EXISTENCE OF AT MOST TWO LIMIT CYCLES FOR SOME NON-AUTONOMOUS DIFFERENTIAL EQUATIONS

ARMENGOL GASULL AND YULIN ZHAO

Abstract. It is know that the non-autonomous differential equations $\mathrm{d}x/\mathrm{d}t = a(t) + b(t)|x|$, where $a(t)$ and $b(t)$ are 1-periodic maps of class $\mathcal{C}^1$, have no upper bound for their number of limit cycles (isolated solutions satisfying $x(0) = x(1)$). We prove that if either $a(t)$ or $b(t)$ does not change sign, then their maximum number of limit cycles is two, taking into account their multiplicities, and that this upper bound is sharp. We also study all possible configurations of limit cycles. Our result is similar to other ones known for Abel type periodic differential equations although the proofs are quite different.

1. Introduction

The problem of knowing the number of limit cycles of general planar vector fields is extremely complicated and many efforts have been dedicated to face it for concrete families of planar differential equations. Some of these families are: quadratic systems, cubic systems, Kolmogorov systems, rigid systems, Liénard type equations, ... For this reason, and to try to consider simpler questions that capture the main difficulties of the problem, people has addressed similar problems for one-dimensional non-autonomous and periodic differential equations.

More concretely, consider $\mathcal{C}^1$ differential equations of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = S(x,t), \tag{1}$$

with $x, t \in \mathbb{R}$, that are 1-periodic in the variable t. We are interested on solutions $x(t)$, defined for all $t \in \mathbb{R}$, and such that $x(0) = x(1)$. We will call them *periodic solutions* because they are closed when we consider (1) on the cylinder $\mathbb{R} \times [0,1]$. Moreover, a periodic solution which is isolated in the set of all the periodic solutions of (1) is called a *limit cycle* of the differential equation.

These differential equations are interesting by themselves, but also appear from some families of planar autonomous polynomial differential equations ([3]), in problems of control theory, see for instance [6] and their references, or in other models of the real world ([1]).

In particular, the study of the maximum number of limit cycles of (1) when $S(x,t)$ is a polynomial of degree n has a long history. We briefly summarize it. This question was proposed by N. G. Lloyd ([12]), V. A. Pliss ([16]), and C. Pugh ([11]). When $n = 1$ it is a *linear equation* and it is well known that it has either a continuum of periodic solutions or at most 1 limit cycle. Similarly, when $n = 2$ it is a periodic *Riccati equation* and it has at most two limit cycles, see for instance [11]. When $n = 3$, it is called *Abel equation* and the situation is much more complicated. In fact, for all $n \geq 3$ it is known

2000 *Mathematics Subject Classification.* Primary: 34C25 Secondary: 34A34, 37C07, 37C27.
Key words and phrases. Non-autonomous differential equation; Limit cycle; Periodic orbit.

that there is no upper bound for the number of limit cycles for periodic Abel differential equations, see [11].

Hence, for $n \geq 3$ to have results that guarantee an upper bound for the number of limit cycles of polynomial differential equation some additional hypotheses need to be added. Next we recall some results where it can be seen that usually these hypotheses fix the sign invariance of one of the coefficients of some of the monomials x^k.

- The 1-periodic differential equation,
$$\frac{\mathrm{d}x}{\mathrm{d}t} = a_n(t)x^n + a_m(t)x^m + a_1(t)x, \tag{2}$$
with $n > m > 1$, where either $a_n(t)$ or $a_m(t)$ does not change sign, has at most 4 (resp. 5) limit cycles when n is even (rep. odd). Moreover both bounds are optimal if $n \geq 4$, see [8]. When $n = 3$ and $a_3(t)$ does not change sign, the (optimal) upper bound for the number of limit cycles is 3, see [10, 11]. Also, when $n = 3$, $a_1(t) \equiv 0$ and $a_2(t)$ does not change sign, it is proved in [10] that again the maximum number of limit cycles is 3.
- The 1-periodic differential equation
$$\frac{\mathrm{d}x}{\mathrm{d}t} = a_n(t)x^n + a_2(t)x^2 + a_1(t)x + a_0(t), \tag{3}$$
with $n \geq 3$, and where $a_n(t)$ does not change sign, has at most 3 limit cycles when n is odd ([14]) and it has no upper bound for its number of limit cycles when n is even ([8, 14]). See also [4] for some refinements when $a_2(t) \equiv 0$.

It is important to notice for instance that when it is assumed in (2) that $a_1(t)$ does not change sign, there is no upper bound for its number of limit cycles. Hence the role of the coefficients corresponding to different x^k can be very different. Moreover, for (3) its role clearly also depends on the parity of n.

In this paper we are interested on the number of limit cycles of the differential equation
$$\frac{\mathrm{d}x}{\mathrm{d}t} = a(t) + b(t)|x|, \tag{4}$$
where $a(t)$ and $b(t)$ are real, 1-periodic and $\mathcal{C}^1$ functions, $t \in \mathbb{R}$. Although, at a first look, it might look simple, it is proved in [2, 5] that there is no upper bound for its number of limit cycles. More concretely, for instance in [5] it is proved that given any natural number $k \geq 2$, for ε small enough, the differential equation
$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2\pi\sin(2\pi t) + \varepsilon\cos(2k\pi t)|x|, \tag{5}$$
has at least $k - 2$ limit cycles.

Therefore it is quite natural to wonder if (4) will have an upper bound for its number of limit cycles when either $a(t)$ or $b(t)$ does not change sign. As we will show, the answer is yes and to prove this fact is the main goal of this work.

Notice that the line $\mathcal{L} = \{x = 0\}$ is the locus where the differential equation (4) is not smooth, but this differential equation is Lipschitz continuous with respect to x and as a consequence the existence and uniqueness of solutions of its associated Cauchy initial value problem is guaranteed. We will denote by $u(t) = u(t, 0, x)$ the solution of (4) with the initial condition $u(0) = x$.

