DISCRETE MELNIKOV FUNCTIONS

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Abstract. We consider non-autonomous N-periodic discrete dynamical systems of the form $r_{n+1} = F_n(r_n, \varepsilon)$, having when $\varepsilon = 0$ an open continuum of initial conditions such that the corresponding sequences are N-periodic. From the study of some variational equations of low order we obtain successive maps, that we call discrete Melnikov functions, such that the simple zeroes of the first one that is not identically zero control the initial conditions that persist as N-periodic sequences of the perturbed discrete dynamical system. We apply these results to several examples, including some Abel-type discrete dynamical systems and some non-autonomous perturbed globally periodic difference equations.

1. Introduction and statement of the main results

The interest on the study non-autonomous periodic discrete dynamical systems has been increasing in the last years, among other reasons, because they are good models for describing the dynamics of biological and ecological systems that vary periodically, either due to external disturbances or for effects of seasonality, see for instance [2, 11, 12, 13, 15, 16, 17] and the references therein.

Consider non-autonomous discrete dynamical systems of the form

$$r_{n+1} = f_n(r_n), \qquad r_n \in \mathbb{R}^d,\, n \in \mathbb{N}, \tag{1}$$

where $d \in \mathbb{N}^+$, and f_n is an N-periodic sequence of real smooth invertible such that $f_m = f_{m+N}$ for all $m \in \mathbb{N}$. Here, $f_n : \mathcal{U} \subset \mathbb{R}^d \to \mathcal{U}$ being $\mathcal{U}$ an open set of $\mathbb{R}^d$. Given an initial condition $r_0 = \rho \in \mathcal{U}$ we will denote by $r_n = \varphi_n(\rho)$ the sequence defined by (1). For convenience, for $n > 0$, we write $f_{n,n-1,\ldots,1,0} = f_n \circ f_{n-1} \cdots \circ f_1 \circ f_0$. Then, for $n > 0$,

$$r_n = \varphi_n(\rho) = f_{n-1,n-2,\ldots,1,0}(\rho). \tag{2}$$

It is well-known that given an N-periodic discrete dynamical system (1), it can be understood via the so called *composition map* $f_{N-1,N-2,\ldots,1,0}$. For instance, if all maps share a common fixed point, the nature of this steady state point can be studied through the nature of this fixed point for $f_{N-1,N-2,\ldots,1,0}$, see [2, 6]. Similarly, the attractor of a periodic discrete dynamical system (1) is the union of attractors of some composition maps, see [13, Thms. 3 and 6].

To find N-periodic solutions of (1) is equivalent to find solutions of $\varphi_N(\rho) = \rho$, or equivalently of the equation $f_{N-1,N-2,\ldots,1,0}(\rho) = \rho$. Usually, is not easy to deal with it. The main goal of this paper is to give an alternative and indirect mechanism to study this problem for a special class of discrete dynamical systems. More specifically, it is

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said that (1) is *globally N-periodic* in $\mathcal{U}$ when for all $r_0 = \rho \in \mathcal{U}$ it holds that $\varphi_N(\rho) = \rho$. We will consider N-periodic perturbed discrete dynamical systems of the form

$$r_{n+1} = F_n(r_n, \varepsilon) = f_n(r_n) + \varepsilon g_n(r_n) + \varepsilon^2 h_n(r_n) + O(\varepsilon^3), \quad n \in \mathbb{N}, \quad r \in \mathbb{R}^d, \qquad (3)$$

where for $\varepsilon = 0$ it is globally N-periodic and $\varepsilon \in \mathbb{R}$ is a small parameter. Here F_n are N-periodic and of class $\mathcal{C}^3(\mathcal{U} \times I)$, for I a small open interval containing 0.

In other words, we want to determined which of the continuum of periodic sequences of the unperturbed dynamical system (3) with $\varepsilon = 0$, remain as isolated periodic sequences of (3) when $\varepsilon \neq 0$ is small enough. With this aim we adapt to this setting the use of variational equations, that was introduced by Poincaré for studying similar questions for ordinary differential equations. As we will see, we obtain some functions of ρ, $M_i(\rho)$, for $i \in \mathbb{N}$ such that their simple zeroes give rise to the searched N-periodic sequences. These functions can be seen as a kind of Melnikov functions for discrete dynamical systems.

Let $\phi_n(\rho, \varepsilon)$ be the sequence solution of (3) such that $\phi_0(\rho, \varepsilon) \equiv \rho \in \mathcal{U}$. It can be written as

$$r_n = \phi_n(\rho, \varepsilon) = \varphi_n(\rho) + u_n(\rho)\varepsilon + v_n(\rho)\varepsilon^2 + O(\varepsilon^3), \qquad (4)$$

for some sequences $u_n(\rho), v_n(\rho)$ such that $u_0(\rho) \equiv v_0(\rho) \equiv 0$. We define $M_1(\rho) = u_N(\rho)$ and call it the first discrete Melnikov function for the discrete dynamical system (3). When $M_1 = 0$, then $M_2(\rho) = v_N(\rho)$ is the second discrete Melnikov function. In a sufficiently enough smooth setting, the j-th discrete Melnikov can be similarly defined when all previous Melnikov functions identically vanish.

