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LIMIT CYCLES FOR A CLASS OF ABEL EQUATIONS*

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Abstract. The number of solutions of the Abel differential equation $dx(t)/dt = A(t)x(t)^3 + B(t)x(t)^2 + C(t)x(t)$ satisfying the condition $x(0) = x(1)$ is studied, under the hypothesis that either $A(t)$ or $B(t)$ does not change sign for $t \in [0, 1]$. The main result obtained is that there are either infinitely many or at most three such solutions. This result is also applied to control the maximum number of limit cycles for some planar polynomial vector fields with homogeneous nonlinearities.

Key words. Abel differential equation, limit cycle, Riccati equation

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1. Introduction and statement of the main results. A problem proposed by Pugh (see [12]) consists of the following: Let $a_0, a_1, \cdots, a_n : \mathbf{R} \to \mathbf{R}$ be smooth functions and consider the differential equation

$$\frac{dx}{dt} = a_n(t)x^n + a_{n-1}(t)x^{n-1} + \cdots + a_1(t)x + a_0(t), \qquad 0 \leqq t \leqq 1. \tag{1}$$

We will say that a solution $x(t)$ of (1) is a *closed solution* or a *periodic solution* if it is defined in the interval $[0, 1]$ and $x(0) = x(1)$. The adjectives "closed" and "periodic" are motivated by the case where $a_0, a_1, \cdots, a_n$ are 1-periodic, in which (1) can be considered in the cylinder and the "closed" solutions really correspond to periodic orbits in the cylinder. An isolated closed solution in the set of all the closed solutions will be called a *limit cycle*. Then the problem is: Does there exist a bound on the number of limit cycles of (1)?

In the case $n = 2$, (1) is called the *Riccati equation* and the problem of determining the number of limit cycles is already known: there are at most two of them (see, for instance, [12], [14]). When $n = 3$, (1) is called the *Abel equation*. Also in [12] it is proved that there is no upper bound for the number of closed solutions for the Abel equations. Hence a more specific problem arises: Give a bound on the number of limit cycles of Abel equations assuming additional hypotheses on $a_3(t)$, $a_2(t)$, $a_1(t)$, and $a_0(t)$.

A problem that is studied in several papers is Pugh's problem for Abel equations when $a_3(t)$ does not change sign (see [7], [12], [18]). In this case the maximum number of closed solutions is three.

The Ricatti equation acquired importance when it was introduced by Jacopo Francesco, Count Riccati of Venice (1676-1754), who worked in acoustics, to help solve second-order ordinary differential equations. Abel's differential equation arose in the context of the studies of N. H. Abel on the theory of elliptic functions.

The aim of this paper is to study the problem of determining the maximum number of limit cycles of Abel equations when $a_0(t) \equiv 0$ and one of the other three functions that define the differential equation does not change sign. For simplicity we write the

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Abel equation with $a_0(t)\equiv 0$ in the following form:

$$\frac{dx}{dt}=A(t)x^3+B(t)x^2+C(t)x. \tag{2}$$

Note that any Abel equation with a periodic orbit $x_1(t)$ can be written in the form (2) by using the new coordinate $\bar{x}=x-x_1(t)$. Observe also that the function $A(t)$ does not change in the new coordinate.

Let L and L' be the straight lines $t=0$ and $t=1$, respectively, defined on the strip $(t, x)\in[0, 1]\times\mathbf{R}$, where part of the flow of (2) lies. We consider the return map $h: L\to L'$ (when it is defined) as follows. If $y\in L$ then $h(y)=x(1, y)$, where $x(t, y)$ denotes the solution of (2) such that $x(0, y)=y$. Note that a periodic solution $x(t, y)$ satisfies $h(y)=y$. The *multiplicity of a limit cycle* $x(t, y)$ is the multiplicity of y as a zero of the function $h(y)-y$. Multiplicity of limit cycles for (2) is studied in [1], [16].

The main results that we prove are stated in the following theorems.

THEOREM A. *Suppose that* $A(t)\not\equiv 0$ *and does not change sign. Then the following hold.*

(a) *The sum of multiplicities of all limit cycles of* (2) *is at most* 3.

(b) *Table* 1 *shows a more precise distribution of the limit cycles* (2) *when* $A(t)\geqq 0$ (*the case* $A(t)\leqq 0$ *has associated the table obtained reversing the inequalities for* c *and* d).

