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Global Asymptotic Stability of Differential Equations in the Plane

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1. INTRODUCTION

The problem of determining the basin of attraction of equilibrium points is of paramount importance for applications of stability theory.

Local conditions which guarantee the existence of small basins of attraction, such as $\operatorname{tr} L < 0$ and $\det L > 0$, where L is the linear part of the planar system at an equilibrium point, are well known.

This paper is concerned with sufficient conditions which guarantee that the basin of attraction of an equilibrium point of a $\mathscr{C}^1$ planar system of differential equations $x' = f(x)$ is the whole x-space $\mathbb{R}^2$.

In this context, the fundamental problem, yet unsolved, is the following: Consider an autonomous system of differential equations

$$x' = f(x) \qquad (' = d/dt), \tag{S}$$

where $x = (x_1, x_2)$ and $f(x) = (f_1(x_1, x_2), f_2(x_1, x_2))$.

Let $\mathscr{F}$ be the class of $\mathscr{C}^1$ maps $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that

(i) the origin $0 = (0, 0)$ is a critical point of (S), i.e., $f(0) = 0$,

(ii) $\operatorname{tr} Df(x) < 0$ on $\mathbb{R}^2$,

(iii) $\det Df(x) > 0$ on $\mathbb{R}^2$,

where $Df(x) = (\partial f_i/\partial x_j)$ is the Jacobian matrix.

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Fundamental Problem on Global Asymptotic Stability. Does $f \in \mathscr{F}$ imply that $x = 0$ is a global asymptotically stable solution of (S)? In other words, does every solution curve of (S) approach 0 as $t \to \infty$?

This problem goes back to Krasowskii [Kr] and Markus and Yamabe [MY]. The last two authors solved it under the additional condition that one of the partial derivatives $\partial f_i/\partial x_j$ $(i, j = 1, 2)$ vanishes identically on $\mathbb{R}^2$. Hartman [Ha] gave another affirmative answer to this problem assuming the stronger condition that $Df(x)$ is negative definite. Other additional conditions are discussed in Section 3 of this paper (see Theorem B). This section is preceded by a study of the equivalence between the Fundamental Problem and other apparently different problems, such as those stated below.

Problem 1. Does $f \in \mathscr{F}$ imply that the mapping f: $\mathbb{R}^2 \to \mathbb{R}^2$ is globally one-to-one?

Problem 2. Does $f \in \mathscr{F}$ imply that there is a natural number K such that for each $p \in \mathbb{R}^2$ the number of solutions of $f(x) = p$ is bounded by K?

Problem 3. Does $f \in \mathscr{F}$ imply that there are two positive constants ρ and r such that $|f(x)| \geqslant \rho > 0$ for $|x| \geqslant r > 0$ (where $|\;|$ denotes the Euclidean norm)?

Problem 4. Does $f \in \mathscr{F}$ imply that

$$\int_0^\infty \left[\min_{|x| = r} |f(x)|\right] dr = \infty?$$

A main result of this paper is the following.

THEOREM A. *The following five statements are equivalent.*

(FP) *The Fundamental Problem has an affirmative answer for all $f \in \mathscr{F}$.*

(Pi) *Problem i has an affirmative answer for all $f \in \mathscr{F}$, where $i \in \{1, 2, 3, 4\}$.*

Olech in [O] proved that FP $\Leftrightarrow$ P1 and that P3 $\Rightarrow$ FP. Hartman and Olech in [HO] showed that P4 $\Rightarrow$ FP, with the additional condition $f(x) \neq 0$ if $x \neq 0$. Recently, Meisters and Olech [MO] proved that the FP and P1 have a positive answer for the class of polynomial maps f: $\mathbb{R}^2 \to \mathbb{R}^2$, i.e., for maps $f=(f_1, f_2)$ such that f_1 and f_2 are real polynomials in two variables. Their proof led us to consider the equivalence of P2 with the other four problems. Theorem C extends the result of [MO] to large class of analytic vector fields.

2. Equivalent Problems

In this section we prove Theorem A stated in Section 1.

