\zeta_j = w_j$ for $j = 1, \dots, k-1$ and $\zeta_j = w_j + A_j\zeta + O_2(\zeta, w_1, \dots, w_{k+\ell-1})$, for $j = k, \dots, k+\ell-1$. We notice that ζ is small enough and we have, near the origin of the parameter space, a curve (parametrized) by ζ of weak-foci of order $k+\ell$ that unfolds exactly, using the Weierstrass Preparation Theorem, $k+\ell-1$ hyperbolic limit cycles of small amplitude bifurcating from the equilibrium point located at the origin. The last limit cycle appears using the trace parameter α in a classical Hopf bifurcation. □

Christopher in [5] comments the generic unfolding of k limit cycles in families of polynomial vector fields when we consider centers on a component of the center variety of codimension k. This is the aim of Theorems 2.2 and 2.4. The above result shows that on some special points on such component the cyclicity can increase. This is the mechanism that we have used in the following sections to improve the known lower bounds for the local cyclicity $M(n)$ for some low values of n. In particular for $n = 3$ and $n = 4$. We think that Giné's conjecture in [10,11] about the lower bound for $M(n) = n^2 + 3n - 7$ can be thought in the sense of generic centers. We remark that, for providing higher values for $M(n)$ for higher degree n, we need to know better center families. Because the known families or have low codimension or they have too many parameters and the computational difficulties, as we will see in the following examples, increase so fast.

The fact that the cyclicity of Hamiltonian families depends on the parameters was previously studied by Han and Yu in [15], but not applied correctly in [21]. Here we extend this result for other types of center families.

4. Bifurcation diagrams for local cyclicity in families of cubic centers

In this section we use Theorem 3.1 to study the bifurcation diagram for some families of cubic centers, lying in components of the center variety of codimension 11, 10, and 9. The first, in Proposition 4.1, is the family labeled $CD_{31}^{(12)}$ that has generically cyclicity 11 and was studied previously by Christopher in [5], for only one parameter value $a = 2$ in (1), and by Yu and Tian in [21]. This proposition proves partially the main Theorem 1.1. The family labeled as $CR_{17}^{(12)}$ in [25], which depends also of a parameter a, is studied in Proposition 4.2. We have studied the local cyclicity for some values of this parameter a of the family $CR_{17}^{(12)}$ up to order 4 in the Lyapunov constants and we have found only 10 limit cycles. But using Theorem 3.1 we can get an extra limit cycle. Up to our knowledge, this is the first time that the cyclicity of this family has been studied. The last cubic family has 2 free parameters, see Proposition 4.4, and we show that generically the origin has cyclicity at least 9 and that there are curves with cyclicity at least 10 and some special points with cyclicity at least 11. According to Gasull, Garijo, and Jarque, in [7], any holomorphic center is also a Darboux center. Liang and Torregrosa in [17] show that for some values of the cubic family the cyclicity is as least 9. Here we explain that the cyclicity will increase depending on the specific center that we select. Up to our knowledge, the studies of the bifurcation diagrams are new for these families. At the end of the section, we show a center family with two parameters and not maximal codimension, labeled as $CD_{10}^{(11)}$ in [25], such that, perturbing with cubic polynomials, bifurcate also 12 limit cycles of small amplitude.

Proposition 4.1. *Consider system (5) with $n = 3$ and the unperturbed center*

$$\begin{cases} \dot{x} = -10(256a^3xy + 384a^3y - 96a^2x^2 - 384a^2y^2 - 16a^2x - 600axy \\ \qquad -480ay + 225x^2 + 900y^2 - 225x)(32a^2x + 48a^2 - 75x + 150), \\ \dot{y} = 16384a^5xy^2 + 24576a^5y^2 - 61440a^4y^3 + 16384a^5x + 56320a^4xy \\ \qquad -76800a^3xy^2 - 7680a^4y - 384000a^3y^2 + 288000a^2y^3 - 32000a^3x \\ \qquad -96000a^2xy + 90000axy^2 - 132000a^2y + 765000ay^2 - 337500y^3 \\ \qquad +168750ax - 84375xy - 337500y, \end{cases} \tag{9}$$

with a such that $32a^2 - 75 > 0$. Then, there exist only six parameter values a^* such that 12 limit cycles of small amplitude bifurcate from the origin. They are approximately ± 2.019925086, ± 7.444369217, and ± 15.62631048.

We have computed the linear part of the first Lyapunov constants for some values of a, different from the a^* stated in the above proposition, obtaining always maximal rank 11. Then, using Theorem 2.2, we can obtain, after perturbation, only 11 limit cycles of small amplitude. We think that this situation will be generic.

Proof of Theorem 4.1. The system corresponding to the rational first integral (1) has a center at the point $(x, y) = (6(8a^2+25)/(32a^2-75), 70a/(32a^2-75))$. Then, translating it to the origin we get system (9).

Let us consider (5) with $b_{30} = 0, b_{12} = 0$, and $b_{03} = 0$. After computing the first 12 Lyapunov constants up to order 1, we have that, generically for every a, $L_1^{(1)}, \ldots, L_{10}^{(1)}$ are linearly independent with respect to the parameters

$$a_{02}, a_{03}, a_{11}, a_{12}, a_{20}, a_{21}, a_{30}, b_{02}, b_{11}, b_{20}.$$

Then, we can write, after a linear change of parameters, $L_k = u_k + O_2(u)$, for $k = 1, \ldots, 10$, where $u_{11} = b_{21}$ and $O_2(u)$ denotes the monomials in u of degree higher than 2 with coefficients rational functions in the parameter a. Moreover, we have that L_j write as (6) with

$$L_{11}^{(1)} = \sum_{l=1}^{10} g_{10,l}(a)u_l + g(a)f_0(a)u_{11}, \quad L_{12}^{(1)} = \sum_{l=1}^{10} g_{11,l}(a)u_l + g(a)f_1(a)u_{11},$$

where f_0 and f_1 are polynomials of degrees 26 and 39 in a^2, respectively, g is a rational function without common factors with f_0 nor f_1. Additionally, the numerator and denominator of g are polynomials of degrees 69 and 90 in a^2 and $g_{10,l}$ and $g_{11,l}$ are also rational functions. All the involved polynomials are polynomials with rational coefficients.