Let $u_+(t)$ be the solution of (4) such that $u(t) \geq 0$ for every $t \in \mathbb{R}$. Then $u(t) = u_+(t)$ satisfies $x' = a(t) + b(t)x$ and is called a *positive solution.* If $u_+(t)$ is 1-periodic then it is called a *positive periodic orbit.* Similarly, let $u_-(t)$ be the solution of (4) such that

$u(t) \leq 0$ for every $t \in \mathbb{R}$. Then $u_-(t)$ satisfies $x' = a(t) - b(t)x$ and is called a *negative solution*. A periodic solution $u_-(t)$ is called a *negative periodic orbit*.

The limit cycles of (4) that take positive and negative values will be called *crossing limit cycles*. This is so because they cross the line $\mathcal{L}$. For instance, the $k-2$ limit cycles of (5) are of crossing type.

As usual, define the *Poincaré return map* as $P(\rho) = u(1, 0, \rho)$ and the *displacement map* $H(\rho) = P(\rho) - \rho$. Clearly, the periodic orbits of (4) are the fixed points of $P(\rho)$, or equivalently, the zeros of $H(\rho)$. Moreover, if some of these values of ρ is a isolated fixed point (or zero) then the corresponding periodic solution $u(t, 0, \rho)$ is a limit cycle. It is proved in [2] that $P(\rho)$ and $H(\rho)$ are of class $\mathcal{C}^1$, see Proposition 2.4. If the limit cycle $x = u(t, 0, \rho)$ is such that $P(\rho) = \rho$ and $P'(\rho) \neq 1$ then it is said that it as a *hyperbolic limit cycle*. A limit cycle that is an atractor from below and a repeller from above (or vice versa) is called *semistable*. If $P(\rho)$ is twice derivable, $P(\rho) = \rho$, $P'(\rho) = 1$ and $P''(\rho) \neq 0$, then the corresponding limit cycle is called *double limit cycle* and it is semistable.

For short we will say that the limit cycles configuration of a differential equation (4) with $a(t) \not\equiv 0$ is $[k, \ell, m]$ if it has respectively, k, ℓ and m negative, crossing and positive limit cycles, taking into account their multiplicities. In Section 4 we will see examples with all possible configurations when either $a(t)$ or $b(t)$ does not change sign. Notice that $x = 0$ can be a limit cycle only if $a(t) \equiv 0$ and in this paper we disregard this trivial case.

Theorem 1.1. *Consider the differential equation* (4)*,*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a(t) + b(t)|x|,$$

where $a(t)$ and $b(t)$ are real, 1-periodic $\mathcal{C}^1$ functions. Then the following statements hold:

(a) *If $a(t) \not\equiv 0$ and does not change sign then it has at most two limit cycles. Moreover they exist if and only if $\operatorname{sign}(a) \int_0^1 b(t)\,\mathrm{d}t < 0$, and then they have configuration $[1, 0, 1]$ and are hyperbolic.*

(b) *If $b(t)$ does not change sign and $a(t)$ has finitely many zeros in $[0, 1]$, then it has at most two limit cycles, taking into account their multiplicities. The limit cycles can only exist when $\operatorname{sign}(b) \int_0^1 a(t)\,\mathrm{d}t < 0$, and if they exist, the total number, taking into account their multiplicities, is two. Moreover, the only possible configurations are $[1, 0, 1]$, $[1, 1, 0]$, $[0, 1, 1]$ and $[0, 2, 0]$ and all are realizable, where when a 2 appears it means that there are or two hyperbolic crossing limit cycles or a double (semistable) crossing limit cycle, and both cases do happen.*

We end of this section with several comments.

The hypothesis that $a(t)$ has finitely many zeros is technical and we have inherited it from [2], where it is also introduced for other reasons. The smoothness conditions on $a(t)$ and $b(t)$ are sufficient conditions to prove that differential equation (4) has at most two limit cycles, taking into account their multiplicities. If we only care about the existence of at most two limit cycles then the conditions on $a(t)$ and $b(t)$ can be relaxed by imposing only their continuity. To show it, suppose that $a(t)$ and $b(t)$ are real, 1-periodic continuous functions. If $b(t)$ does not change sign, we assume $b(t) \geq 0$ without loss of generality, see the explanation in the proof of Theorem 1.1 in Section 3 below.

Notice that if $\mathcal{S}(x,t) = a(t) + b(t)|x|$ then $\mathcal{S}(x,t)$ satisfies the following properties

$$\frac{\mathcal{S}(x_2,t) - \mathcal{S}(x_1,t)}{x_2 - x_1} \leq \frac{\mathcal{S}(x_3,t) - \mathcal{S}(x_1,t)}{x_3 - x_1}$$

if $x_1 < x_2 < x_3$. Moreover, the inequality is strict if $x_1 < 0 < x_2$. By using a classical convexity method as in [13, Sec. 3], a simpler proof that equation (4) has at most two limit cycles follows. Notice that this result is different from the one in (b) of Theorem 1.1 which shows that equation (4) has either no limit cycle, or two limit cycles (taking also into account their multiplicities). Our proof depends on the analysis of the convexity of Poincaré map, and hence we get some more information about the number for the limit cycles.

2. Preliminary results

In this paper when a continuous function $w(t)$ is non-negative, but not identically zero, that is $w(t) \geq 0, w(t) \not\equiv 0$, we will say that its sign is positive. Similarly we define negative sign. Moreover, for short, when we say that the sign of some function gives the stability of a periodic orbit we will mean that when it is positive (resp. negative) then the orbit is an attractor (resp. a repeller).