Our first result gives a closed expression for M_1 and M_2. Given a smooth function $F\colon \mathbb{R}^d \to \mathbb{R}^d$ we will denote by DF its Jacobian matrix and D^2F its Hessian matrix. We will use the following notation: given $x \in \mathbb{R}^d$ and a $d \times d$ matrix $C = (c_{i,j})$ then $x^T C x = \sum_i \sum_j c_{i,j} x_i x_j$, where x^T is the transpose vector of x.

Theorem 1. *Let $r_n = \phi_n(\rho, \varepsilon)$ be the sequence* (4) *generated by* (3). *Assume that $\varphi_N(\rho) \equiv \rho$ for all $\rho \in U$ and all f_n are invertible in $\mathcal{U}$. Then the first two discrete Melnikov functions associated to* (3) *are*

$$M_1(\rho) = \sum_{j=0}^{N-1} \big(D\varphi_j(\rho)\big)^{-1} g_j(\varphi_j(\rho)),$$

$$M_2(\rho) = \sum_{j=0}^{N-1} \big(D\varphi_j(\rho)\big)^{-1} \Big(\frac{1}{2} u_j^T(\rho) D^2 f_j(\varphi_j(\rho)) u_j(\rho) + Dg_j(\varphi_j(\rho)) u_j(\rho) + h_j(\varphi_j(\rho))\Big),$$

where $u_0(\rho) = 0$ and for $j > 0$,

$$u_j(\rho) = D\varphi_j(\rho) \left(\sum_{k=0}^{j-1} \big(D\varphi_k(\rho)\big)^{-1} g_k(\varphi_k(\rho)) \right).$$

Moreover, for ε sufficiently small and any $j \in \{1, 2\}$, each simple zero of M_j, $\rho = \rho^* \in \mathcal{U}$ gives rise to an isolated N-periodic sequence of (3). *This periodic sequence has an initial condition $r_0 = R(\varepsilon)$ such that $\lim_{\varepsilon \to 0} R(\varepsilon) = \rho^*$.*

After looking to our proof it will clear that the same approach allows to get similar expressions for some more M_j, $j > 2$. We do not include their expressions for the sake of shortness. In the particular case $f_n(r) \equiv r$ for all n, then $\varphi_n(\rho) \equiv \rho$ and the expressions of M_i are much simpler. In the following proposition we give them for $i = 1, \dots, 3$.

Proposition 2. *Consider the class $\mathcal{C}^4$, N-periodic discrete dynamical system*

$$r_{n+1} = r_n + \varepsilon g_n(r_n) + \varepsilon^2 h_n(r_n) + \varepsilon^3 k_n(r_n) + O(\varepsilon^4), \quad n \in \mathbb{N}, \quad r \in \mathcal{U} \subset \mathbb{R}^d,$$

with ε small enough. Then

$$M_1(\rho) = \sum_{j=0}^{N-1} g_j(\rho),$$
$$M_2(\rho) = \sum_{j=0}^{N-1} \Big(Dg_j(\rho) u_j(\rho) + h_j(\rho) \Big),$$
$$M_3(\rho) = \sum_{j=0}^{N-1} \Big(\frac{1}{2} u_j^T(\rho) D^2 g_j(\rho) u_j(\rho) + Dg_j(\rho) v_j(\rho) + Dh_j(\rho) u_j(\rho) + k_j(\rho) \Big),$$

where $u_0(\rho) = v_0(\rho) = 0$ and for $j > 0$,

$$u_j(\rho) = \sum_{k=0}^{j-1} g_k(\rho) \quad \textit{and} \quad v_j(\rho) = \sum_{k=0}^{j-1} \Big(Dg_k(\rho) u_k(\rho) + h_k(\rho) \Big).$$

Moreover, including also M_3, the same conclusions that in Theorem 1 hold.

In fact, this paper can be seen as an extension of the results and ideas introduced in [9] for studying isolated periodic solutions of T-periodic non-autonomous differential equations to N-periodic discrete dynamical systems. As an example of the similarities of the results obtained we recall the first Melnikov function given there when studying the T-periodic in time differential equation

$$\frac{dr}{dt} = f(t, r) + \varepsilon g(t, r) + O(\varepsilon^2).$$

It is

$$M_1(\rho) = \int_0^T \big(D_\rho \varphi(t, \rho) \big)^{-1} g(t, \varphi(t, \rho))\, dt,$$

where $\varphi(t, \rho)$ is the solution of the above differential equation when $\varepsilon = 0$ and such that $\varphi(0, \rho) \equiv \rho$. Compare with the expression of M_1 given in Theorem 1. It is clear, that both could be obtained simultaneously by using all the machinery of time scales, see again [3] and [4, 5].