The proof follows applying Theorem 3.1. To do that, we need to check that f_0 has real simple zeros and that the resultant of f_0 and f_1 with respect to a is a nonzero rational number. So, there should be at least a special value $a = a^*$ such that $f_0(a^*) = 0$, $f_0'(a^*) \neq 0$, and $f_1(a^*) \neq 0$. Finally, it can be checked that there are only six possible values for a^*. The numerical approximation values for a^* are the ones given in the statement. $\square$

In the proof of the existence of the extra limit cycle done in [21] the computations of $L_k^{(1)}$ are the same that we obtain. As we have understood, the mistake is that their proof is not based directly on a result like Theorem 3.1 which we have perfectly identified the perturbation parameters and we have restricted the perturbation in order to apply it. Their proof is based in the fact that $L_{11}^{(1)}$ vanishes and $L_{12}^{(1)}$ not. This is not enough because the terms of order 2 of L_{11} can appear and the weak-focus order does not increase. In fact, if we only consider $f_0(a^*) = 0$ then $L_{11}^{(2)} = u_{11}^2\, g_1(a)/g_2(a)$, with g_1 and g_2 polynomials of degree 66 and 103 in a^2. Moreover, in [21] the control of the number of relevant parameters as we have commented in Remark 3.2 is not clear.

Proposition 4.2. *Let $a \notin \{0, -1/6\}$ a real parameter. Consider the system*

$$\begin{cases} \dot{x} &= (x - ay + a + 2)(2\eta - 3\eta_y + 3x^2 + 6x + 6) - 3\eta\eta_y \\ & \quad -9x^2\eta_y + 9(2ax^2 + (2a-1)x + 2a), \\ \dot{y} &= 3(y(x - ay + a + 2)(-3x + y + 2) + 3(x^2 + x - 2)), \end{cases} \tag{10}$$

with $\eta = xy - ay^2 + 2x + 2(1+a)y + 1 - a$, $-1/6 < a < 0$, $1/3 < a < 1$, or $1 < a$. Then, it has a center at

$$(x^*, y^*) = \left(\frac{3(a-1)}{6a+1}, -\frac{3a^2 - 4a + 1}{a(6a+1)}\right),$$

and the next properties hold.

(i) *If $g(a) \neq 0$ and $f_0(a) \neq 0$ the local cyclicity, perturbing with polynomials of degree 3, is at least 10.*
(ii) *If $g(a) \neq 0$, $f_0(a) = 0$ the local cyclicity, perturbing with polynomials of degree 3, is at least 11. Moreover, f_0 has only 4 simple roots in the considered intervals. The numerical approximation is $\{-0.12245, 0.39672, 0.61983, 2.70517\}$.*

The expressions of polynomials f_0 and g are

$$\begin{aligned} f_0(a) = {} & 115567116089031205520a^{26} - 82791934329314091672a^{25} \\ & + 228195405046186847010a^{24} + 90491533122780174 24a^{23} \\ & - 157081144205847844 3464a^{22} + 335918075048147398 2039a^{21} \end{aligned}$$

$$
\begin{aligned}
& - 315147810716332642 7694a^{20} - 3259553243992338297 96a^{19} \\
& + 1421137122046938900 7506a^{18} - 3867036728366971062 1611a^{17} \\
& + 5686893498266503626 5406a^{16} - 5437717932617864400 6963a^{15} \\
& + 3080390878407350690 7336a^{14} - 9019277045696632383 477a^{13} \\
& - 6649229967375681687 78a^{12} + 2963892390472140000 813a^{11} \\
& - 1762296309778946693 076a^{10} + 4083431892496963319 43a^{9} \\
& - 5342376894194351959 2a^{8} + 3688723106531530364 7a^{7} \\
& - 1326383678363391115 2a^{6} + 1484165815203151098a^{5} \\
& + 85191877643707008a^{4} - 114163404746428485a^{3} \\
& + 1130289090405930a^{2} + 1973552231555520a + 103574370739840, \\
g(a) = {} & 4413012875799720164 2800a^{31} - 2525013156212545596 84000a^{30} \\
& + 5679972508489162450 20180a^{29} - 8137938285118733498 37180a^{28} \\
& + 2399279362949988891 138690a^{27} - 2777203364308983128 745270a^{26} \\
& - 1117982977709921462 9608785a^{25} + 5110034312827876920 1023051a^{24} \\
& - 9672273456885616905 5589531a^{23} + 1010724142371470731 55782098a^{22} \\
& - 8191116789244198181 2923273a^{21} + 9154373799722590388 1665763a^{20} \\
& - 1234642089357580685 86525599a^{19} + 1353853355799434724 06867144a^{18} \\
& - 1074703166613425094 76035270a^{17} + 5932298567720321123 8176126a^{16} \\
& - 2246844350391022929 3603606a^{15} + 6323085724047239916 867708a^{14} \\
& - 1656039645590378761 238526a^{13} + 3513462751671847804 34730a^{12} \\
& + 1240755469220636887 1724a^{11} - 2921779219862791558 9278a^{10} \\
& + 3200041670276393240 067a^{9} + 9330954664808213433 99a^{8} \\
& - 8196465110717287287 9a^{7} - 2355432176480659687 8a^{6} \\
& - 526449753238950189a^{5} + 210455326225541295a^{4} + 20323154636412705a^{3} \\
& + 375301845557100a^{2} - 28137453964620a - 1083684121520.
\end{aligned}
$$

Different phase portraits of system (10) are given in Fig. 1.

Remark 4.3. We notice that, although the family (10) is considered of codimension 12 by Żołądek in [25], we have not found more than 10 limit cycles of small amplitude as it is stated in the above result for $a = 2$ and computing up to order 10. We think that the same will happen for other values of a except the ones in Proposition 4.2 such that f_0 vanishes.