Theorem A(a), as we said before, is already known. The new contribution consists of the additional information given in Table 1. The proof of the results stated in this table will use ideas similar to those of [7].

THEOREM B. *Assume* $B(t)\not\equiv 0$ *and does not change sign. Then the following hold*:

(a) *The sum of multiplicities of all limit cycles of* (2) *is at most* 3.

(b) *Table* 2 *shows a more precise distribution of the limit cycles of* (2) *when* $B(t)\geqq 0$ (*the case* $B(t)\leqq 0$ *has associated the table obtained reversing the inequalities for* c).

TABLE 1

Maximum number of limit cycles of equation (2) *when* $A(t)\geqq 0$. *Here* $c=\int_0^1 C(t)dt$, $d=\int_0^1 B(t)\, e^{\int_0^t C(s)\,ds}\, dt$.

		$c=0$			$c>0$		
	$c<0$	$d<0$	$d=0$	$d>0$	$d<0$	$d=0$	$d>0$
Maximum number of limit cycles in the half-strip $x>0$ taking into account their multiplicity	1	1	0	0	2	0	0
Multiplicity of the limit cycle $x=0$	1	2	3	2	1	1	1
Maximum number of limit cycles in the half-strip $x<0$ taking into account their multiplicity	1	0	0	1	0	0	2

Theorem B improves Proposition 2.3 of [17]. Its proof also uses the ideas utilized in the proof of Theorem A plus some geometrical results associated with the change of coordinates $x\to -x$.

The results of these theorems are the best ones in the following sense: The maximum number of limit cycles stated in the two tables are realizable for Abel equations when either A or B does not change sign. It is enough to consider $A(t)$, $B(t)$, and $C(t)$ constant functions.

Table 2 could be improved by introducing a new parameter (similar to the parameter d in Table 1) that would give us a maximum number of limit cycles such that their sum in the whole strip was always at most 3. Unfortunately, we have not found this parameter.

TABLE 2
Maximum number of limit cycles of equation (2) *when* $B(t) \geqq 0$. *Here* $c = \int_0^1 C(t)\, dt$.

	$c<0$	$c=0$	$c>0$
Maximum number of limit cycles in the half-strip $x>0$ taking into account their multiplicity	2	1	1
Multiplicity of the limit cycle $x=0$	1	2	1
Maximum number of limit cycles in the half-strip $x<0$ taking into account their multiplicity	1	1	2
The sum of the multiplicities is at most	3	3	3

Similar results to those of Theorems A and B are not possible when we consider that $C(t)$ does not change sign. In fact, the example of an Abel equation with an arbitrary number of limit cycles, which we mentioned before, can be constructed with $C(t)$ a constant function, as was shown in [12]. That example is of the form

$$\frac{dx}{dt} = \varepsilon f(t)x^3 + a(t)x^2 + \delta x,$$

where $|\delta|$ is small, $a(t)$ is a polynomial of degree 1, and $f(t)$ is a polynomial of degree $2n$, and it can have at least $n+3$ limit cycles for suitable a and f.

In fact if we find a bound on the number of limit cycles of (2) with $C(t)$ a constant function in terms of A and B, we could give a bound on the number of limit cycles that a quadratic system has. Theorems A and B can be used in any way to study the limit cycles of planar polynomial vector fields with homogeneous nonlinearities. We consider two-dimensional autonomous systems of differential equations

$$\dot{x} = \lambda x - y + P_n(x, y), \qquad \dot{y} = x + \lambda y + Q_n(x, y), \tag{3}$$

where P_n and Q_n are real homogeneous polynomials of degree $n \geqq 2$. These systems for arbitrary $n \geqq 2$ have been studied in [2]-[5], [17]. When $n=2$ we have a subclass of quadratic systems which has been studied in [7], [8], [12]. System (3) with $P_n(x, y) = (ax+by)R_{n-1}(x, y)$ and $Q_n(x, y) = (cx+dy)R_{n-1}(x, y)$, where R_{n-1} is a homogeneous polynomial of degree $n-1$, has been studied in [6], [9]-[11].