The proof of Theorem 1 is given in Section 2. The proof of Proposition 2 is similar and it is omitted for the sake of shortness, see Section 2.1. In Section 3 we present several examples of application, including Abel type discrete dynamical systems and some perturbed globally periodic difference equations.

2. Proof of the main results

Proof of Theorem 1. Write $\phi_n(\rho,\varepsilon) = \varphi_n(\rho) + \varepsilon p_n(\rho,\varepsilon)$, as in (4), where $p_n(\rho,\varepsilon) = u_n(\rho) + \varepsilon v_n(\rho) + O(\varepsilon^2)$. By imposing that $r_n = \phi_n(\rho,\varepsilon)$ satisfies (3) we get the identity

$$\begin{aligned}\phi_{n+1}(\rho,\varepsilon) =& F_n\big(\phi_n(\rho,\varepsilon),\varepsilon\big) = f_n\big(\phi_n(\rho,\varepsilon)\big) + g_n\big(\phi_n(\rho,\varepsilon)\big)\varepsilon + h_n\big(\phi_n(\rho,\varepsilon)\big)\varepsilon^2 + O(\varepsilon^3)\\ =& f_n(\varphi_n(\rho)) + \varepsilon D_r f_n(\varphi_n(\rho))p_n(\rho,\varepsilon) + \frac{\varepsilon^2}{2}p_n^T(\rho,\varepsilon)D_{rr}f_n(\varphi_n(\rho))p_n(\rho,\varepsilon) + O(\varepsilon^3)\\ &+ \Big(g_n(\varphi_n(\rho)) + \varepsilon D_r g_n(\varphi_n(\rho))p_n(\rho,\varepsilon) + O(\varepsilon^2)\Big)\varepsilon\\ &+ \big(h_n(\varphi_n(\rho)) + O(\varepsilon)\big)\varepsilon^2 + O(\varepsilon^3).\end{aligned}$$

By collecting the terms in ε^0, ε and ε^2 into the above expression, we obtain the unperturbed difference equation (1) and two new linear N-periodic difference equations (the variational equations of first and second order) with unknowns $u_n(\rho)$ and $v_n(\rho)$ and initial conditions $u_0(\rho) \equiv v_0(\rho) \equiv 0$,

$$\begin{aligned}\varphi_{n+1}(\rho) &= f_n(\varphi_n(\rho)),\\ u_{n+1}(\rho) &= Df_n(\varphi_n(\rho))u_n(\rho) + g_n(\varphi_n(\rho)), \qquad (5)\\ v_{n+1}(\rho) &= Df_n(\varphi_n(\rho))v_n(\rho) + \frac{1}{2}u_n^T(\rho)D^2 f_n(\varphi_n(\rho))u_n(\rho)\\ &\quad + Dg_n(\varphi_n(\rho))u_n(\rho) + h_n(\varphi_n(\rho)). \qquad (6)\end{aligned}$$

Let us recall how to solve the non-autonomous linear difference equation

$$x_{n+1} = A_n x_n + b_n, \quad x_0 = 0,$$

where $x_n, b_n \in \mathbb{R}^d$ and A_n are $d\times d$ real matrices, see for instance [10]. By direct substitution, for $n > 0$,

$$x_n = \sum_{j=0}^{n-1} A_{n-1}A_{n-2}\cdots A_{j+2}A_{j+1}b_j,$$

where by the sake of notation, when $j = n-1$, the term in the above sum is simply b_{n-1}. If we assume that all A_j are invertible, then

$$x_n = A_{n-1}A_{n-2}\cdots A_1A_0\left(\sum_{j=0}^{n-1}\big(A_jA_{j-1}\cdots A_1A_0\big)^{-1}b_j\right).$$

Let us apply the above formula to (5). Notice that $A_j = Df_j(\varphi_j(\rho))$. Thus, by the chain rule,

$$\begin{aligned}A_{n-1}A_{n-2}\cdots A_1A_0 &= Df_{n-1}(\varphi_{n-1}(\rho))Df_{n-2}(\varphi_{n-2}(\rho))\cdots Df_1(\varphi_1(\rho))Df_0(\varphi_0(\rho))\\ &= Df_{n-1}(f_{n-2,n-3,\ldots,1,0}(\rho))Df_{n-2}((f_{n-3,n-4,\ldots,1,0}(\rho))\cdots Df_1(f_0(\rho))Df_0(\rho)\\ &= Df_{n-1,n-2,\ldots,1,0}(\rho) = D\varphi_n(\rho).\end{aligned}$$

As a consequence
$$u_n(\rho) = D\varphi_n(\rho)\left(\sum_{j=0}^{n-1}\big(D\varphi_j(\rho)\big)^{-1} g_j(\varphi_j(\rho))\right).$$
In particular for $n = N$, since $\varphi_N(\rho) \equiv \rho$,
$$M_1(\rho) = u_N(\rho) = \sum_{j=0}^{N-1}\big(D\varphi_j(\rho)\big)^{-1} g_j(\varphi_j(\rho)),$$
as we wanted to prove.