Proof of Proposition 4.2. Doing a translation in order that the center (x^*, y^*) of system (10) moves to the origin, we get

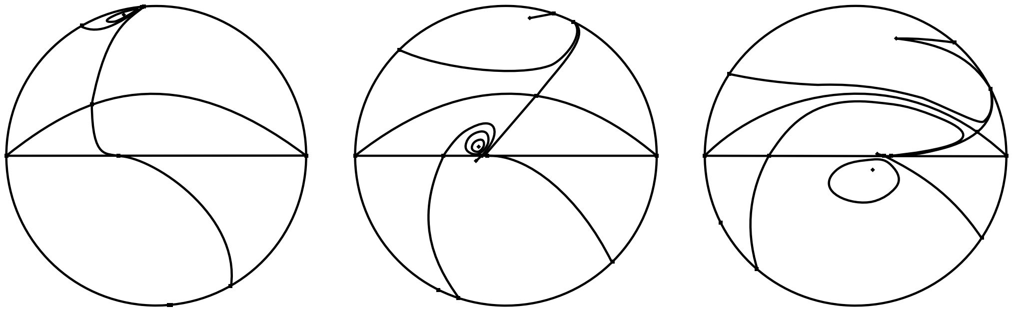

Fig. 1. Phaseportraits in the Poincaré disk of the center (10) for $a = -1/12$, $a = 1/2$, and $a = 2$.

$$\begin{cases} \dot{x} &= -(6a+1)(648a^7y^3 - 1944a^7y^2 - 2430a^6x^2y - 1944a^6xy^2 + 216a^6y^3 \\ &\quad +1458a^7y + 729a^6x^2 - 2916a^6xy - 3564a^6y^2 + 972a^5x^3 - 1701a^5x^2y \\ &\quad -1296a^5xy^2 + 18a^5y^3 - 972a^6y + 2187a^5x^2 + 486a^5xy - 1404a^5y^2 \\ &\quad +1134a^4x^3 - 216a^4x^2y - 270a^4xy^2 - 486a^5y + 1053a^4x^2 + 1782a^4xy \\ &\quad -144a^4y^2 + 486a^3x^3 + 72a^3x^2y - 18a^3xy^2 + 54a^4y - 162a^3x^2 \\ &\quad +594a^3xy + 90a^2x^3 + 18a^2x^2y - 36a^3y - 189a^2x^2 + 54a^2xy \\ &\quad +6x^3a + ax^2y - 18a^2y - 33ax^2 - x^2), \\ \dot{y} &= 3(3a+1)^4(6a^2y - 3a^2 + ay + 4a - 1)(6a^2y - 9a^2 + ay + 3a - 1)x. \end{cases}$$

Then we can consider equation (5). The proof that this family has a center follows from a rational symmetry and it can be found in [22,25].

Next step is the computation of $L_k^{(1)}$, for $k = 1, \ldots, 9$ and we consider them as linear functions depending on $a_{02}, a_{03}, a_{11}, a_{12}, a_{20}, a_{21}, b_{02}, b_{03}, b_{20}$. Hence, we write, after a linear change of coordinates adding $b_{21} = u_{10}$,

$$L_j = \quad u_j + O_2(u), \text{ for } j = 1, \ldots, 9.$$

The other parameter values in (5) have been taken as zero. In $O_2(u)$ appear some denominators in a which are nonzero under the hypotheses of the statement. In particular, the condition $g(a) \neq 0$ appears solving the above linear change. It can be seen also in the following expressions of the next two Lyapunov constants. After using the Implicit Function Theorem to vanish the first nine Lyapunov constants, the simplified expressions of the next two, are, except nonzero multiplicative constants,

$$L_{10} = \frac{(3a+1)^{13}(6a+1)^{18}}{a^3(9a^2-3a+1)^{10}(a-1)^9}\frac{f_0(a)}{g(a)}u_{10} + O_2(u_{10}),$$
$$L_{11} = \frac{(3a+1)^{13}(6a+1)^{19}}{a^4(3a-1)(9a^2-3a+1)^{12}(a-1)^{11}}\frac{f_1(a)}{g(a)}u_{10} + O_2(u_{10}),$$

where f_0 and g are defined in the statement and f_1 is

$$\begin{aligned} f_1(a) = {} & 72453647760857223788088 0a^{32} + 640589325778944777413782 80a^{31} \\ & - 610144481859757586223401556a^{30} + 2360973008978454210093841374a^{29} \end{aligned}$$

$$
\begin{aligned}
&- 3106072481972105279560942206a^{28} - 7847548346783924455871215944a^{27}\\
&+ 3735046528119834043057366657 5a^{26} - 6591294991279570315334914158 3a^{25}\\
&+ 5521383491291137923493255888 5a^{24} + 6562489081413077400265003107 0a^{23}\\
&- 3860975104162813854835688571 75a^{22} + 8522590404897635458648691244 60a^{21}\\
&- 1193508332460445900562643584016a^{20} + 1139964285706711135528711455009a^{19}\\
&- 730726233625740844361877322266a^{18} + 280817510225041089315898703766a^{17}\\
&- 21487202084536712499526119540a^{16} - 54304219150060860608252108112a^{15}\\
&+ 42894589880370044683717289676a^{14} - 16098081186021186459359445174a^{13}\\
&+ 2841857092329161976333442044a^{12} - 401707003814285433422087250a^{11}\\
&+ 278519520884076892704921201a^{10} - 89922626165488742408968047a^{9}\\
&- 21943437319421124107681 7a^{8} + 5240045076877491398959122a^{7}\\
&- 1141062554605305892208985a^{6} - 151240365953281702155 96a^{5}\\
&+ 4292814380007330375309 0a^{4} - 982397541469926950 55a^{3}\\
&- 5760057501860990359 50a^{2} - 2874706116185852256 0a + 216470434846265 60.
\end{aligned}
$$

Clearly $9a^2 - 3a + 1$ is nonvanishing and, with the restriction on a given in the statement, all the rational functions are well defined.

