Three essays on Machin's type formulas

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Abstract

We study three questions related to Machin's type formulas. The first one gives all two terms Machin formulas where both arctangent functions are evaluated 2-integers, that is values of the form $b/2^a$ for some integers a and b. These formulas are computationally useful because multiplication or division by a power of two is a very fast operation for most computers. The second one presents a method for finding infinitely many formulas with N terms. In the particular case $N = 2$ the method is quite useful. It recovers most known formulas, gives some new ones, and allows to prove, in an easy way, that there are two terms Machin formulas with Lehmer measure as small as desired. Finally, we correct an oversight from previous result and give all Machin's type formulas with two terms involving arctangents of powers of the golden section.

Keywords: Machin's type formulas, Lehmer measure, computation of π, golden section.

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1 Introduction

In 1706, John Machin found the identity

$$4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \frac{\pi}{4}. \tag{1}$$

In conjunction with the arctan expansion

$$\arctan x = \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} x^{2m+1}, \qquad |x| < 1, \tag{2}$$

discovered by Gregory in 1671, Machin used (1) to compute 100 digits of π.

In the mathematical literature there are many formulas similar to (1), that is, combinations of arctan functions that, in some way, generate π. Besides (1), the following are the most classical formulas

$$\arctan(1/2) + \arctan(1/3) = \pi/4, \tag{3}$$
$$2\arctan(1/2) - \arctan(1/7) = \pi/4, \tag{4}$$
$$2\arctan(1/3) + \arctan(1/7) = \pi/4, \tag{5}$$