Notice that derivating with respect to ρ, $\varphi_{n+1}(\rho) = f_n(\varphi_n(\rho))$, and using that $\varphi_N(\rho) \equiv \rho$, we arrive to
$$D\varphi_{n+1}(\rho) = Df_n(\varphi_n(\rho))D\varphi_n(\rho), \quad \text{with} \quad D\varphi_0(\rho) = \mathrm{Id},$$
proving that $D\varphi_n(\rho)$ is a fundamental matrix of (5) and providing an equivalent way to obtain $u_n(\rho)$.

It is clear that doing similar computations we can obtain the solution of (6) and the expression of $M_2(\rho)$, given in the statement. We omit the details.

To end the proof, let us show that each simple zero of $M_1(\rho)$ gives rise, for ε sufficiently small, to an isolated N-periodic solution of system (3). From (4), we have that $\varphi_N(\rho,\varepsilon) = \rho + \varepsilon M_1(\rho) + O(\varepsilon^2)$ and let ρ^* be such that $M_1(\rho^*) = 0$ and $\det\big(DM_1(\rho^*)\big) \neq 0$. Then
$$\Pi(\rho,\varepsilon) = \frac{\varphi_N(\rho,\varepsilon) - \rho}{\varepsilon} = M_1(\rho) + O(\varepsilon).$$
We know that
$$\Pi(\rho^*, 0) = 0 \quad \text{and} \quad D\Pi(\rho, 0))\Big|_{\rho=\rho^*} = DM_1(\rho^*).$$
Then, since $\det\big(DM_1(\rho^*)\big) \neq 0$, from the implicit function theorem the result follows.

When $M_1(\rho) \equiv 0$ for we can apply the same argument to M_2 by considering
$$\widehat{\Pi}(\rho,\varepsilon) = \frac{\varphi_N(\rho,\varepsilon) - \rho}{\varepsilon^2} = M_2(\rho) + O(\varepsilon).$$
This completes the proof. □

2.1. Relation between Theorem 1 and Proposition 2.

Consider the discrete dynamical system (3). Let Φ_n be a family of invertible smooth diffeomorphism and define the new sequence $s_n, n \geq 0$, by the equation $r_n = \Phi_n(s_n)$. Then straightforward computations give that
$$\begin{aligned} s_{n+1} =& \Phi_{n+1}^{-1}\big(F_n(\Phi_n(s_n),\varepsilon)\big) = \Phi_{n+1}^{-1}\big(f_n(\Phi_n(s_n)) + \varepsilon g_n(\Phi_n(s_n)) + O(\varepsilon^2)\big) \\ =& \Phi_{n+1}^{-1}\big(f_n(\Phi_n(s_n)\big) + \varepsilon D\Phi_{n+1}^{-1}(f_n\big(\Phi_n(s_n)\big)g_n\big(\Phi_n(s_n)\big) + O(\varepsilon^2). \end{aligned}$$

If we take as special family of diffeomorphisms the solution of the unperturbed dynamical system, that is $\Phi_n = \varphi_n$, notice that by (2),
$$\begin{aligned} \Phi_{n+1}^{-1}\big(f_n(\Phi_n(s_n)\big) =& \varphi_{n+1}^{-1}\big(f_n(\varphi_n(s_n)\big) = (f_{n,n-1,\ldots,1,0})^{-1}(f_n(f_{n-1,n-2,\ldots,1,0}(s_n))) \\ =& (f_{n,n-1,\ldots,1,0})^{-1}(f_{n,n-1,\ldots,1,0}(s_n)) = s_n. \end{aligned}$$

Hence (3) can always be written as

$$s_{n+1} = s_n + \varepsilon D\varphi_{n+1}^{-1}\big(f_n\big(\varphi_n(s_n)\big)g_n\big(\varphi_n(s_n)\big) + O(\varepsilon^2)\big)$$

and Theorem 1 and Proposition 2 can be deduced one from the other.

3. Applications

3.1. **Abel type discrete dynamical systems.** It is well-know, see for instance [3], that the 1-dimensional discrete dynamical version of the Riccati differential equation $\frac{dr}{dt} = a(t) + b(t)r + c(t)r^2$ is

$$r_{n+1} = \frac{a_n + b_n r_n}{c_n + d_n r_n}.$$

The main reason is that in the two cases, the continuous and discrete flows are given by Möbius transformations. For this cause, both equations, assuming T-periodicity in the first case or that the four sequences are N-periodic in the second case, have at most 2 isolated T or N periodic solutions, respectively. In fact, both cases can be simultaneously studied by using the time scales setting, see again [3].