Statement (i) follows from Theorem 2.2. Statement (ii) follows as the proof of Proposition 4.1. That is, computing the resultant of f_0 and f_1 and the resultant of f_0 and f_0' with respect to a, and checking that f_0 has real zeros, which will be simple, such that f_1 does not vanish at them. From Theorem 3.1 we know that for the values of a such that f_0 vanishes we have 11 limit cycles of small amplitude bifurcating from the origin. □

The next result provides a complete bifurcation diagram for all holomorphic cubic centers having the coefficient of z^2 nonvanishing. In this case, it is not restrictive, rescaling if necessary, to assume that it is 1. In complex coordinates they write as

$$\dot{z} = \mathrm{i}\, z + z^2 + (a + \mathrm{i}\, b) z^3. \tag{11}$$

Proposition 4.4. *Consider system (5) with $n = 3$ and the unperturbed center*

$$
\begin{cases}
\dot{x} = ax^3 - 3axy^2 - 3bx^2y + by^3 + x^2 - y^2 - y,\\
\dot{y} = 3ax^2y - ay^3 + bx^3 - 3bxy^2 + 2xy + x,
\end{cases}
$$

for every value of the parameters $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ and the polynomials

$$
\begin{aligned}
f_0(a, b) =\,& 8a^6 + 24a^4b^2 + 24a^2b^4 + 8b^6 + 282a^4b + 564a^2b^3 + 282b^5 - 37569a^4\\
&- 45954a^2b^2 - 8385b^4 - 91924a^2b - 162484b^3 - 646020a^2 - 37860b^2,\\
f_1(a, b) =\,& 2448a^6b + 7344a^4b^3 + 7344a^2b^5 + 2448b^7 + 3208a^6 + 95916a^4b^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 182208a^2b^4 + 89500b^6 - 12055032a^4b - 15179760a^2b^3 - 3124728b^5\\
&- 19489169a^4 - 64437898a^2b^2 - 66540089b^4 - 285166044a^2b\\
&- 92688444b^3 - 310735620a^2 - 18210660b^2,\\
f_2(a,b) =\, &145864a^8 - 3776a^6b^2 - 886512a^4b^4 - 1178240a^2b^6 - 441368b^8\\
&+ 3892522a^6b - 9022362a^4b^3 - 29722290a^2b^5 - 16807406b^7\\
&- 708522105a^6 + 1379959497a^4b^2 + 2743262973a^2b^4 + 654781371b^6\\
&+ 8068743920a^4b + 18906063664a^2b^3 + 16016705984b^5\\
&- 5202830396a^4 + 86382442952a^2b^2 + 48606733828b^4\\
&+ 185131413648a^2b + 33791194128b^3 + 93466173600a^2 + 5477584800b^2,\\
g(a,b) =\, &27936a^6 + 83808a^4b^2 + 83808a^2b^4 + 27936b^6 - 162180a^4b\\
&- 324360a^2b^3 - 162180b^5 - 199825a^4 - 227714a^2b^2 - 27889b^4\\
&- 347172a^2b - 23172b^3 + 30636a^2 - 5364b^2.
\end{aligned}
$$

Then,

(i) *if* $f_0(a,b)g(a,b) \neq 0$ *there are* 9 *limit cycles of small amplitude bifurcating from the origin;*
(ii) *if* $f_0(a,b) = 0$ *and* $f_1(a,b)g(a,b) \neq 0$ *there are* 10 *limit cycles of small amplitude bifurcating from the origin;*
(iii) *if* $f_0(a,b) = f_1(a,b) = 0$ *and* $f_2(a,b)g(a,b) \neq 0$ *there are* 11 *limit cycles of small amplitude bifurcating from the origin.*

Moreover, there exist only two transversal intersection points of the curves $f_0(a,b) = 0$ *and* $f_1(a,b) = 0$ *which are* $(\pm a^*, b^*) \approx (\pm 69.66852455, -6.617950485)$.

The above result provides the bifurcation diagram for the local cyclicity of the 2-parameter holomorphic family (11). The curves f_0, f_1, and f_2 are drawn in Fig. 2 in red, green, and blue, respectively. Generically, the local cyclicity is 9. On the red curve, generically, the cyclicity is 10 and in the intersection point of the curves red and green, the cyclicity is 11.

Proof of Proposition 4.4. After a change of sign if necessary we can restrict our analysis to $a > 0$. For every (a,b) different from $(0,0)$ and taking zero the parameters $b_{20}, b_{11}, b_{30}, b_{12}$, we compute, with the parallelized algorithm described in [13], the linear terms of the first 11 Lyapunov constants, with respect to the relevant parameters $a_{20}, a_{11}, a_{02}, b_{02}, a_{30}, a_{21}, a_{12}, a_{03}, b_{21}$. If $g(a,b) \neq 0$ then, up to a linear change of parameters, we can write $L_j^{(1)} = u_j$, for $j = 1, \ldots, 8$, and

$$
L_9^{(1)} = \frac{(81a^2 + (9b+2)^2)(a^2+b^2)^3}{g(a,b)} f_0(a,b)u_9,
$$
$$
L_{10}^{(1)} = \frac{(81a^2 + (9b+2)^2)(a^2+b^2)^3}{g(a,b)} f_1(a,b)u_9,
$$