P3 $\Rightarrow$ P4. It is trivial. ▮

P4 $\Rightarrow$ FP. Let A denote the set of points in $\mathbb{R}^2$ whose ω-limit set with respect to system (S) is the origin. From (i), (ii), and (iii) the origin is a sink, so A is a nonempty open set. To prove that the Fundamental Problem has an affirmative solution for system (S) we have to show that $A=\mathbb{R}^2$. Suppose that $A \neq \mathbb{R}^2$. Then the boundary of A, ∂A, is nonempty and at least a solution curve of (S), $\varphi_t(x)$, through x is contained in ∂A, for all t on the maximal interval (α, β). Note that the orbit $L=\{\varphi_t(x);\, t \in (\alpha, \beta)\}$ is a closed set. In fact from (i), (ii), and (iii) and the Poincaré–Bendixson Theorem it follows that the α and ω-limit sets of this curve are empty.

Consider the space $E=A \cup L$. Denote by $l_{xy}(c)$ the f-arc lenght of a curve c: $[a, b] \rightarrow E$, joining x and y. That is: $l_{xy}(c)=\int_a^b |f(c(s))|\, |c'(s)|\, ds$.
Then assuming $|x| \leqslant |y|$, we have

$$l_{xy}(c) \geqslant \int_{|x|}^{|y|} [\inf_{|u|=r} |f(u)|]\, dr. \tag{1}$$

To verify this, take $r=|c(s)|$ then $dr/ds=\langle c(s)/|c(s)|, c'(s)\rangle$, where $\langle\, ,\rangle$ denotes the Euclidean inner product. Therefore $|c'(s)| \geqslant dr/ds$. Integration finishes the argument.

The integral hypothesis in P4 will be used to guarantee that a family of curves, one of whose extremes go to infinity while the others remain bounded, has unbounded f-arc length, as follows from (1).

Together with (S), we consider the orthogonal system

$$x' = f^{\perp}(x), \tag{S$^\perp$}$$

where $f^{\perp}$ is one of $(-f_2, f_1)$ or $(f_2, -f_1)$. The appropriate choice is made to ensure that $f^{\perp}$, on L, points into A. Denote by $\psi_s(q)$ the solution curve of (S$^\perp$) through q. Write $\gamma_t=\{\psi_s(\varphi_t(x));\, s \in [0, \beta_t)\}$ to denote the maximal positive orbit of $(S^{\perp})$ through $\varphi_t(x)$. Let $\delta = \int_0^{d(L,0)} [\inf_{|u|=r} f(u)|]\, dr$, where d denotes the Euclidean distance.

Take $y \in \gamma_0 \cap E$ such that the f-arc length of γ_0 between x and y is less than or equal to $\delta/2$ and write $M = \{\varphi_t(y);\ t \in [0, \infty)\}$.

The following two assertions, proved below, will lead to a contradiction with the assumption $\partial A \neq \phi$.

ASSERTION 1. *If γ_t cuts M at a point y_t then the f-arc length of γ_0 between x and y is larger than or equal to the f-arc length of γ_t between $\varphi_t(x)$ and y_t. That is*

$$l_{x,y}(\gamma_0) \geqslant l_{\varphi_t(x),\, y_t}(\gamma_t).$$

ASSERTION 2. *The curve γ_t cuts M for all $t \in [0, \beta)$ at a point denoted by $y_t = \varphi_{\tau(t)}(y)$. Furthermore $\tau(t) \uparrow \infty$ as $t \uparrow \beta$.*

Conclusion. From Assertions 1 and 2, and the choice of y, we have that

$$l_{\varphi_t(x),\, y_t}(\gamma_t) \leqslant \frac{\delta}{2}, \qquad t \in [0, \beta).$$

Since $|y_t| \to 0$ when $t \uparrow \beta$, and $\varphi_t(x) \in L$ we have from (1) and the choice of δ that $\frac{3}{4}\delta \leqslant l_{\varphi_t(x),\, y_t}(\gamma_t)$, for t near β, a contradiction with the inequality above. Therefore $A = \mathbb{R}^2$.