For similarity, we consider as a discrete version of the Abel differential equation $\frac{dr}{dt} = a(t) + b(t)r + c(t)r^2 + d(t)r^3$ the discrete dynamical system

$$r_{n+1} = \frac{a_n + b_n r_n}{c_n + d_n r_n + e_n r_n^2}. \tag{7}$$

In this section, as an application of Theorem 1, we reproduce some results of [3]. We prove that when the five sequences defining (7) are N-periodic there are examples of this dynamical system having at least $N-1$ isolated N-periodic orbits. As a consequence, in contrast of what happens for the Riccati case, there is no upper bound for the number of isolated periodic sequences generated by general discrete dynamical systems of the form (7). This result is a natural extension of a similar property that holds for T-periodic Abel differential equations, proved in the pioneering work [14].

We will prove for (7) the above result by studying the first order discrete Melnikov function M_1 associated to the particular family

$$r_{n+1} = \frac{r_n}{1 + d_n r_n + \varepsilon e_n r_n^2}, \tag{8}$$

with d_n and e_n, N-periodic sequences and such that $D_N = 0$, where for $k > 0$, $D_k = \sum_{j=0}^{k-1} d_j$, $D_0 = 0$ and ε is a small parameter.

When $\varepsilon = 0$, (8) is a Riccati equation and the sequences generated by it such that $r_0 = \rho$, are given by the Möbius maps

$$r_n = \varphi_n(\rho) = \frac{\rho}{1 + D_n\rho}, \quad n \geq 0.$$

Notice that this unperturbed dynamical system is globally periodic in a neighborhood of 0, because $\varphi_N(\rho) = \frac{\rho}{1+D_N\rho} = \rho$. Moreover,

$$r_{n+1} = F_n(r_n, \varepsilon) = \frac{r_n}{1 + d_n r_n + \varepsilon e_n r_n^2} = \frac{r_n}{1 + d_n r_n} - \varepsilon\frac{e_n r_n^3}{(1 + d_n r_n)^2} + O(\varepsilon^2).$$

Hence, in the notation of Theorem 1,

$$f_n(r)=\frac{r}{1+d_nr},\quad g_n(r)=-\frac{e_nr^3}{(1+d_nr)^2}.$$

Thus, since $\varphi_n'(\rho)=(1+D_n\rho)^{-2}$,

$$\begin{aligned}M_1(\rho)&=\sum_{j=0}^{N-1}\big(\varphi_j'(\rho)\big)^{-1}g_j(\varphi_j(\rho))=-\sum_{j=0}^{N-1}(1+D_j\rho)^2\,\frac{e_jr^3}{(1+d_jr)^2}\bigg|_{r=\varphi_j(\rho)}\\&=-\rho^3\sum_{j=0}^{N-1}e_j\frac{1+D_j\rho}{(1+D_{j+1}\rho)^2}.\end{aligned}$$

Therefore, the number of simple isolated zeroes of the above function in a suitable open interval $\mathcal{U}=(0,R)$ will give the same number of periodic sequences of (8), for ε small enough. Although it is not difficult to study this number (bellow we will recall a well-known procedure to have lower bounds for it) we prefer to take advantage of the idea introduced in Section 2.1 to simplify the problem. With this aim, and following the notation of that section we introduce the new variables s_n as $r_n=\Phi(s_n)=1/s_n$. Notice that here, $\Phi(s)=1/s$ is independent on n. With these variables (8) writes as

$$s_{n+1}=s_n+d_n+\frac{e_n}{s_n},\tag{9}$$

and the initial condition is $s_0=1/\rho=\varrho$. Using again the notation of Theorem 1 we obtain that

$$s_n=\varphi_n(\varrho)=\varrho+D_n,\quad f_n(s)=s+d_n\quad\text{and}\quad g_n(s)=e_n/s.$$

In these new variables, the discrete Melnikov function, say $\widehat{M_1}$, is

$$\widehat{M_1}(\varrho)=\sum_{j=0}^{N-1}\big(\varphi_j'(\varrho)\big)^{-1}g_j(\varphi_j(\varrho))=\sum_{j=0}^{N-1}e_j\frac{1}{\varrho+D_j}$$

and we are interested in its zeroes in a suitable neighborhood of infinity. Notice that M_1 and $\widehat{M_1}$ are quite similar, but different due to the action of the diffeomorphism Φ. We continue our study with the second one because it is simpler. It is convenient to use the following lemma, proved in [8]:

Lemma 3. *Let $h_j:\mathcal{U}\subset\mathbb{R}\to\mathbb{R}$, $j=0,1,\dots N-1$, be N linearly independent functions.*

(i) Given $N-1$ arbitrary values $x_j\in\mathcal{U}$, $j=1,2,\dots N$ there exist N constants c_j, $j=0,1,\dots,N-1$ such that $h(x)=\sum_{j=0}^{N-1}c_jh_j(x)$ is not the zero function and $h(x_j)=0$ for $j=1,2,\dots,N-1$.

(ii) Furthermore, if all h_j are analytic in $\mathcal{U}$ and there exits one of them that has constant sign in $\mathcal{U}$, it is possible to obtain an h of the above form such that it has at least $N-1$ simple zeroes in $\mathcal{U}$.

