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EXISTENCE OF AT MOST TWO LIMIT CYCLES FOR SOME NON-AUTONOMOUS DIFFERENTIAL EQUATIONS

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Abstract. It is know that the non-autonomous differential equations $\mathrm{d}x/\mathrm{d}t = a(t)+b(t)|x|$, where $a(t)$ and $b(t)$ are 1-periodic maps of class $\mathcal{C}^1$, have no upper bound for their number of limit cycles (isolated solutions satisfying $x(0) = x(1)$). We prove that if either $a(t)$ or $b(t)$ does not change sign, then their maximum number of limit cycles is two, taking into account their multiplicities, and that this upper bound is sharp. We also study all possible configurations of limit cycles. Our result is similar to other ones known for Abel type periodic differential equations although the proofs are quite different.

1. **Introduction.** The problem of knowing the number of limit cycles of general planar vector fields is extremely complicated and many efforts have been dedicated to face it for concrete families of planar differential equations. Some of these families are: quadratic systems, cubic systems, Kolmogorov systems, rigid systems, Liénard type equations, ... For this reason, and to try to consider simpler questions that capture the main difficulties of the problem, people has addressed similar problems for one-dimensional non-autonomous and periodic differential equations.

More concretely, consider $\mathcal{C}^1$ differential equations of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = S(x,t), \tag{1.1}$$

with $x, t \in \mathbb{R}$, that are 1-periodic in the variable t. We are interested on solutions $x(t)$, defined for all $t \in \mathbb{R}$, and such that $x(0) = x(1)$. We will call them *periodic solutions* because they are closed when we consider (1.1) on the cylinder $\mathbb{R} \times [0,1]$.

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