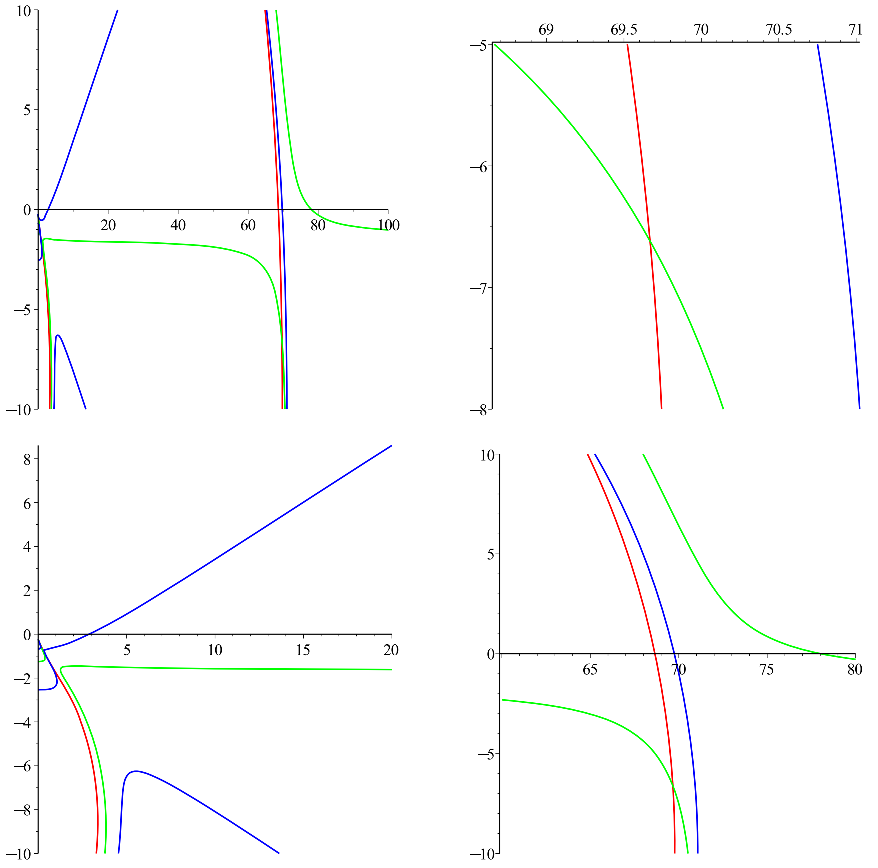

Fig. 2. The curves, with some zooms, $f_0(a,b)=0$, $f_1(a,b)=0$, and $f_2(a,b)=0$ given in Proposition 4.4 in red, green, blue, respectively. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

$$L_{11}^{(1)} = \frac{(81a^2+(9b+2)^2)(a^2+b^2)^3}{g(a,b)} f_2(a,b)u_9.$$

To simplify we have divided, if necessary, by nonzero multiplicative constants.

Computing the resultants of the pairs (f_0, f_1) and (f_0, f_2) with respect to a, we get

$$\begin{aligned}
&b^6(4b-9)^2(9b-59)^2(512192700b^4+13330993797b^3+61034982291b^2\\
&-33028358509b-10270019239)^2,\\
&b^6(4b-9)^2(9b-59)^2(570698912585670507000b^7+2299097628123738749501 4b^6\\
&+3688157828483981431708 5b^5-4478880915283836703764940b^4\\
&-950522720315380276649 2979b^3+384766091398809370306 5912b^2\\
&+1335195418811908515140 5788b+2696188868201530577480960)^2.
\end{aligned}$$

Removing the common factors, the above two polynomials in b of degrees 4 and 7 have no common roots, because it resultant, with respect to b is nonvanishing. Then, any intersection point of the curves $f_0 = 0$ and $f_1 = 0$ is not in the curve $f_2 = 0$. Then straightforward computations show that the curves $\{f_0 = 0, f_1 = 0\}$ have only one real intersection point $(a^*, b^*) \approx (69.66852455, -6.617950485)$. Moreover, it is a transversal intersection and $f_2(a^*, b^*) \neq 0$.

The proof follows using Theorem 3.1 in each item in the statement. □

Finally, we remark that the existence of centers with local cyclicity bigger than or equal to 12 is not restricted to families of codimension 12, some of them given in [25]. With the technique developed in this work, we can approach the lower bounds for the local cyclicity problem in families with more than one parameter. But the systems of polynomial equations that appear have a very high degree and the difficulties in solving them are beyond the reach of our computers. As an example, we provide a new family of cubic centers that we have proved, only numerically, this lower bound. This family is inspired by the one labeled by $CD_{10}^{(11)}$ in [25] with codimension 11. In particular, it is defined by the rational first integral

$$H(x, y) = \frac{(9x^3 + 9x^2 + 9bx + 9ay + 6a^2 + 6b^2 - b)^7}{(h_0(x) + h_1(x)y + h_2(x)y^2)^3}$$

where

$$\begin{aligned}
h_0(x) = {} & 2187bx^7 + 5103bx^6 + 1701b(3b + 2)x^5 + 189b(18a^2 + 18b^2 + 33b + 2)x^4 \\
& + 63b(72a^2 + 126b^2 + 6b - 1)x^3 + 21b(216a^2b + 216b^3 + 36a^2 + 54b^2 \\
& - 18b + 1)x^2 + 21b(18a^2 + 18b^2 + 6b - 1)(6a^2 + 6b^2 - b)x \\
& + (18a^2 + 18b^2 + 6b - 1)(6a^2 + 6b^2 - b)^2, \\
h_1(x) = {} & 5103abx^4 + 6804abx^3 + 1134ab(6b + 1)x^2 + 252ab(18a^2 + 18b^2 + 6b - 1)x \\
& + 21a(18a^2 + 18b^2 + 6b - 1)(6a^2 + 6b^2 - b), \\
h_2(x) = {} & 1134a^2b(3x + 1).
\end{aligned}$$

Using similar arguments as in the previous proofs, up to a linear change of parameters, we can write the linear developments of the Lyapunov constants as $L_j^{(1)} = u_j$, for $j = 1, \ldots, 9$, and $L_{j+10}^{(1)} = f_j(a, b)/g(a, b)\, u_{10}$, for $j = 0, 1, 2$. Where the f_0, f_1, f_2, and g are polynomials of degrees 254, 266, 278, and 288, respectively. The resultants of the noncommon factors of f_0, f_1 and f_0, f_2 with respect to a factorize in some polynomials. The noncommon factors of both are polynomials in a^2 of degrees 506 and 771, respectively, that we denote them by $f_{01}(a)$ and $f_{02}(a)$. Similarly for the resultants with respect to b having also degrees 506 and 771, and we denote them by $\bar{f}_{01}(b)$ and $\bar{f}_{02}(b)$. We find numerically the real zeros of $f_{01}(a)$ and $\bar{f}_{01}(b)$ and check which pairs (a, b) provide the transversal intersection points (a^*, b^*) of the curves $f_0(a, b) = f_1(a, b) = 0$ such that $f_2(a^*, b^*) \neq 0$. We have found 13 pairs with $a^* > 0$. One of them is $(a^*, b^*) \approx (0.393618957, 0.738793590)$. The transversality condition is guaranteed checking that $\mathrm{Jac}(f_0, f_1)$ is nonvanishing at (a^*, b^*). The numerical computations have been made with more than 1000 digits, seeing the stabilization of the nonzero values when we increase the precision.

5. Bifurcation diagrams for local cyclicity in families of quartic centers

This section is devoted to proving the second part of the statement of our main result, Theorem 1.1. It follows from the next proposition. We provide the bifurcation diagram of the local cyclicity of the cubic center given by Bondar and Sadovskiĭ in [4] adding a straight line of equilibria. This problem can be studied to get 19 limit cycles. Here show a curious fact, the cyclicity depends on the selected straight line. We work with two parameters (a, b), showing the existence of a curve with cyclicity at least 20 and a point with at least 21. Our servers need around one day to get the expressions of all necessary Lyapunov constants. Moreover, the size of each text file containing them has a size higher than 170 MB. We use a Computer Assisted Proof using the Poincaré–Miranda Theorem (Theorem 2.5), the Gershgorin Theorem (Theorem 2.6), and technical Lemmas 7.1 and 7.2.