If we take all D_j, $j=0,1,\dots,N-1$, different it is not difficult to see that all the functions $h_j(x)=1/(x+D_j)$ are linearly independent. By applying Lemma 3 to $\widehat{M_1}$ we obtain that we can take suitable e_j such that the corresponding function has $N-1$ simple zeroes in a given neighborhood of infinity, as we wanted to prove. By Theorem 1, these zeroes give rise to the desired $N-1$ isolated N-periodic sequences for discrete dynamical systems of type (7).

3.2. **Polynomial type discrete dynamical systems.** Recall that by Proposition 2 the formulas of M_j are simpler when the unperturbed discrete dynamical system is $r_{n+1}=r_n$. Moreover, when $d=1$ and F is a polynomial in r of degree ℓ the computation of the discrete Melnikov functions simplify even more. Consider

$$r_{n+1}=F_n(r_n,\varepsilon)=r_n+\varepsilon g_n(r_n)+\varepsilon^2 h_n(r_n)+\varepsilon^3 k_n(r_n)+O(\varepsilon^4)$$

where all $g_n(r),h_n(r)$ and $k_n(r)$ are polynomials of degree ℓ and the three sequences are N-periodic.

By Proposition 2 we obtain that M_1,M_2 and M_3 are polynomials of degree at most ℓ, $2\ell-1$ and $3\ell-2$, respectively. These values are in consequence the maximum number of isolated N-periodic sequences that can be constructed with our approach. Let us discuss if these maximum values can be achieved.

It is clear that there are examples where M_1 has exactly degree ℓ and it has ℓ simple real roots, because $M_1(\rho)=\sum_{j=0}^{N-1} g_j(\rho)$ and all the g_j are polynomials in ρ.

To study M_2 recall that we have to assume that $M_1=0$. We write each polynomial g_j as $g_j(\rho)=\overline{g}_j r^\ell+Q_j(\rho)$, for some polynomials Q_j of degree at most $\ell-1$. Hence, in particular, $\sum_{j=0}^{N-1}\overline{g}_j=0$. Moreover, from the expression of M_2 we have that

$$M_2(\rho)=\sum_{j=0}^{N-1}\left(g_j'(\rho)\sum_{k=0}^{j-1}g_k(\rho)+h_j(\rho)\right)=\ell\sum_{j=0}^{N-1}\left(\overline{g}_j\sum_{k=0}^{j-1}\overline{g}_k\right)\rho^{2\ell-1}+R_{2\ell-2}(\rho),$$

where $R_{2\ell-2}$ is a polynomial of degree at most $2\ell-2$. Hence it is clear that the values $\overline{g}_j$ can be taken such that $M_1=0$ and M_2 has exactly degree $2\ell-1$. It can be seen that to have an example such that M_2 has exactly $2\ell-1$ real roots if suffices to consider g_j and h_j, $j=0,1,\ldots N-1$, of the form

$$\begin{aligned} g_j(r)=&\overline{g}_j r^\ell+c_{\ell-1}r^{\ell-1}+c_{\ell-2}r^{\ell-2}+\cdots+c_2r^2+c_1r+c_0,\\ h_j(r)=&d_\ell r^\ell+d_{\ell-1}r^{\ell-1}+d_{\ell-2}r^{\ell-2}+\cdots+d_2r^2+d_1r+d_0,\end{aligned}$$

for some suitable constants.

To study the actual degree of M_3 we need a preliminary result.

Lemma 4. *Given any sequence $f_j, j\in\mathbb{N}$ and $0<n\in\mathbb{N}$ it holds that*

$$\sum_{j=0}^{n-1}f_j^2+2\sum_{j=0}^{n-1}f_j\sum_{k=0}^{j-1}f_k=\left(\sum_{j=0}^{n-1}f_j\right)^2,$$

where by the way of notation $\sum_{k=0}^{-1}f_k=0$.

Proof. Consider an arbitrary sequence $F_j, j\in\mathbb{N}$. As usual, $\Delta F_j=F_{j+1}-F_j$. Clearly,

$$(\Delta F_j)^2+2F_j\Delta F_j=F_{j+1}^2-F_j^2.$$

Summing both sides of the above equality from $j = 0$ until $j = n-1$ and taking into account that the right-hand side is a telescopic sum, we obtain that

$$\sum_{j=0}^{n-1} (\Delta F_j)^2 + 2\sum_{j=0}^{n-1} F_j \Delta F_j = F_n^2 - F_0^2.$$

By considering $F_0 = 0$ and for $j > 0$, $F_j = \sum_{k=0}^{j-1} f_k$ we have that $\Delta F_j = f_j$. Replacing these values in the above equality we obtain the result of the statement. □