Next result characterizes the positive or negative periodic orbits. For simplicity we introduce the following quantities:

$$\begin{aligned} A &= \exp\left(\int_0^1 a(t)\,\mathrm{d}t\right) > 0, \qquad & C &= \int_0^1 a(t)\exp\left(\int_t^1 b(s)\mathrm{d}s\right)\mathrm{d}t \\ B &= \exp\left(\int_0^1 b(t)\,\mathrm{d}t\right) > 0, \qquad & D &= \int_0^1 a(t)\exp\left(-\int_t^1 b(s)\mathrm{d}s\right)\mathrm{d}t \end{aligned} \tag{6}$$

Proposition 2.1. *(i) The maximum number of positive limit cycles of* (4) *is one and a positive limit cycle exists if and only if* $B \neq 1$ *and* $x = u^+(t,0,C/(1-B)) \geq 0$, *where* u^+ *is the solution of* $x' = a(t) + b(t)x$, *with the given initial condition, and* B *and* C *are given in* (6). *Moreover, it is hyperbolic and its stability is given by the sign of* $B - 1$.

(ii) The maximum number of negative limit cycles of (4) *is one and a negative limit cycle exists if and only if* $B \neq 1$ *and* $x = u^-(t,0,BD/(B-1)) \leq 0$, *where* u^- *is the solution of* $x' = a(t) - b(t)x$, *with the given initial condition, and* B *and* D *are given in* (6). *Moreover, it is hyperbolic and its stability is given by the sign of* $1 - B$.

Proof. (i) Linear differential equations can be solved analytically and its associated Poincaré map can obtained explicitly in terms of the values introduced in (6). The Poincaré map for $x' = a(t) + b(t)x$ is $P(\rho) = B\rho + C$. Hence the necessary and sufficient conditions for $x = u^+(t,0,\rho)$, solution of this linear differential equation, to be a positive limit cycle for (4) are:

$$B \neq 1, \quad B\rho + C = \rho \quad \text{and} \quad u(t,0,C/(1-B)) \geq 0 \quad \text{for all} \quad t \in \mathbb{R}.$$

When $u^+(t,0,\rho)$ is strictly positive the $P'(\rho) = B$ and the hyperbolicity and stability of the limit cycle follows because $B \neq 1$. When $x = u^+(t,0,\rho) \geq 0$ and the solution tangentially touches $x = 0$ by the results of [2] we know that the Poincaré map for (4) is of class $\mathcal{C}^1$. As a consequence, its derivative at the initial condition corresponding to this periodic orbit coincides with the derivative from the right that is again B, and the same results follows.

(ii) The proof of this item is similar and we skip the details. In this case the Poincaré map is $P(\rho) = \rho/B + D$. □

Remark 2.2. Notice that the simple case $a(t) \equiv 0$ can be trivially integrated and has always the solution $x = 0$ as a periodic orbit and it is a (hyperbolic) limit cycles if and only if $B = 1$. Similarly, the case $b(t) \equiv 0$ has either no periodic solutions when $A \neq 1$, or otherwise, it has a continuum of periodic solutions.

Next proposition adapts the ideas of the theory of rotated vector fields ([15]) to our type of non-autonomous differential equations.

Proposition 2.3. *Consider the 1-parameter family of $\mathcal{C}^0$ differential equations of the form*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = S(x,t) + k, \tag{7}$$

with $k, x, t \in \mathbb{R}$, that are 1-periodic in the variable t. Let γ be a periodic orbit of (7) when $k = K$. Then the following statements hold:

(i) *Let Γ be a periodic orbit of of (7) when $k \neq K$, then $\Gamma \cap \gamma = \emptyset$.*
(ii) *If γ is an attractive limit cycle then, when $k \gtrsim K$ (resp. $k \lesssim K$), it moves up (resp. down) to another attractive limit cycle.*
(iii) *If γ is a repulsive limit cycle then, when $k \gtrsim K$ (resp. $k \lesssim K$), it moves down (resp. up) to another repulsive limit cycle.*
(iv) *If γ is semistable then, when $k \approx K, k \neq K$, it disappears when k increases and breaks into an attractive and a repulsive limit cycle when k decreases, or viceversa.*

Proof. This differential equation can be seen as the autonomous differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}s} = S(x,t) + k, \quad \frac{\mathrm{d}t}{\mathrm{d}s} = 1,$$

on the cylinder. Then the angle of its associated vector field $X_k(x,t) = (S(x,t) + k, 1)$ varies monotonically with k and all the properties of the families of rotated vector fields on the plane ([15]) can be adapted to this setting. In particular the properties given in the statement hold. □

We also will need the following result proved in [2].

Proposition 2.4. ([2]) *Consider the scalar piecewise differential equation*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = S(x,t) = \begin{cases} f(x,t), & x \geq 0, \\ g(x,t), & x \leq 0, \end{cases}$$

where $f, g : \mathbb{R}^2 \to \mathbb{R}$ are T-periodic with respect to t, f, g are $\mathcal{C}^1$-functions, $f(t,0) = g(t,0) = c(t)$. Suppose that c has finitely many zeros in $[0,T]$. Then its Poincaré map is a $\mathcal{C}^1$-function and

$$P'(\rho) = \exp\left(\int_0^T \frac{\partial S}{\partial x}(u(t,0,\rho),t)\,\mathrm{d}t\right),$$

where

$$\frac{\partial S}{\partial x}(x,t) = \begin{cases} \dfrac{\partial f}{\partial x}(x,t), & x \geq 0, \\ \dfrac{\partial g}{\partial x}(x,t), & x \leq 0. \end{cases}$$

We will use the following corollary:

Corollary 2.5. *Let $P(x)$ be the Poincaré map associated to* (4)*. If $a(t)$ has finitely many zeros, then it is of class $\mathcal{C}^1$ and*

$$P'(\rho) = \exp\left(\int_0^1 \chi(\rho,t)\,\mathrm{d}t\right), \tag{8}$$

where

$$\chi(\rho,t) = \begin{cases} b(t), & \text{if } u(t,0,\rho) \geq 0, \\ -b(t), & \text{if } u(t,0,\rho) \leq 0. \end{cases} \tag{9}$$

We end this section by a simple, but useful result on non-existence of periodic orbits.