Finally, we do a partial study of the bifurcation diagram of the local cyclicity of the holomorphic center of degree $n = 4$, depending on 4 parameters. We prove the existence of a holomorphic center with 20 limit cycles of small amplitude bifurcating from the origin. We have a strong numerical evidence that there are values of the parameters such that 21 limit cycles bifurcate from the origin, but the calculus is hard and an analytical proof has been impossible to be obtained. For the moment and for the perturbation of this family, we only present the analytical proof for the bifurcation of 20 limit cycles.

Proposition 5.1. *Consider equation (5) for $n = 4$ with the unperturbed system*

$$\begin{cases} \dot{x} &=& -y(1183x^2 - 68x + 1)(1 - ax - by), \\ \dot{y} &=& (672x^3 + 1484x^2y - 945xy^2 - 84y^3 - 58x^2 - 44xy + 30y^2 + x)\times \\ & & (1 - ax - by). \end{cases} \tag{12}$$

Then, there exists a pair $(a^*, b^*) \approx (-0.8159251773700849, 0.55062996428210239)$ such that, for parameters α, λ small enough in (5), at least 21 limit cycles of small amplitude bifurcate from origin.

Proof. System (12), without the straight line of equilibria, has a center at the origin because it has a rational first integral, see [4]. We restrict our study to $b_{11} = b_{21} = b_{30} = b_{31} = b_{40} = 0$ in (5). After a linear change of coordinates we move from $a_{02}, a_{03}, a_{04}, a_{11}, a_{12}, a_{13}, a_{20}, a_{21}, a_{22}, a_{30}, a_{31}, a_{40}, b_{02}, b_{03}, b_{04}, b_{11}, b_{12}, b_{13}$ to $u_1, \ldots, u_{18}$ and write $L_k^{(1)} = u_k$, for $k = 1, \ldots, 18$. As we have done in the previous proofs, writing $b_{20} = u_{19}$, and removing the common factors, which are rational functions in (a, b), in the linear development of the next Lyapunov constants we can write

$$L_{19}^{(1)} = f_0(a, b)u_{19}, \quad L_{20}^{(1)} = f_1(a, b)u_{19}, \quad L_{21}^{(1)} = f_2(a, b)u_{19}. \tag{13}$$

The numerators of f_0, f_1, f_2 are polynomials with rational coefficient of degrees 180, 182, and 184, respectively. The total number of monomials are, respectively, 16329, 16694, and 17063. We have not added here the expressions because of their size.

Numerically we can find the solution (a^*, b^*) in the statement for the algebraic system $\{f_0 = 0, f_1 = 0\}$. Moreover, the intersection is transversal because the determinant of the Jacobian matrix at the intersection point is $-8.7569521108153076 \cdot 10^{570}$. At this point we have $f_2(a^*, b^*) = -1.7191356490086216 \cdot 10^{290}$.

To get an analytic proof we will use a Computer Assisted Proof with the help of Lemmas 7.1 and 7.2. We use Theorem 2.5 for the existence of the intersection point of f_0 and f_1 and Theorem 2.6 to prove the transversality. The technical lemmas also are used to check that at point f_2 is nonvanishing. We fix a square $\mathcal{Q} = [-h, h]^2$ with $h = 10^{-12}$ and we do a rational affine change of coordinates such that a good rational approximation of (a^*, b^*) be inside $\mathcal{Q}$. This affine change of variables is chosen such that the Taylor series of degree 1 of f_0 and f_1 at (a^*, b^*) will be the new coordinates. Then

$$
\begin{aligned}
\tilde{f}_0(S_0^-) &\subset [-1.31146 \times 10^{-12}, -8.44847 \times 10^{-13}],\\
\tilde{f}_0(S_0^+) &\subset [1.15471 \times 10^{-12}, 6.89142 \times 10^{-13}],\\
\tilde{f}_1(S_0^-) &\subset [-1.15545 \times 10^{-12}, -6.90604 \times 10^{-13}],\\
\tilde{f}_1(S_0^+) &\subset [1.30878 \times 10^{-12}, 8.44982 \times 10^{-13}],\\
\tilde{f}_2(\mathcal{Q}) &\subset [0.9035737600, 1.096426240],
\end{aligned}
$$

and we have proved the existence of (a^*, b^*) such that f_2 is nonvanishing. In the computations we have worked with rational numbers with numerators and denominators of around 15000 digits. To simplify the computations we have worked with the functions $\tilde{f}_j(a, b) = f_j(a, b)/f_j(0, 0)$.

The last part is to check the transversality. Instead of compute the determinant of the Jacobian matrix of (f_0, f_1) with respect to (a, b), we use the technical lemmas to get that the elements in the Jacobian matrix for the transformed variables are, varying in $\mathcal{Q}$, $A_{11}, A_{22} \subset (0.84568065, 1.15431935)$, $A_{12} \subset (-0.15535611, 0.15535611)$, and $A_{21} \subset (-0.15498852, 0.15498852)$. Then with Theorem 2.6, both eigenvalues are positive and belong to the interval $(0.74, 1.25)$. Therefore, the determinant is different from zero. □

Remark 5.2. We remark the computational difficulties of the numeric in the above result. We should work with very high precision. In fact, working with 1000 digits the evaluations of f_0 and f_1 at (a^*, b^*) are $-2.19920305995245 \cdot 10^{-397}$ and $3.595005930091451 \cdot 10^{-390}$, respectively. Moreover, the necessary affine change of variables has need more than one computation day. Finally, the curves in Fig. 3 has been drawn computing the points one by one working with very high precision and then using polynomial interpolation. In fact, the first time that we got (a^*, b^*) was from the intersection of these polynomial interpolation curves.