Proof of Assertion 1. For every $u \in \gamma_0$ between x and y, the Poincaré map $\pi\colon \gamma_0 \to \gamma_t$ is defined by the flow of (S). Setting $\pi(u) = u_t$, Poincaré formula gives

$$\left.\frac{ds_1}{ds_0}\right|_{s_0=0} = \frac{|f(u)|}{|f(u_t)|} \exp\left[\int_0^{t(u)} \operatorname{tr} Df(\varphi_t(u))\, dt\right],$$

where s_0 (resp. s_1) is the Euclidean arc length parameter on γ_0 (resp. γ_t), with origin at u (resp. u_t). From (ii) it follows that

$$\left.\frac{ds_1}{ds_0}\right|_{s_0=0} < \frac{|f(u)|}{|f(u_t)|}.$$

Equivalently,

$$\left.\frac{|f(u_t)|\, ds_1}{|f(u)|\, ds_0}\right|_{s_0=0} < 1,$$

which, after integration, gives Assertion 1. ∎

Proof of Assertion 2. Consider the following subset of $[0, \beta)$,

$$B=\left\{t \in [0, \beta); \gamma_t \cap M = y_t \text{ and } l_{\varphi_t(x),\, y_t}(\gamma_t) \leqslant \frac{\delta}{2}\right\}.$$

By the choice of M, $B \neq \varnothing$. By the continuous dependence of solutions with respect to initial conditions and Assertion 1, it follows that B is an open set.

To prove that B is closed, consider $\{t_n\}$ in B, such that $t_n \to \bar{t} < \beta$; then $\varphi_{t_n}(x) \to \varphi_{\bar{t}}(x) = z$.

If either $\gamma_{\bar{t}}$ leaves E or tends to infinity, it is clear that $\gamma_{\bar{t}} \in B$, because M must intersect $\gamma_{\bar{t}}$ to reach the origin.

So we can assume that $\gamma_{\bar{t}} \subset E$ and is contained in a compact set. So the ω-limit set of the solution curve of $(\mathrm{S}^{\perp})$ through z is either the origin or a periodic orbit surrounding it. In both cases, by (1) and the definition of δ, we can choose $\bar{s}$ such that $l_{z, \psi_{\bar{s}}(z)}(\gamma_{\bar{t}}) \geqslant 3\delta/4$. Hence, by continuity with respect to the initial conditions, it follows that, for n large $\{\psi_s(\varphi_{t_n}(x)); s \geqslant 0\}$ is $\mathscr{C}^1$ uniformly close to $\{\psi_s(z); s \geqslant 0\}$, on arcs of f-arc length $3\delta/4$. Therefore M must intersect $\gamma_{\bar{t}}$. From this fact and Assertion 1 we have that $\bar{t} \in B$. Then $B = [0, \beta)$.

To finish the proof it is enough to show that $\tau(t) \uparrow \infty$, when $t \uparrow \beta$. Otherwise we could take $t_n \uparrow \beta$ such that $\tau(t_n) \to T < \infty$, and we could construct curves γ_{t_n} joining points $\varphi_{t_n}(x) \in L$, which tend to infinity, with points $y_n \in M$, which tend to $\varphi_T(y)$. Furthermore, from Assertion 1, the f-arc length of these curves would be less than or equal to $\delta/2$, contradicting the divergence of the integral in P4. ∎

FP $\Rightarrow$ P1. Suppose that f is not globally one-to-one on $\mathbb{R}^2$. That means that there are $y, z \in \mathbb{R}^2$ such that $y \neq z$ and $f(y) = f(z) = a$. Then the function $g(x) = f(x + y) - a$ satisfies assumptions (i), (ii), and (iii). That g satisfies all assumptions of the FP and at the same time that the system $x' = g(x)$ has two critical points implies a contradiction. ∎

P1 $\Rightarrow$ P2. It is immediate. ∎

P2 $\Rightarrow$ P3. Let p be a point in $\mathbb{R}^2$ for which $f(x) = p$ has the maximum possible number K of solutions. We denote by x_i, for $i = 1, ..., K$, the solutions of $f(x) = p$. Assumption (iii) implies that f is a local diffeomorphism. Therefore there is a $\rho > 0$ and an open bounded neighborhood V_i of each x_i such that $f|_{V_i}$ is a diffeomorphism, $V_i \cap V_j = \varnothing$ if $i \neq j$, and $f(V_i) = \{x: |x - p| < \rho\} = B$ for each $i = 1, ..., K$.