Proposition 2.6. *Consider the differential equation* (4) *and assume that $b(t) \not\equiv 0$ and does not change sign. Then if $a(t) \not\equiv 0$ and $\operatorname{sign}(b)\int_0^1 a(t)\,\mathrm{d}t \geq 0$ it has not periodic orbits.*

Proof. We prove the result by contradiction. Assume that the differential equation has a periodic orbit $x = u(t)$. Then it holds that $\int_0^1 u'(t)\,\mathrm{d}t = u(1) - u(0) = 0$. Moreover, since $a(t) \not\equiv 0$, we know that $u(t) \not\equiv 0$. Hence

$$0 = \int_0^1 u'(t)\,\mathrm{d}t = \int_0^1 a(t)\,\mathrm{d}t + \int_0^1 b(t)|u(t)|\,\mathrm{d}t,$$

or, equivalently, $0 \neq \int_0^1 b(t)|u(t)|\,\mathrm{d}t = -\int_0^1 a(t)\,\mathrm{d}t$. In consequence, $\operatorname{sign}(b)\int_0^1 a(t)\,\mathrm{d}t < 0$ in contradiction with our hypothesis. □

3. Proof of Theorem 1.1

Proof. (a) First we observe that there are no crossing periodic orbits. This is so, simply because the line $\mathcal{L} = \{x = 0\}$ is without contact by the flow of (4) because $\mathrm{d}x/\mathrm{d}t|_{x=0} = a(t)$ that does not change sign. Hence all periodic orbits are either positive and negative and by Proposition 2.1 the upper bound for the number of limit cycles in each of the regions $\{x \geq 0\}$ or $\{x \leq 0\}$ is one as we wanted to prove. Their hyperbolicity is also proved in that proposition. Moreover, the positive (resp. negative) limit cycle exists if $B \neq 1$ and $C/(1-B) > 0$ (resp. $BD/(B-1) < 0$), where B, C and D are given in (6). Since $\operatorname{sign}(a) = \operatorname{sign}(C) = \operatorname{sign}(D)$ and $B > 0$, it follows that either both limit cycles exist simultaneously when $(1-B)\operatorname{sign}(a) > 0$ and otherwise, none of them exists. The result follows because $\operatorname{sign}(1-B) = -\operatorname{sign}(\int_0^1 b(t)\,\mathrm{d}t)$.

(b) If $b(t) \leq 0$, then by the change $y = -x$, equation (4) can transform to $\mathrm{d}y/\mathrm{d}t = -a(t) + \bar{b}(t)|y|$, where $\bar{b}(t) = -b(t) \geq 0$. Therefore, without loss of generality, we assume $b(t) \geq 0$ in the rest of this proof.

Let $u(t, t_0, x_0)$ be the solution of (4) with the initial condition $u(t_0, t_0, x_0) = x_0$. Assume that $a(t)$ has n zeros in $[0,1]$. Inspired by the proof of Proposition 2.4 of [2], denote by $\tau_1 < \tau_2 < \cdots < \tau_n$ the zeros of $a(t)$ in $[0,1]$. For each $\tau_i, i = 1, ...n$, set $x_i = u(0, \tau_i, 0), x_{n+1} = 0, x_{n+2} = u(0,1,0)$. One can reorder the initial conditions x_i so that

$$-\infty = x_0 < x_1 < ... < x_r < x_{r+1} = +\infty,\ 1 \leq r \leq n+2.$$

We also rename, accordingly the values τ_j associated to these new x_j. As an illustration see Figure 1.

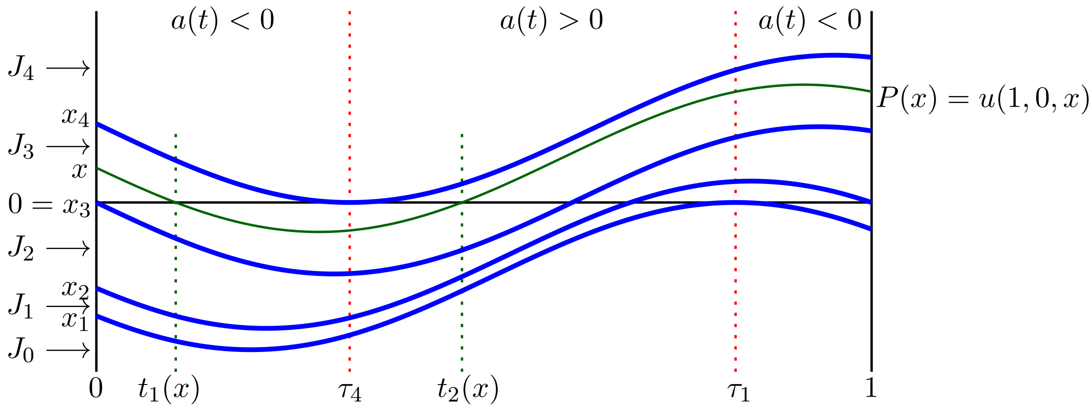

FIGURE 1. For a given differential equation (4), example of the values x_i and τ_i. Also, for a particular solution $u(t,0,x)$, with $n=2$ and $r=4$, we show the values $t_1(x)$ and $t_2(x)$. Notice that for x in each of the intervals J_i the number of zeroes of $u(t,0,x)$ is constant. In particular, for $x\in J_3$, this number is $k_3=2$.