System (3) in polar coordinates can be written in the form

$$\dot{r} = \lambda r + r^n f(\theta), \qquad \dot{\theta} = 1 + r^{n-1} g(\theta), \tag{4}$$

with

$$f(\theta) = \cos\theta P_n(\cos\theta, \sin\theta) + \sin\theta Q_n(\cos\theta, \sin\theta),$$

$$g(\theta) = \cos\theta Q_n(\cos\theta, \sin\theta) - \sin\theta P_n(\cos\theta, \sin\theta).$$

It is known that the periodic orbits surrounding the origin of system (4) do not intersect the curve $\dot{\theta} = 0$ (see the Appendix of [4]). Therefore, these periodic orbits can be studied by making the transformation introduced by Cherkas [5], $T(r, \theta) = (\rho, \theta)$, where

$$\rho = r^{n-1}/(1 + r^{n-1} g(\theta)). \tag{5}$$

In the new coordinates (ρ, θ), system (4) becomes the following Abel equation

$$\frac{d\rho}{d\theta} = A(\theta)\rho^3 + B(\theta)\rho^2 + (n-1)\lambda\rho, \tag{6}$$

where $A = (n-1)g(\lambda g - f)$, and $B = (n-1)(f - 2\lambda g) - g'$.

In short, by studying all the periodic solutions $\rho(\theta)$ of (6) we study all the periodic solutions of system (3) surrounding the origin. Then by using Theorems A and B we can prove the following result.

THEOREM C. (a) *Suppose that either A or B does not change sign,* $B \not\equiv 0$, *and* $A \not\equiv 0$. *Then system* (3) *has at most two limit cycles surrounding the origin.*

(b) *If either* $A \equiv 0$ *or* $B \equiv 0$ *system* (3) *has at most one limit cycle surrounding the origin.*

Examples of system (3) with the maximum number of limit cycles given in the above theorem are given in [2], [3], [9]. For a more detailed study of the number of limit cycles of system (3), see Propositions 7–9 of § 4.

Note that if $B \not\equiv 0$ in (6), then it changes sign when n is even.

The rest of the paper is organized in the following way. In § 2 we state some auxiliary results that we will need in the proofs of Theorems A and B, which are given in § 3. Lastly, the cases $A \equiv 0$, $B \equiv 0$, and the proof of Theorem C are found in § 4.

2. Preliminary results. We will need the following results.

PROPOSITION 1 (see [15]). *If* $h(y)$ *is the return map associated with the differential equation* $dx/dt = S(x, t)$, $0 \leqq t \leqq 1$, *then*

(a) $h'(y) = \exp \int_0^1 \frac{\partial S}{\partial x}(x(t, y), t)\, dt$,

(b) $h''(y) = h'(y)\left[\int_0^1 \frac{\partial^2 S}{\partial x^2}(x(t, y), t) \exp\left\{\int_0^t \frac{\partial S}{\partial x}(x(s, y), s)\, ds\right\} dt\right]$,

(c) $$h'''(y) = h'(y)\left[\frac{3}{2}\left(\frac{h''(y)}{h'(y)}\right)^2 + \int_0^1 \frac{\partial^3 S}{\partial x^3}(x(t, y), t) \exp\left\{2\int_0^t \frac{\partial S}{\partial x}(x(s, y), s)\, ds\right\} dt\right],$$

where $x(t, y)$ *denotes the solution of the differential equation such that* $x(0, y) = y$.

LEMMA 2. *The first derivative of the return map associated with a periodic orbit* $x(t)$ *of* (2) *is*

$$\exp \int_0^1 C(t)\, dt \quad \textit{if } x(t) \equiv 0,$$

or

$$\exp\left[-\int_0^1 \{B(t)x(t) + 2C(t)\}\, dt\right] = \exp \int_0^1 [A(t)x^2(t) - C(t)]\, dt \quad \textit{if } x(t) \not\equiv 0.$$

Proof. For any periodic orbit $x(t)$ we know from Proposition 1 that the first derivative of the return map is

$$\exp\left(\int_0^1 (3Ax^2(t) + 2Bx(t) + C)\, dt\right),$$

so if $x(t)\equiv 0$, the lemma follows. If $x(t)\not\equiv 0$ then from (2) we know that

$$\frac{x'(t)}{x(t)}=Ax^2(t)+Bx(t)+C,$$

and integrating between zero and 1 we obtain

$$0=\int_0^1 (Ax^2(t)+Bx(t)+C)\,dt. \tag{7}$$

Hence, by multiplying (7) by -3 or -2 and adding it to $\int_0^1 (3Ax^2(t)+2Bx(t)+C)\,dt$, the lemma follows. □