Recall that M_3 is only defined when $M_1 = M_2 = 0$. Then it holds that

$$\sum_{j=0}^{N-1} \overline{g}_j = 0 \quad \text{and} \quad \sum_{j=0}^{N-1} \left(\overline{g}_j \sum_{k=0}^{j-1} \overline{g}_k \right) = 0.$$

Hence, applying the above lemma to $f_j = \overline{g}_j, j = 0, 1, \ldots N-1$, we obtain that $\sum_{j=0}^{n-1} \overline{g}_j^2 = 0$. Thus all $\overline{g}_j = 0$ and, as a consequence, the actual degree of all g_j is $\ell - 1$. Hence, by using its expression in Proposition 2 we get that the degree of M_3 is at most $3\ell - 5$. Therefore, to obtain more isolated N-periodic sequences that by using M_2, the degree ℓ should be at least 5. Even for a fixed ℓ, the construction of examples exhibiting the maximum number of isolated N-periodic sequences is tedious. We prefer do not give more details.

3.3. **A 2-dimensional Abel type dynamical system.** In this section we will see how the ideas introduced in the Section 3.1, together with Theorem 1 can be used to study the following 2-dimensional discrete dynamical system:

$$\begin{aligned} x_{n+1} =& \frac{x_n}{1 + a_n x_n + \varepsilon\big(b_n x_n^2 + c_n x_n y_n\big)}, \\ y_{n+1} =& \frac{y_n}{1 + d_n y_n + \varepsilon\big(e_n x_n y_n + h_n y_n^2\big)}, \end{aligned}$$

where $r_n = (x_n, y_n)$, the 6 sequences are N-periodic and ε is a small parameter. Following again the idea introduced in Section 2.1 we perform the change of variables $r_n = (x_n, y_n) = \Phi(s_n) = \Phi(u_n, v_n) = (1/u_n, 1/v_n)$. By using it, the above system is transformed into

$$\begin{aligned} u_{n+1} =& u_n + a_n + \varepsilon \left(\frac{b_n}{u_n} + \frac{c_n}{v_n} \right), \\ v_{n+1} =& v_n + d_n + \varepsilon \left(\frac{e_n}{u_n} + \frac{h_n}{v_n} \right). \end{aligned}$$

When $\varepsilon = 0$, the sequences generated by it, with initial condition $r_0 = (u_0, v_0) = (\rho, \varrho)$, are

$$(u_n, v_n) = (\rho + A_n, \varrho + D_n), \quad \text{where} \quad A_0 = B_0 = 0, A_n = \sum_{j=0}^{n-1} a_j,\ D_n = \sum_{j=0}^{n-1} d_j.$$

Hence they are N-periodic when $A_N = D_N = 0$. Moreover, in the notation of Theorem 1,

$$g_n(u, v) = \left(\frac{b_n}{u} + \frac{c_n}{v}, \frac{e_n}{u} + \frac{h_n}{v} \right).$$

Therefore, the first discrete Melnikov function is

$$M_1(\rho,\varrho)=\left(\sum_{j=0}^{N-1}\left(\frac{b_j}{\rho+A_j}+\frac{c_j}{\varrho+D_j}\right),\sum_{j=0}^{N-1}\left(\frac{e_j}{\rho+A_j}+\frac{h_j}{\varrho+D_j}\right)\right)$$

and it is defined in $\mathcal{U}$, a neighborhood of infinity in $\mathbb{R}^2$. Its simple zeroes give rise to N-periodic sequences defined by the 2-dimensional N-periodic discrete dynamical system.

3.4. **Perturbations of autonomous globally periodic systems.** When the unperturbed system in (3) is N-globally periodic and autonomous, that is when all $f_n = f$, the expressions M_j in Theorem 1 are simpler. If $f^0 = \mathrm{Id}$ and $f^k = f\circ f^{k-1}$, then $f^N = \mathrm{Id}$, $r_n = \varphi_n(\rho) = f^n(\rho)$ and, for instance,

$$\begin{aligned}M_1(\rho)&=\sum_{j=0}^{N-1}\big(D\varphi_j(\rho)\big)^{-1}g_j(\varphi_j(\rho))\\&=\sum_{j=0}^{N-1}\big(Df^j(\rho)\big)^{-1}g_j(f^j(\rho))=\sum_{j=0}^{N-1}Df^{N-j}(f^j(\rho))g_j(f^j(\rho)),\end{aligned}\tag{10}$$

because derivating $f^N(\rho)=f^{N-j}(f^j(\rho))=\rho$ we obtain that

$$Df^{N-j}(f^j(\rho))Df^j(\rho)=\mathrm{Id}\,.$$

Moreover, in the particular case that f is linear, that is $f(r) = Ar$ where A is a $d\times d$ matrix such that $A^N = \mathrm{Id}$, we obtain that

$$M_1(\rho)=\sum_{j=0}^{N-1}A^{N-j}g_j(A^j\rho),\tag{11}$$

and, similarly,

$$M_2(\rho)=\sum_{j=0}^{N-1}A^{N-j}\left(Dg_j(A^j\rho)\sum_{k=0}^{j-1}A^{j-k}g_k(A^k\rho)+h_j(A^j\rho)\right),$$

where the second sum, when $j = 0$ is defined as zero, because it corresponds to $u_0(\rho) = 0$.