By definition of x_i and the uniqueness of solution of equation (4), we know that $u(t,0,x_i)=u(t,\tau_i,0)$. Hence $u(t,0,x_i)$ has a zero at $t=\tau_i$. It follows from (4) that

$$\frac{\partial}{\partial t}u(\tau_i,0,x_i)=a(\tau_i)+b(\tau_i)|u(\tau_i,0,x_i)|=a(\tau_i)=0.$$

Let us study the functions $P(x)$ and $H(x)$ on each of the intervals $J_i:=(x_i,x_{i+1})$, $i=0,1,\ldots,r$. Because of the above construction, the number of zeros of each solution $t\to u(t,0,x)$ for each $x\in J_i$ is $0\le k_i\le n+1$ and only depends on i, see again [2]. Fix one of these intervals, say J_i with $k_i\ge 1$. When $k_i\ge 1$, for shortness set $\ell=k_i$ and denote all these zeroes by $t_1(x)$, $t_2(x),\cdots,t_\ell(x)$, and $t_j(x)<t_{j+1}(x)$, $j=1,2,\cdots,\ell-1$. In fact all them are simple zeros and all the functions $t_m(x)$ are of class $\mathcal{C}^1$ for $x\in J_i$. This is so, because each one of them is defined implicitly by the equation $u(t_m(x),0,x)=0$ and since $\partial u(t_m(x),0,x)/\partial t=a(t_m(x))\neq 0$, because $x\in J_i$, it holds that

$$\frac{\partial}{\partial t}u(t_m(x),0,x)t_m'(x)+\frac{\partial}{\partial x}u(t_m(x),0,x)=0,$$

which implies

$$t_m'(x)=-\frac{\frac{\partial}{\partial x}u(t_m(x),0,x)}{\frac{\partial}{\partial t}u(t_m(x),0,x)}=-\frac{\frac{\partial}{\partial x}u(t_m(x),0,x)}{a(t_m(x))},$$

where notice that we have used that $\partial u(t_m(x),0,x)/\partial x>0$, fact proved also in [2] and consequence that $x\to u(t,0,x)$ is a flow and preserves orientation. In short, for all m, $\operatorname{sign}(t_m'(x))=-\operatorname{sign}(a(t_m(x)))$.

By using Corollary 2.5, let us compute the derivative of the Poincaré map at $x\in J_i$. First notice that since $\partial u(t,0,x)/\partial t|_{u=0}=a(t)$ and that since $t_m(x)$ and $t_{m+1}(x)$ are two consecutive simple zeros of $u(t,0,x)$, the monotonicity of $u(t,0,x)$ at $t=t_m(x)$ and $t=t_{m+1}(x)$ are different and $u_t(t_m(x),0,x)u_t(t_{m+1},0,x)=a(t_m(x))a(t_{m+1}(x))<0$, which implies $t_m'(x)t_{m+1}'(x)<0$. Set $\sigma_m=-\operatorname{sign}(a(t_m(x)))$. Then $\sigma_m=(-1)^{m-1}\sigma_1$.

Before approaching to the general situation, for the sake of clarity, let us compute $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^1\chi(s)\,\mathrm{d}s\right)$ for the particular value $x\in J_3$ and the particular situation given in

Figure 1, where $\sigma_1 = +1$, $\ell = 2$, $t_1'(x) > 0$ and $t_2'(x) < 0$. Notice that, by Corollary 2.5,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^1 \chi(s)\,\mathrm{d}s\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{t_1(x)} \chi(s)\,\mathrm{d}s + \int_{t_1(x)}^{t_2(x)} \chi(s)\,\mathrm{d}s + \int_{t_2(x)}^{1} \chi(s)\,\mathrm{d}s\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{t_1(x)} b(s)\,\mathrm{d}s - \int_{t_1(x)}^{t_2(x)} b(s)\,\mathrm{d}s + \int_{t_2(x)}^{1} b(s)\,\mathrm{d}s\right)\\ &= 2b(t_1(x))t_1'(x) - 2b(t_2(x))t_2'(x) > 0,\end{aligned}$$

because both terms, $t_1'(x)$ and $-t_2'(x) > 0$ are positive.

In general, it holds that

$$\begin{aligned}\frac{P''(x)}{P'(x)} &= \frac{\mathrm{d}}{\mathrm{d}x}\big(\ln(P'(x))\big) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^1 \chi(s)\,\mathrm{d}s\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sigma_1\int_0^{t_1(x)} b(s)\,\mathrm{d}s + \sum_{m=1}^{\ell-1}\sigma_{m+1}\int_{t_m(x)}^{t_{m+1}(x)} b(s)\,\mathrm{d}s + \sigma_{\ell+1}\int_{t_\ell(x)}^{1} b(s)\,\mathrm{d}s\right)\\ &= \sigma_1\Bigg(b(t_1(x))t_1'(x) + \sum_{m=1}^{\ell-1}(-1)^m\big(b(t_{m+1}(x))t_{m+1}'(x) - b(t_m(x))t_m'(x)\big)\\ &\qquad\qquad + (-1)^{\ell-1}b(t_\ell(x))t_\ell'(x)\Bigg)\\ &= 2\sigma_1\left(\sum_{m=1}^{\ell}(-1)^{m-1}b(t_m(x))t_m'(x)\right) > 0.\end{aligned}$$

Notice that the above quantity is positive because all terms have the same sign and $\sigma_1 b(t_1(x))t_1'(x) > 0$. This is so, because $b(t) \geq 0$, for each m, $t_m'(x)t_{m+1}'(x) < 0$, and also $\operatorname{sign}(t_m'(x)) = -\operatorname{sign}(a(t_m(x))) = \sigma_m$. In particular $\sigma_1 t_1'(x) > 0$.

The above equality implies

$$H''(x) = P''(x) = \exp\left(\int_0^1 \chi(s)\mathrm{d}s\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^1 \chi(s)\mathrm{d}s\right) > 0, \quad x \in J_i,$$

provided that $u(t, 0, x)$ has at least one zero in $[0, 1]$. Notice that in particular this implies that H' is increasing on J_i.

On the other hand, by (8) and (9), we have that $P' > 0$ and H' are constant if $x \in J_i$, where J_i is any of the intervals such that when $x \in J_i$ then $u(t, 0, x)$ has not zeros in $[0, 1]$. Even more, by the proof of Proposition 2.1, $H'(x) \equiv (1 - B)/B$, when $u(t, 0, x) < 0$ and $H'(x) \equiv B - 1$, when $u(t, 0, x) > 0$, where B is given in (6), and since we are assuming that $b(t) \geq 0$, $B > 1$. Furthermore, clearly $H''(x) = P''(x) = 0$ for x in these intervals J_i.