LEMMA 3. *It is not restrictive in the study of the number of limit cycles of* (2) *to consider* $-B$ *instead of* B, *or* $-A$ *and* $-C$ *instead of* A *and* C, *respectively.*

Proof. By using one of the following three changes of variables, $(x,t)\to(-x,t)$, $(x,t)\to(x,1-t)$, or $(x,t)\to(-x,1-t)$, the lemma follows. □

LEMMA 4. *Solutions of* (2) *in the region* $x>0$ (*respectively*, $x<0$) *can be studied in the region* $y\equiv x^{-2}>0$ *as solutions of the differential equation* (8) (*respectively*, (9)).

$$\frac{dy}{dt}=-2A(t)-2B(t)y^{1/2}-2C(t)y, \tag{8}$$

$$\frac{dy}{dt}=-2A(t)+2B(t)y^{1/2}-2C(t)y. \tag{9}$$

The proof follows easily.

LEMMA 5. *Assume that* $B(t)\geqq 0$ *and does not vanish identically. If* $x(t)$ *is a periodic orbit of* (2), *then the flow of* (2) *in the strip* $[0,1]\times\mathbf{R}$ *moves upward across the curve* $(t,-x(t))$.

Proof. If $x(t)$ is a periodic orbit of (2), then $x'(t)=Ax^3(t)+Bx^2(t)+Cx(t)$. Hence the tangent of the curve $(t,-x(t))$ has the direction $(1,-Ax^3(t)-Bx^2(t)-Cx(t))$. Since we know that the vector field given by (2) at the point $(t,-x(t))$ is $(1,-Ax^3(t)+Bx^2(t)-Cx(t))$, the lemma follows. □

3. Proof of Theorems A and B.

Proof of Theorem A. By Lemma 3 we can assume that $A(t)\geqq 0$. Since for (2) $(\partial^3S/\partial x^3)(t,x)=6A(t)\geqq 0$, from Proposition 1 we know that $h'''(x)\geqq 0$ for all x for which h is defined. So, by Rolle's theorem, the maximum number of limit cycles of (2) taking into account their multiplicities is three.

To show that Table 1 is right we use more information about h. From Proposition 1 and Lemmas 2 and 3 we have $h(0)=0$, $h'(0)=\exp c$, $h''(0)=2dh'(0)$, where $c=\int_0^1 C(t)\,dt$ and

$$d=\int_0^1 B(t)\exp\left\{\int_0^t C(s)\,ds\right\}dt.$$

Furthermore

$$h'(x(0))=\exp\int_0^1 (A(t)x^2(t)-C(t))\,dt \quad \text{when } x(0)\neq 0. \tag{10}$$

Assume now that $c<0$. Then from (10) for any fixed point $x\neq 0$ of h, $h'(x)>1$, and Table 1 follows.

Consider the case $c\geqq 0$. In this case we can assume that $d\geqq 0$, since the case $d<0$ follows from this one and Lemma 3. We define $H(x):=h(x)-x$. For H we know that $H(0)=0$, $H'(0)=e^c-1\geqq 0$, and $H''(0)=2de^c\geqq 0$. Now we are going to prove that there are no limit cycles in the half-strip $x>0$. Assume that $x=x_0$ gives the initial condition for the closest positive periodic solution to $x\equiv 0$. Then $H'(x_0)\leqq 0$ because $x\equiv 0$ is unstable. Hence from Rolle's theorem there exists y, $0<y<x_0$ such that $H'(y)=0$ and $H''(y)\leqq 0$. Note that conditions $H''(0)\geqq 0$ and $H''(y)\leqq 0$ with $0<y$ are in contradiction with the fact $H'''(x)=h'''(x)>0$. The rest of Table 1 follows from part (a) except when $c>0$ and $d=0$, in the half-strip $x<0$. Lemma 3 reduces this last case to the same case but in the half-strip $x>0$. □

Proof of Theorem B. From Lemma 4 we have that (2) is equivalent to either (8) or (9). By Lemma 3 we can take B of suitable sign, so that in the y coordinates the return map satisfies

$$h''(y)=\pm\frac{h'(y)}{2}\int_0^1 B(t)y^{-3/2}(t)\exp\left\{\int_0^t(\mp B(s)y^{-1/2}(s)-2C(s))\right\}dt>0.$$

Hence, by Rolle's theorem, we have proved that the sum of the multiplicities of the limit cycles of (2) in any half-strip, $x>0$, or $x<0$, is at most 2.