We claim that $f^{-1}(B) = \bigcup_{i=1}^{K} V_i$. Clearly $f^{-1}(B) \supset \bigcup_{i=1}^{K} V_i$. Suppose the inclusion in the other direction does not hold. That means that there is a point y not in $\bigcup_{i=1}^{K} V_i$ such that $f(y) = z \in B$. But for each $i = 1, ..., K$ there is a point y_i in V_i such that $f(y_i)=z$ also. Since $V_i \cap V_j = \varnothing$ for $i \neq j$ it follows that $y_i \neq y_j$ when $i \neq j$. Furthermore these points y_i are all different from y. Hence the equation $f(x)=z$ has $K+1$ distinct solutions, in contradiction with the maximality of K. Thus $f^{-1}(B) = \bigcup_{i=1}^{K} V_i$.

We may now choose $r' > 0$ so large that the ball of radius r' centered at the origin 0 contains $\bigcup_{i=1}^{K} V_i$. For this r' and the previously chosen ρ we have obtained that

$$|f(x) - p| \geqslant \rho > 0 \qquad \text{if} \quad |x| \geqslant r' > 0. \tag{2}$$

Therefore the function $g(x) = f(x + x_1) - p$ satisfies all assumptions of P3 with $r = r' + |x_1|$, and since P3 $\Rightarrow$ P4 $\Rightarrow$ FP the origin 0 is a global asymptotically stable critical point for the system $x' = g(x)$. But then this system can have no other critical points. This means that $K = 1$. In other words, the function f: $\mathbb{R}^2 \to \mathbb{R}^2$ is globally one-to-one. Hence we can assume $p = 0$, and from (2) follows P3. ∎

Remark. Notice that the proof FP $\Rightarrow$ P1 has been achieved by changing the given f by another function g inside $\mathscr{F}$. It is not known if this change can be avoided.

3. Additional Sufficient Conditions

In this section we shall consider additional hypotheses that together with (i), (ii), and (iii) imply for a given f that the Fundamental Problem has a positive answer.

PROPOSITION 1. *Assume that f satisfies hypotheses* (i), (ii), *and* (iii) *and the condition*

$$\text{(C4)} \qquad \int_0^\infty \left[\min_{|x|=r} |f(x)|\right] dr = \infty.$$

Then 0 *is global asymptotically stable for* $x' = f(x)$.

Proof. It follows from the proof of P4 $\Rightarrow$ FP. ∎

PROPOSITION 2. *If f satisfies hypotheses* (i), (ii), *and* (iii) *and one of the following conditions*:

(C1) *f is globally one-to-one, or*

(C3) *there exists* ρ, r *such that*

$$|f(x)| \geqslant \rho > 0, \qquad \text{for} \quad |x| \geqslant r > 0,$$

then (C4) *holds and so* 0 *is global asymptotically stable for* $x' = f(x)$.

Proof. It is clear that (C1) implies (C3). So since (C3) implies (C4) the proof follows from Proposition 1. ∎

There are several conditions that imply that f is globally one-to-one. We consider some of them in the following result.

PROPOSITION 3. *Assume that f satisfies hypotheses* (i), (ii), *and* (iii) *and one of the following conditions*

(C2) *there is a positive integer K such that for each p in $\mathbb{R}^2$ the number of solutions of the system $f(x) = p$ is bounded by K;*

(C5) *$\partial f_1/\partial x_1$ and $\partial f_2/\partial x_2$ do not change sign;*

(C6) *$\partial f_1/\partial x_2$ and $\partial f_2/\partial x_1$ do not change sign;*

(C7) *there are some real numbers p and q such that $p(\partial f_1/\partial x_1) + q(\partial f_2/\partial x_1)$ and $p(\partial f_1/\partial x_2) + q(\partial f_2/\partial x_2)$ do not change sign and one of them does not vanish;*

(C8) *for all $v \in \mathbb{R}^2$, $|v| = 1$ the solution of the initial value problem*

$$x' = (Df(x))^{-1} v, \qquad x(0) = 0, \tag{S_v}$$

is defined for all $t \geqslant 0$.