We recall that $H(x)$ and $P(x)$ are of class $\mathcal{C}^1$, see Proposition 2.4. Their second derivatives can be discontinuous at the points x_i, $i = 0, 1, \ldots, r$, but they are well defined on the intervals J_i. As a simple example, see the map P for the differential equation $\mathrm{d}x/\mathrm{d}t = -1 + |x|$, in Section 4.1. In short, from all the above properties we get that in general $H'(x)$ is negative and constant in J_0, positive and constant in J_r, strictly increasing in $\mathbb{R} \setminus (J_1 \cup J_r)$, and continuous. As a consequence $H'(x)$ has at most one zero, and by Rolle's theorem $H(x)$ has at most two zeros, as we wanted to prove. This fact is equivalent to say that (4) has at most two limit cycles. Because of our proof we also know that two is the maximum number of limit cycles, taking into account their multiplicities.

That the limit cycle can exist only when $\text{sign}(b)\int_0^1 a(t)\,\mathrm{d}t < 0$ is a consequence of Proposition 2.6.

Let us prove that the total number of limit cycles is either 0 or 2. Consider the straight lines $x = \pm R$, $R > 0$. Since $b(t) \geq 0$ (and not identically zero), for R big enough both lines are without contact by the flow the differential equation. In particular, this implies that the number of zeros of H must be even, as this result follows.

To end the proof we only need to show that examples with two limit cycles do exist and study all their possible configurations. Recall that in Proposition 2.1 it is proved that there are never two positive (or two negative) limit cycles. Hence we know that the only possible configurations are $[1, 0, 1]$, $[1, 1, 0]$, $[0, 1, 1]$ and $[0, 2, 0]$. In next section we prove that all them exist. Hence, the theorem follows. □

4. Examples of all possible configurations

In this section we collect several examples to illustrate the different configurations of limit cycles of the differential equation considered in Theorem 1.1.

4.1. **Examples with configuration** $[1, 0, 1]$**.** The differential equation

$$\frac{dx}{\mathrm{d}t} = -1 + |x| \tag{10}$$

has two limit cycles $x = 1$ and $x = -1$ and both $a(t) \equiv -1$ and $b(t) \equiv 1$ do not change their signs.

In fact (10) is a very simple example for which is not difficult to get the Poincaré map explicitly. It is

$$P(\rho) = \begin{cases} \dfrac{\rho + 1}{\mathrm{e}} - 1, & \rho \leq 0, \\ \dfrac{1}{\mathrm{e}(1-\rho)} - 1, & 0 \leq \rho \leq 1 - 1/\mathrm{e}, \\ \mathrm{e}(\rho - 1) + 1, & \rho \geq 1 - 1/\mathrm{e}. \end{cases}$$

From the above expressions it is easy to see that $\rho = \pm 1$ are the only two fixed point of P, that correspond to the two limit cycles $x = \pm 1$. Moreover, as it is proved in [2] for the general differential equation (4), P' is continuous, $P'(-1) = 1/\mathrm{e} < 1$ and $P'(1) = \mathrm{e} > 1$. Hence, both limit cycles are hyperbolic and with different stabilities. Finally, if $H = P - \mathrm{Id}$ is the displacement map, $H'' = 0$ outside $(-1, 1)$ and $H'' > 0$ on the interval $(-1, 1)$, as it is proved in Section 3 when b does not change sign. Notice that P'' and H'' are discontinuous.

In fact, imbedding the differential equation (10) into the 1-parameter family of differential equations,

$$\frac{dx}{\mathrm{d}t} = -1 + k + |x|,$$

we have a nice illustration of the results proved in Proposition 2.3. For $k < 1$ it has two hyperbolic limit cycles, $x = \pm(1 - k)$, that collapse into the double (semistable) limit cycle $x = 0$ when $k = 1$. For the rest of values of k the equation has not periodic orbits.

Although the limit cycles of (10) are explicit and simple, it is not difficult to construct examples under each of the hypotheses of Theorem 1.1, with two limit cycles (one positive and one negative), which in general have much more involved expressions. For instance we can consider

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -1 + \varepsilon f(t) + (1 + \varepsilon g(t))|x|,$$

where f and g are arbitrary 1-periodic functions of $\mathcal{C}^1$ and ε is small enough. The two limit cycles exist because for $\varepsilon = 0$ they exist and are both hyperbolic and, as a consequence, for ε small enough and near each of one them, a hyperbolic limit cycle remains. Moreover, for suitable small ε it is clear that both $-1 + \varepsilon f(t)$ and $1 + \varepsilon g(t)$ do not change sign.

4.2. **Examples with configuration** $[0, 2, 0]$**.** It is clear that if we impose that a 1-periodic differentiable function $x = u(t)$ is solution of a differential equation (4) we get that $u'(t) = a(t) + b(t)|u(t)|$. Hence, for any 1-periodic function b it holds that $x = u(t)$ is a 1-periodic solution of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u'(t) - b(t)|u(t)| + b(t)|x|.$$

When $u(t) > 0$ or $u(t) < 0$ the above differential equation is of class $\mathcal{C}^1$ and so, is of the type of equations considered in this paper. Here we are more interested to construct $\mathcal{C}^1$ differential equations (4) for which $x = u(t)$ is a crossing limit cycle. To force that $b(t)|u(t)|$ is of class $\mathcal{C}^1$, and that we are under the hypotheses of Theorem 1.1, a natural way is to assume that all zeros of u are simple, and moreover to choose $b = u^2$. Then we have the $\mathcal{C}^1$ differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u'(t) - u^2(t)|u(t)| + u^2(t)|x|.$$

To simplify the computations, take u such that $u(0) = 1$, positive for $t \in [0, \tau_1) \cup (\tau_2, 1]$, negative for $t \in (\tau_1, \tau_2)$ and with simple zeros at $t = \tau_1$ and $t = \tau_2$. Then, $\rho = 1$ is the initial condition of the periodic orbit, and by Corollary 2.5, the derivative of the Poincaré map is

$$P'(1) = \exp\left(\int_0^{\tau_1} u^2(t)\,\mathrm{d}t - \int_{\tau_1}^{\tau_2} u^2(t)\,\mathrm{d}t + \int_{\tau_2}^{1} u^2(t)\,\mathrm{d}t\right).$$

Therefore it is obvious that we can choose $u(t)$ such that $P'(1)$ takes any positive value, obtaining in particular cases where $x = u(t)$ is a hyperbolic and stable, or unstable, crossing limit cycle. Because we have proved in Theorem 1.1 that when there are limit cycles its exact number is two, we know that in these situations the differential equation has another limit cycle. We do not know if it is of crossing type or not.