To show the final result we have to consider more information about the stability and relative position of the possible limit cycles.

Again by Lemma 3 it is not restrictive to consider $c\geqq 0$ and $B\geqq 0$. From Lemma 2 we have that for any initial condition x_0 of a periodic orbit of (2) in the half-strip $x>0$, $h'(x_0)=\exp\left(-\int_0^1(Bx_0(t)\,dt+2C(t))\,dt\right)<1$. Hence there is at most one limit cycle in this region. If $c=0$ the study in the half-strip $x<0$ follows in the same way.

So in order to finish the proof of this theorem it only remains to show that the maximum number of limit cycles in the whole strip is three, taking into account their multiplicities.

We consider the case $c>0$; the case $c=0$ follows in a similar way. Assume that there is a limit cycle with initial condition $x_0>0$ and two limit cycles (or a double one) with initial conditions $0>x_1\geqq x_2$. Assume that $x_1>x_2$. The case $x_1=x_2$ follows by using the same kind of arguments. From the results proved until now we know that the three limit cycles are hyperbolic and we know also their stabilities. So since the origin is a repellor limit cycle, we have by Lemma 5 that $x_0>|x_2|>|x_1|$, because x_0 has to be different from $|x_1|$ and $|x_2|$ and if $x_0<|x_2|$ then another positive limit cycle would exist between x_0 and $|x_2|$. But, again by Lemma 5, between $-x_0$ and x_2 system (2) would have another limit cycle and this is not possible. So, either the limit cycle with initial condition x_0 or the limit cycle with initial condition x_2 does not exist. □

4. Cases $A\equiv 0$, $B\equiv 0$, and proof of Theorem C. When either $A\equiv 0$ or $B\equiv 0$, (2) is of Bernoulli type, and it is well known how to integrate it. Hence in these cases we can know exactly the trajectories of all periodic solutions. Their initial conditions are given in the following lemma.

LEMMA 6. *Set*

$$c=\int_0^1 C(t)\,dt,\quad d=\int_0^1 B(t)\exp\{\textstyle\int_0^t C(s)\,ds\}\,dt,\quad d'=2\int_0^1 A(t)\exp\{2\int_0^t C(s)\,ds\}\,dt.$$

Then the following hold.

(a) *If $A\equiv 0$ and $c=d=0$ all trajectories of (2) in a neighbourhood of $x\equiv 0$ are periodic.*

(b) *If $A\equiv 0$ and $|c|+|d|\neq 0$, (2) has at most two periodic solutions. Furthermore, these solutions are the solutions with initial conditions*

$$x(0)=0,\qquad x(0)=\frac{1-e^c}{d},$$

defined for all t between zero and 1.

(c) *If* $B \equiv 0$ *and* $c = d' = 0$, *all trajectories of* (2) *in a neighbourhood of* $x \equiv 0$ *are periodic.*

(d) *If* $B \equiv 0$ *and* $|c| + |d'| \neq 0$, *equation* (2) *has at most three closed solutions. Furthermore, these solutions are the solutions with initial condition,*

$$x(0) = 0, \qquad x(0) = \pm\sqrt{\frac{1 - e^{2c}}{d'}},$$

defined for all t between zero and 1.

The proof follows by direct computations.

Before applying Theorems A and B and the above result on the Abel equation (6) associated with system (3), we state some elementary results that can be found in [3] and [4]. Note that for the Abel equation (6) associated with (3) we are interested in 2π-periodic solutions. It is not difficult to translate all our results to this case. It is enough to consider instead of θ the new parameter $t := \theta/2\pi$.

(R1) In the region $\dot{\theta} > 0$ the flow of system (3) is diffeomorphic (preserving the orientation) to the flow of the Abel equation (6) contained in the half-cylinder R_1 defined by $0 \leqq \rho < 1/g(\theta)$ where this last inequality only works when $g(\theta) > 0$; see Fig. 1.

(R2) In the region $\dot{\theta} < 0$ the flow of system (3) is diffeomorphic (reversing the orientation) to the flow of (6) contained in the region $R_2 = \{\rho < 0\} \cap \{\rho < 1/g(\theta)$ when $g(\theta) < 0\}$; see Fig. 1.