Proposition 5.3. *Consider equation* (5) *for* $n = 4$ *with the unperturbed system written in complex coordinates,* $z = x + \mathrm{i}\, y$*, as*

$$
\dot{z} = \mathrm{i}\, z + z^2 + (a_1 + \mathrm{i}\, a_2) z^3 + (a_3 + \mathrm{i}\, a_4) z^4. \tag{14}
$$

If $a_1 = 1$ *and* $a_3 = 3$*, there exist two algebraic curves* $f_0(a_2, a_4)$ *and* $f_1(a_2, a_4)$ *such that, generically on* $f_0(a_2, a_4)$*, there are small parameters* λ *for which* (5) *has at least* 19 *limit cycles of small amplitude bifurcating from the origin. Moreover, there are at least three transversal intersection points,*

$$
(a_2^*, a_4^*) \in \{(-6.788836, 2.856062), (-4.387174, 4.549274), (-4.619905, -4.565876)\},
$$

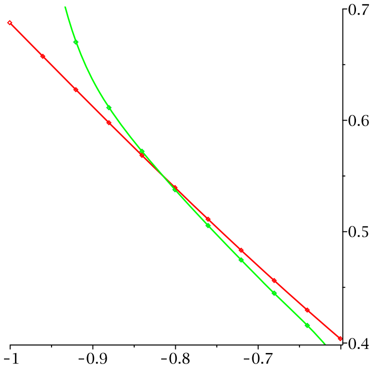

Fig. 3. Drawing the zero level sets of f_0 and f_1 in (13) in red and green, respectively.

of f_0 and f_1 for which there are perturbations of degree four of (5) *such that at least* 20 *limit cycles of small amplitude bifurcate from the origin.*

Proof. In Cartesian coordinates, taking $a_1 = 1$ and $a_3 = 3$, system (14) writes as

$$\begin{cases} \dot{x} &= 3x^4 - 4a_4x^3y - 18x^2y^2 + 4a_4xy^3 + 3y^4 + x^3 - 3a_2x^2y - 3xy^2 \\ & \quad +a_2y^3 + x^2 - y^2 - y, \\ \dot{y} &= a_4x^4 + 12x^3y - 6a_4x^2y^2 - 12xy^3 + a_4y^4 + a_2x^3 + 3x^2y - 3a_2xy^2 \\ & \quad -y^3 + 2xy + x. \end{cases}$$

We will restrict our analysis to $b_{11} = b_{20} = b_{21} = b_{30} = b_{31} = b_{40} = 0$. The Lyapunov constants up to order 1, with the algorithm explained in [13] and similarly as the proof of Proposition 5.1, can be computed and written as $L_k^{(1)} = u_k$, for $k = 1, \ldots, 17$. Here we have done a linear change of coordinates in the parameter space changing the linear independent parameters

$$a_{12}, a_{02}, a_{03}, a_{04}, a_{11}, a_{13}, a_{20}, a_{21}, a_{22}, a_{30}, a_{31}, a_{40}, b_{02}, b_{03}, b_{04}, b_{12}, b_{13},$$

by $u_1, \ldots, u_{17}$. Changing the last one b_{22} to u_{18} we have, as in the previous proofs and except a multiplicative rational function in a_2, a_4 as a common factor,

$$L_{18}^{(1)} = f_0(a_2, a_4)u_{18}, \quad L_{19}^{(1)} = f_1(a_2, a_4)u_{18}, \quad L_{20}^{(1)} = f_2(a_2, a_4)u_{18}. \tag{15}$$

The proof follows similarly as the proof of Theorem 4.4 to get the transversal intersection points in the statement. Computing the necessary resultants with respect to a_2 and a_4 to apply Theorem 3.1.

In Fig. 4, we have drawn the algebraic curves $f_k(a_2, a_4) = 0$, for $k = 0, 1, 2$, in red, blue, and green, respectively. Notice, that in the pictures it is clear the existence of transversal intersections of $f_0 = 0$ and $f_1 = 0$ where f_2 is nonvanishing. □

Remark 5.4. Following the same procedure as for the Lyapunov constants given in (15) we compute also the next Lyapunov constant that can be written as

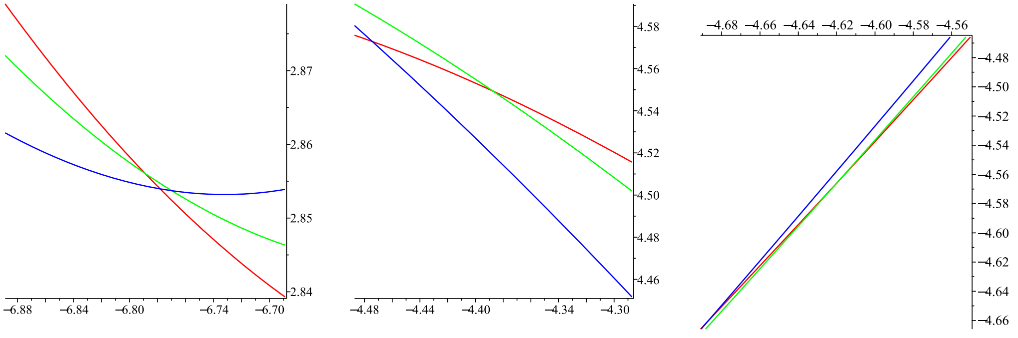

Fig. 4. Drawing the zero level sets of f_0, f_1, and f_2 in (15) in red, green, and blue, respectively.

$$L_{21}^{(1)} = f_3(a_2, a_4)u_{18}.$$

Now taking $a_1 = 1$ in (14) we can compute the corresponding algebraic functions f_0, f_1, f_2, f_3 depending only on (a_2, a_3, a_4). They have around 10^5 monomials and degrees 100, 101, 102, 103, respectively. Then, we can solve numerically with high precision the first three obtaining

$$a^* = (a_2^*, a_3^*, a_4^*) \approx (0.26423354653702, 2.06583351382191, 2.26983478766641).$$

The evaluation at this point gets

$$f_0(a^*) \approx 4.35 \cdot 10^{-281}, \qquad f_1(a^*) \approx 3.2 \cdot 10^{-275},$$
$$f_2(a^*) \approx 3.67 \cdot 10^{-272}, \qquad f_3(a^*) \approx 1.091295989718 \cdot 10^{126},$$

and the determinant of the Jacobian matrix of (f_0, f_1, f_2) with respect to a at the intersection point a^* is $-3.82703230760 \cdot 10^{363}$. This gives a numerical evidence that the holomorphic family of degree 4 exhibits also 21 limit cycles of small amplitude bifurcating from the origin. The stabilization of the digits of the nonvanishing values of f_3 and the determinant has been obtained working with enough precision.