For the case where $P'(1) = 1$ (for instance this happens taking $u(t) = \cos(2\pi t)$, and then $\tau_1 = 1/4, \tau_2 = 3/4$) we know from the proof of Theorem 1.1 that $P''(1) > 0$ and hence $x = u(t)$ is a double (semistable) crossing limit cycle. Therefore, if we consider the 1-parameter family of differential equations,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = k - 2\pi\sin(2\pi t) - \cos^2(2\pi t)|\cos(2\pi t)| + \cos^2(2\pi t)|x|, \tag{11}$$

it holds that:

- It has only a limit cycle, that is double and of crossing type when $k = 0$.
- If $k < 0$ and $|k|$ small enough, it has exactly two limit cycles, that are hyperbolic, of crossing type and with different stabilities.

In fact, in the second item, the existence of both limit cycles is a consequence of Proposition 2.3 and all the other facts, consequence of Theorem 1.1. Notice that differential equations (11) prove that the two possibilities with all limit cycles of crossing type are realizable for (4) under the hypotheses of item (b) of Theorem 1.1.

4.3. **Examples with configurations** $[0,1,1]$**,** $[1,1,0]$ **and** $[1,0,1]$**.** It is clear that if a differential equation (4) has configuration $[0,1,1]$, by changing x by $-x$ we have another one of the same form but with configuration $[1,1,0]$. Hence we focus our attention to find an example with configuration $[0,1,1]$.

To construct our example, we start imposing that $u(t) \geq 0$ is a periodic orbit of the differential equation (4) with $b=1$. Then it writes as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u'(t) - u(t) + |x|. \tag{12}$$

We fix $u(t) = \sin^4(2\pi t) + p$, with $p \geq 0$. Then, the solutions of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u'(t) - u(t) + x.$$

are

$$x(t,0,x_0) = u(t) + (x_0 - p)\exp(t) = \sin^4(2\pi t) + p + (x_0 - p)\exp(t), \quad x_0 \in \mathbb{R},$$

and clearly $x = u(t) = x(t,0,p)$ is a positive limit cycle. Hence its limit cycle configuration is $[k,\ell,1]$ with $k+\ell=1$, because from Theorem 1.1 we know that when b does not vanish and the differential equation (4) has some limit cycle, it has exactly 2 and, moreover, that there are never two positive or two negative limit cycles.

We claim that when $p \in [0,P)$, with $P \approx 0.24823$, where P is the positive root of a polynomial of degree 4 given below,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u'(t) - u(t) - x, \tag{13}$$

has not negative periodic orbits.

Assuming this claim, for these values of p, the differential equation (12) has the positive hyperbolic limit cycle $u(t)$, no negative limit cycle, and a crossing limit cycle, that is it has the limit cycle configuration $[0,1,1]$, as we wanted to show. This is so, because if (12) would have a negative periodic orbit it also would be a negative periodic orbit of (13).

Let us prove the claim. We only need to find the limit cycle of (13) and prove that it is not negative. After some tedious computations we get it and it is $x = v_p(t)$, where

$$\begin{aligned} v_p(t) &= \frac{1}{2(16\pi^2+1)}\big(8\pi\sin(4\pi t) - (16\pi^2-1)\cos(4\pi t)\big) \\ &= \frac{1}{8(64\pi^2+1)}\big((64\pi^2-1)\cos(8\pi t) - 16\pi\sin(8\pi t)\big) - \frac{3}{8} - p. \end{aligned}$$

To control the sign of $v_p(t)$ we will use the usual trick to reduce the problem to a polynomial one. This can be done by introducing a rational parameterization of this function, see for instance [9] for more examples of use of rational parameterizations. Let s be such that

$$\sin(4\pi t) = \frac{2s}{1+s^2}, \quad \cos(4\pi t) = \frac{1-s^2}{1+s^2}.$$

Since $\sin(8\pi t) = 2\sin(4\pi t)\cos(4\pi t)$ and $\cos(8\pi t) = 2\cos^2(4\pi t) - 1$, it holds that

$$v_p(t(s)) = \frac{-V_p(s)}{(1+s^2)^2(16\pi^2+1)(64\pi^2+1)},$$

where

$$\begin{aligned} V_p(s) = {} & \left(1024\pi^4 p - 256\pi^4 + 80\pi^2 p + 48\pi^2 + p + 1\right) s^4 - 16\pi\left(40\pi^2+1\right) s^3 \\ & + 2\left(16\pi^2+1\right)\left(64\pi^2 p + 48\pi^2 + p\right) s^2 - 384\pi^3 s \\ & + 1024\pi^4 p + 768\pi^4 + 80\pi^2 p + p. \end{aligned}$$

It is easy to see that the polynomial of degree 4, $V_0(s)$ has two simple real roots. Therefore $v_0(t)$ changes sign as we wanted to see. In fact, this also can be seen directly because $v_0(0)v_0(1/4) < 0$. We need the rational parameterization to know until which value of p the same property is satisfied. To prove that the same holds for all $p \in [0, P)$ we use the method detailed in the appendix of [7, App. II]. In a few words, these two real roots remain until a value of p, say $p = P$, for which they collide into a double zero. This value is one of the values such that V_p has a multiple zero of V_p. By the properties of the resultant this value of p has to be a zero of

$$R(q) = \operatorname{Res}(V_p(s), V_p'(s), s) = 256(16\pi^2+1)^2(64\pi^2+1)^4 R_1(q)R_4(q),$$

where Res denotes the resultant of both polynomials, see again [**?**] for more details of how to compute it. Here,

$$\begin{aligned} R_1(q) = {} & \left(16\pi^2+1\right)\left(64\pi^2+1\right) p - 256\pi^4 + 48\pi^2 + 1, \\ R_4(q) = {} & \left(64\pi^2+1\right)^2 \left(16\pi^2+1\right)^4 p^4 \\ & + \left(64\pi^2+1\right)\left(32768\pi^6 + 2304\pi^4 + 96\pi^2 + 1\right)\left(16\pi^2+1\right)^2 p^3 \\ & + 18432\pi^6\left(64\pi^2+1\right)\left(16\pi^2+1\right)^2 p^2 - 28311552\pi^{12}. \end{aligned}$$