As an example, consider $r_n = (x_n, y_n)$ defined by

$$(x_{n+1},y_{n+1})=(y_n,-x_n)+\varepsilon(g_n(x_n,y_n),h_n(x_n,y_n))+O(\varepsilon^2),$$

with g_n and h_n, 4-periodic sequences of functions and $r_0 = (x_0, y_0) = (\rho,\varrho)$. We have that

$$A=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad A^2=\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\quad A^3=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\quad\text{and}\quad A^4=\mathrm{Id}\,.$$

Hence to find 4-periodic sequences generated to the discrete dynamical system we can study the zeroes of its first associated discrete Melnikov function. By using (11), some straightforward computations give that

$$\begin{aligned}M_1(\rho,\varrho)=\big(&g_0(\rho,\varrho)-h_1(\varrho,-\rho)-g_2(-\rho,-\varrho)+h_3(-\varrho,\rho),\\&h_0(\rho,\varrho)+g_1(\varrho,-\rho)-h_2(-\rho,-\varrho)-g_3(-\varrho,\rho)\big).\end{aligned}$$

3.5. **Some applications to difference equations.** Consider second order difference equations of the form

$$x_{n+2} = V_n(x_n, x_{n+1}).$$

As usual, they can be written as the discrete dynamical system (1) simply by taking $r_n = (x_n, x_{n+1})$ and $f_n(x, y) = (y, V_n(x, y))$.

We will apply Theorem 1 to find isolated N-periodic sequences of second order difference equations of the form

$$x_{n+2} = V(x_n, x_{n+1}) + \varepsilon W_n(x_n, x_{n+1}) + O(\varepsilon^2),$$

where $x_{n+2} = V(x_n, x_{n+1})$ is globally N-periodic in a suitable domain $\mathcal{U} \subset \mathbb{R}^2$ and W_n is a N-periodic set of functions. By using the above reduction we can write the above difference equation as (3) where

$$f(x, y) = (y, V(x, y)) \quad \text{and} \quad g_n(x, y) = (0, W_n(x, y)).$$

Several examples are given in Table 1, with their corresponding N, see for instance [1, 7]. In particular the case with period 5 corresponds to the celebrated Lyness difference equation. Notice that in the first four cases $\mathcal{U} = \{(x, y) \in \mathbb{R}^2 \,:\, x > 0, y > 0\}$.

N	4	6	5	3	4	6
$V(x,y)$	$\dfrac{1}{x}$	$\dfrac{y}{x}$	$\dfrac{1+y}{x}$	$\dfrac{1}{xy}$	$\dfrac{xy+y^2+p}{x-y}$	$\dfrac{xy+p}{x-y}$

TABLE 1. Examples of globally N-periodic difference equations. Here, p is a real parameter.

As an illustration of the method we choose the simplest case, corresponding to $N = 3$. Then

$$f(x, y) = \Big(y, \frac{1}{xy}\Big), \quad f(f(x, y)) = \Big(\frac{1}{xy}, x\Big), \quad f(f(f(x, y))) = (x, y)$$

and $g_n(x, y) = (0, W_n(x, y))$. In order to obtain the expression of M_1 given in (10) we need to calculate

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ g_0 \end{pmatrix} + \begin{pmatrix} -\rho/\varrho & -\rho^2\varrho \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ g_1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -\rho\varrho^2 & -\varrho/\rho \end{pmatrix} \begin{pmatrix} 0 \\ g_2 \end{pmatrix},$$

where $g_0 = g_0(\rho, \varrho)$, $g_1 = g_1(\varrho, 1/(\rho\varrho))$, $g_2 = g_2(1/(\rho\varrho), \rho)$ and with $r_0 = (x_0, x_1) = (\rho, \varrho)$. Hence

$$M_1(\rho, \varrho) = \bigg(-\rho^2\varrho g_1\Big(\varrho, \frac{1}{\rho\varrho}\Big) + g_2\Big(\frac{1}{\rho\varrho}, \rho\Big), g_0(\rho, \varrho) - \frac{\varrho}{\rho} g_2\Big(\frac{1}{\rho\varrho}, \rho\Big)\bigg).$$

Obviously, the same approach can be extended to study higher order difference equations. For instance we could apply it to study the 8-periodic sequences of the 3rd order perturbed difference equation

$$x_{n+3} = \frac{1 + x_{n+1} + x_{n+2}}{x_n} + \varepsilon W_n(x_n, x_{x+1}, x_{n+2}) + O(\varepsilon^2),$$

taking $r_n = (x_n, x_{x+1}, x_{n+2})$ because when $\varepsilon = 0$ it is a 8-periodic difference equation, called sometimes Todd difference equation.

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