Then f is globally one-to-one.

Proof. When condition (C2) holds it follows from the proof of $P2 \Rightarrow P3$.

In [GN, Sect. 7] it is proved that one of the conditions (C5), (C6), or (C7) implies that f is globally one-to-one.

Condition (C8) gives the inverse function of f

$$f^{-1}(y) = \varphi\left(|y|, \frac{y}{|y|}\right),$$

where $\varphi(t, v)$ denotes the solution of (S_v) such that $\varphi(0, v) = 0$ (see [W]). ∎

Proposition 3 generalizes Theorem 4 of [O].

PROPOSITION 4. *If one of the following conditions holds then the system (S_v) has a solution definited for all $t \geqslant 0$.*

(C9) $|(Df(x))^{-1}| \leqslant a|x| + b$ *for* $a, b \in \mathbb{R}$.

(C10) $\int_0^\infty [\inf_{|x|=r} [\inf_{|v|=1} |Df(x)v|]]\, dr = \infty$.

Proof. It is clear that (C9) implies (C10) because

$$\inf_{|v|=1} |Df(x)v| = |(Df(x))^{-1}|^{-1} \geqslant \frac{1}{a|x|+b}.$$

In [H] it is proved that (C10) implies that the solution of (S_v) is defined for all $t \geqslant 0$. ∎

The result of Proposition 4 also holds in $\mathbb{R}^n$, and (C10) can also be written in the form

$$\int_0^\infty [\inf_{|x|=r} \mu(x)^{1/2}]\, dr = \infty, \tag{3}$$

where $\mu(x)$ denotes the smallest eigenvalue of $(Df(x))(Df(x))^T$.

COROLLARY 5. *When $n=2$ condition* (C10) *can be written as*

$$\int_0^\infty \inf_{|x|=r} \left\{\frac{e(x) - [e^2(x) - 4(\det Df(x))^2]^{1/2}}{2}\right\}^{1/2} dr = \infty,$$

where $e(x) = |\nabla f_1(x)|^2 + |\nabla f_2(x)|^2$.

Proof. The proof follows from (3) because

$$(Df(x))(Df(x))^T = \begin{pmatrix} |\nabla f_1(x)|^2 & \langle \nabla f_1(x), \nabla f_2(x)\rangle \\ \langle \nabla f_1(x), \nabla f_2(x)\rangle & |\nabla f_2(x)|^2 \end{pmatrix}. \quad ∎$$

The results obtained so far in this section can be synthesized in the following theorem.

THEOREM B. *Assume that f satisfies hypotheses* (i), (ii), *and* (iii) *and one of the conditions from* (C1) *to* (C10); *then* 0 *is global asymptotically stable for* $x' = f(x)$.

DEFINITION. We shall say that a function f: $\mathbb{R}^2 \to \mathbb{R}^2$ is a *planar Khovansky function,* and write $f \in K_{m_1,m_2,n}$ if $f = (f_1, f_2)$ where $f_1, f_2 \in \mathbb{R}[x, y]$ are two polynomials with degrees m_1 and m_2, $x = (x_1, x_2)$, $y = (y_1, ..., y_n)$, and $y_i = e^{\langle a_i, x\rangle}$ with $a_i \in \mathbb{R}^2$.

Note that when $n=0$, $K_{m_1,m_2,0}$ is the set of planar polynomial functions.

THEOREM C. *Assume that f is a planar Khovansky function in $K_{m_1,m_2,n}$ satisfying* (i), (ii), *and* (iii). *Then $x=0$ is global asymptotically stable for the system* $x' = f(x)$.

Proof. From (iii) we know that all solutions of the system $(f_1, f_2) = a$ for any $a \in \mathbb{R}^2$ are nondegenerate. Hence from [K] (see also [R]) we obtain that the maximum number of solutions of that system are finite and bounded by

$$K = m_1 m_2 (1 + m_1 + m_2)^n \, 2^{n(n-1)/2}.$$

Then the theorem follows by using condition (C2) of Theorem B. ∎

A simple example satisfying the hypotheses of Theorem C is given by $f = (1 - e^x, \, 1 - e^y e^{-2x})$.

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