The polynomial R_1 has only a simple and explicit real root $p_1 = (256\pi^4 - 48\pi^2 - 1)/((16\pi^2+1)(64\pi^2+1)) \approx 0.24331$. The polynomial R_4 has two real roots, a negative one, $p_2 \approx -0.74410$, and another one positive, say $p_3 \approx 0.24823$. Clearly, we are only interested on values of $p > 0$, because otherwise $u(t)$ would change sign. To see that the value $p = p_1$ is not interesting for our purposes it suffices to see that for a given fixed $\overline{p} \in (0, p_1)$ and another one in $\widehat{p} \in (p_1, p_3)$ it holds that $v_{\overline{p}}(t)$ and $v_{\widehat{p}}(t)$ do change sign in $[0, 1]$. Hence, the value that gives the desired property is $p = p_3 = P$, which is the value given above. Because R_4 has degree 4, the value P can be obtained in closed algebraic form.

Finally, notice that the limit cycle configuration for differential equation (12) with $u(t) = \sin^4(2\pi t) + p$, is $[0, 1, 1]$ when $p \in [0, P)$ and $[1, 0, 1]$ when $p \geq P$.

Acknowledgements

The authors wants to express their thanks to the referee for his/her useful suggestions and valuable comments. The first author is supported by Spanish State Research Agency, through the projects PID2019-104658GB-I00 grant and the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M) and by the grant 2017-SGR-1617 from AGAUR, Generalitat de Catalunya. The second author is supported by the NSF of China (No. 11971495) Guangdong-Hong Kong-Macau Applied Math Center (No. 2020B1515310014) and Guangdong Basic and Applied Basic Research Foundation (No. 2022A1515012105).

Competing interests declaration

The authors declare none.

References

[1] D. M. Benardete, V. W. Noonburg and B. Pollina. *Qualitative tools for studying periodic solutions and bifurcations as applied to the periodically harvested logistic equation,* Amer. Math. Monthly **115** (2008), 202–219.

[2] J. Bravo, M. Fernàndez and A. Tineo, *Periodic solutions of a periodic scalar piecewise ODE*, Communication on Pure and Applied Analysis **6**(2007), 213–228.

[3] J. Devlin, N. G. Lloyd and J. M. Pearson. *Cubic systems and Abel equations,* J. Differential Equations **147** (1998), 435–454.

[4] M. Chamberland and A. Gasull, *Chini equations and isochronous centers in three-dimensional differential systems.* Qual. Theory Dyn. Syst. **9** (2010), 29–38.

[5] B. Coll, A. Gasull and R. Prohens, *Simple non-autonomous differential equations with many limit cycles*, Communication on Applied Nonlinear Analysis **15** (2008), 29-34.

[6] E. Fossas-Colet and J.M. Olm-Miras. *Asymptotic tracking in dc-to-dc nonlinear power converters*, Discrete and Continuous Dynamical Systems-Series B **2** (2002), 295–307.

[7] J. D. García-Saldaña, A. Gasull and H. Giacomini. *Bifurcation values for a family of planar vector fields of degree five*, Discrete Contin. Dyn. Syst. **35** (2015), 669–701.

[8] A. Gasull and A. Guillamon. *Limit cycles for generalized Abel equations* Internat. J. Bifur. Chaos Appl. Sci. Engrg. **16** (2006), 3737–3745.

[9] A. Gasull, J. T. Lázaro and J. Torregrosa. *Rational parameterizations approach for solving equations in some dynamical systems problems*, Qual. Theory Dyn. Syst. **18** (2019), 583–602.

[10] A. Gasull and J. Llibre. *Limit cycles for a class of Abel equations,* SIAM J. Math. Anal. **21** (1990), 1235–1244.

[11] A. Lins Neto. *On the number of solutions of the equation* $dx/dt = \sum_{j=0}^{n} a_j(t)x^j, 0 \le t \le 1$ *for which* $x(0) = x(1)$*,* Inv. Math. **59** (1980), 67–76.

[12] N. G. Lloyd. *The number of periodic solutions of the equation* $\dot z = z^N + p_1(t)\, z^{N-1} + \cdots + p_N(t)$*,* Proc. London Math. Soc. **27** (1973), 667–700.

[13] J. Mawhin, *First order ordinary differential equations with several periodic solutions*, Z. Angew. Math. Phys. **38** (1987), no. 2, 257-265.

[14] A. A. Panov. *The number of periodic solutions of polynomial differential equations,* Math. Notes **64** (1998), 622–628.

[15] L.M. Perko, *Rotated vector fields and the global behavior of limit cycles for a class of quadratic systems in the plane* J. Differential Equations **18** (1975), 63–86.

[16] V.A. Pliss. Non local problems of the theory of oscillations, Academic Press, New York, 1966.

Dept. de Matemàtiques, Universitat Autònoma de Barcelona, Edifici C 08193 Cerdanyola del Vallès (Barcelona), Spain; and Centre de Recerca Matemàtica, Edifici Cc, Campus de Bellaterra, 08193 Cerdanyola del Vallès (Barcelona), Spain

Email address: gasull@mat.uab.cat

School of Mathematics (Zhuhai), Sun Yat-sen University, Zhuhai Campus, Zhuhai, 519082, China

Email address: mcszyl@mail.sysu.edu.cn