6. Final comments

The computations in this work are quite high although basically we have worked only with developments of order 1 in the Lyapunov constants but center depending on parameters. This is because the existence of parameters in the unperturbed centers makes the things more complicated. Before the simplifications, the polynomials appearing as coefficients of the perturbation parameters are of very high degree and with rational coefficients with a high number of digits. In fact, this is why we have only considered vector fields of degrees $n = 3$ and $n = 4$. The mistakes found in the proof of the main theorem of [21] and corrected here are also made in the later work [20]. This other work can also be corrected using the same techniques developed here.

It is possible to find high order focus especially for systems of degree even. The high order of this focus does not imply that automatically we can find more limit cycles. What really happens is that the unfolding of the possible limit cycles is not guaranteed and it is usually not possible. For instance, from the work [20], it can be seen that there are homogeneous perturbations of degree four of a linear type center exhibiting a weak foci curve of order 24. However, the complete unfolding of these weak foci in the general class of degree four vector fields, that has only 24 parameters, is not guaranteed. In fact, for degree six class, there exist weak foci curves of order larger than the number of parameters of the perturbation.

In [17] the holomorphic centers are considered and it is proved that for low degree $4 \le n \le 13$ the cyclicity of each center is at least n^2+n-2 and for $n=3$ it is at least 9. The results of this paper provide higher values of the local cyclicity but only for $n=3$ and $n=4$. Obtaining as new lower bounds in this class 11 and 21, respectively, even though this last value has a proof which is not completely analytic. We have also worked with other holomorphic centers, $n=5,6,7$ but only with one parameter. In all cases, we have found at least one extra limit cycle than the ones obtained in [17]. But as the obtained lower bounds for $M(n)$ are worse than others obtained in [13] we have not added here.

In all the proofs it is very important to restrict our studies to exactly the number of parameters k and ℓ in Theorem 3.1. Then we will always have only lower bound for the cyclicity in the considered families. This restriction ensures that the higher-order terms do no affect the expressions of the first-order developments.

7. Accurate interval analysis

Next two technical results will help us to find upper and lower bounds for a polynomial of n variables in a n-dimensional cube. The proofs of them can be found in [6].

Lemma 7.1 *([6]). Consider $h>0$, $p>0$, q real numbers such that $p \in [\underline{p}, \overline{p}]$, with $\underline{p}\,\overline{p} > 0$, and $q \in [\underline{q}, \overline{q}]$, with $\underline{q}\,\overline{q} > 0$.*

(i) *Then, $\sigma^\ell(q,p) \le qp \le \sigma^r(q,p)$, where*

$$\sigma^\ell(q,p) = \begin{cases} q\underline{p}, & \text{if } q>0,\\ q\overline{p}, & \text{if } q<0, \end{cases} \qquad \sigma^r(q,p) = \begin{cases} q\overline{p}, & \text{if } q>0,\\ q\underline{p}, & \text{if } q<0. \end{cases}$$

(ii) *If $u_j \in [-h,h]$, for $j=1,\dots,n$ and denoting $u^i = u_1^{i_1}\cdots u_n^{i_n}$, for the multiindex $i=(i_1,\dots,i_n) \ne 0$, we have $\mathcal{X}^\ell(q,u^i) \le qu^i \le \mathcal{X}^r(q,u^i)$, where*

$$\mathcal{X}^\ell(q,u^i) = \begin{cases} 0, & \text{if } q>0 \text{ and } i_k \text{ even for all } k=1,\dots,n,\\ -\overline{q}h^{i_1+\cdots+i_n}, & \text{if } q>0 \text{ and } i_k \text{ odd for some } k=1,\dots,n,\\ \underline{q}h^{i_1+\cdots+i_n}, & \text{if } q<0, \end{cases}$$

and

$$\mathcal{X}^r(q,u^i) = \begin{cases} -\underline{q}h^{i_1+\cdots+i_n}, & \text{if } q<0 \text{ and } i_k \text{ odd for all } k=1,\dots,n,\\ 0, & \text{if } q<0 \text{ and } i_k \text{ even for some } k=1,\dots,n,\\ \overline{q}h^{i_1+\cdots+i_n}, & \text{if } q>0. \end{cases}$$

Furthermore, $\mathcal{X}^\ell(q,1) = \underline{q}$ and $\mathcal{X}^r(q,1) = \overline{q}$.

Lemma 7.2 *([6]). Let $h > 0$ and p_j be a positive nonrational numbers such that $p_j \in [\underline{p}_j, \overline{p}_j]$ with $\underline{p}_j, \overline{p}_j$ rational numbers satisfying $\underline{p}_j, \overline{p}_j > 0$, for $j = 1, \ldots, m$. Consider the polynomial*

$$\mathcal{U}(u_1, \ldots, u_n) = \sum_{i_1+\cdots+i_n=0}^{M} \left(\sum_{j=1}^{m} U_{j,i} p_j \right) u^i$$

with $u^i = u_1^{i_1} \cdots u_n^{i_n}$, $i = (i_1, \ldots, i_n)$ and $U_{j,i}$ rational numbers. Then

$$U_i^\ell \leq \sum_{j=1}^{m} U_{j,i} p_j \leq U_i^r,$$

with $U_i^\ell = \sum_{j=1}^{m} U_{j,i} \sigma^\ell(U_{j,i}, p_j)$ and $U_i^r = \sum_{j=1}^{m} U_{j,i} \sigma^r(U_{j,i}, p_j)$. Moreover, if $u_j \in [-h, h]$, for $j = 1, \ldots, n$ and $U_i^\ell \cdot U_i^r > 0$ then

$$\underline{\mathcal{U}} = \sum_{i_1+\cdots+i_n=0}^{M} \mathcal{X}^\ell(U_i^\ell, u^i) \leq \mathcal{U}(u_1, \ldots, u_n) \leq \sum_{i_1+\cdots+i_n=0}^{M} \mathcal{X}^r(U_i^r, u^i) = \overline{\mathcal{U}}.$$

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