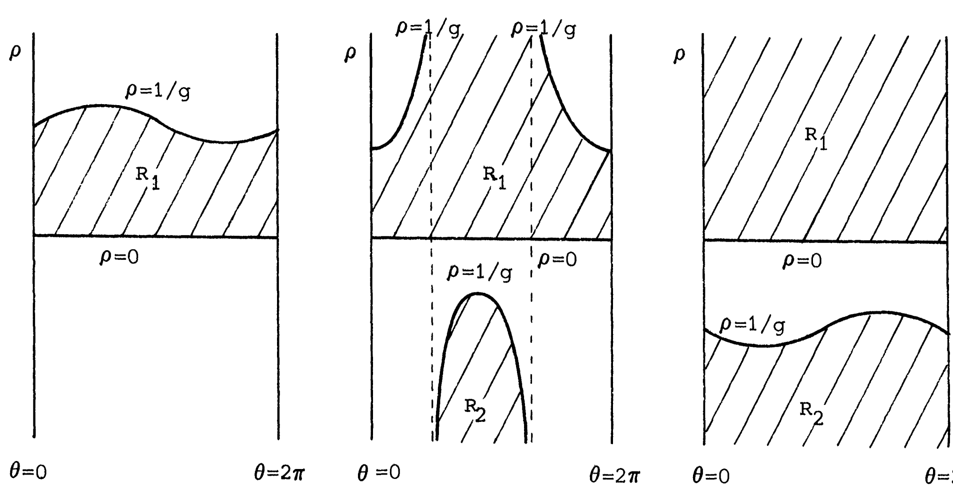

FIG. 1. *Some examples of regions* R_1 *and* R_2 *on the cylinder* (ρ, θ).

(R3) A periodic orbit of system (3) surrounding the origin is a periodic orbit of the Abel equation (6) contained in R_1 or R_2, and vice versa. Moreover, a periodic orbit can be contained in R_2 only if g is negative.

(R4) For the values of θ such that g does not vanish, the curve $\rho = 1/g$ is formed by solutions of the Abel equation (6).

(R5) If g does not vanish then the curve $\rho = 1/g$ is a periodic solution of the Abel equation (6).

(R6) The curve $\rho = 1/g$ for the Abel equation (6) corresponds to the equator of the Poincaré sphere of system (3) without the critical points.

(R7) Transformation T given in (5) sends the subsets $\dot{\theta} = 0$ to $\rho = \infty$, $r = 0$ to $\rho = 0$, and $r = \infty$ to $\rho = 1/g$.

(R8) If n is even and $\rho(\theta)$ is a solution of (6) then $-\rho(\theta+\pi)$ is also a solution of (6).

PROPOSITION 7. *Set* $d=\int_0^{2\pi}((n-1)(f-2\lambda g)-g')\,e^{(n-1)\lambda\theta}\,d\theta$, *and* $d'=2\int_0^{2\pi}(n-1)g(\lambda g-f)\,e^{2(n-1)\lambda\theta}\,d\theta$; *then the following hold.*

(a) *If* $\lambda g-f\equiv 0$ (*so* $A\equiv 0$) *and*

(1) $\lambda=0$, *then the origin of system* (3) *is a center.*

(2) $\lambda\neq 0$, *then system* (3) *has no limit cycles surrounding the origin.*

(b) *If* $g\equiv 0$ (*so* $A\equiv 0$) *and*

(1) $\lambda=d=0$, *then the origin of system* (3) *is a center.*

(2) *either* $\lambda=0$ *and* $d\neq 0$ *or* $\lambda\neq 0$, *then system* (3) *has at most one limit cycle surrounding the origin. Furthermore, if this limit cycle exists its initial condition is* $\rho(0)=(1-e^{\lambda(n-1)2\pi})/d$.

(c) *If* $(n-1)(f-2\lambda g)-g'\equiv 0$ *and*

(1) $\lambda=0$, *then the origin of system* (3) *is a center.*

(2) $\lambda\neq 0$, *then system* (3) *has at most one limit cycle surrounding the origin. Furthermore if this limit cycle exists its initial condition is* $\rho(0)=-\operatorname{sign}(g(0))\sqrt{(1-e^{\lambda(n-1)4\pi})/d'}$ *and coincides with the function* $\rho(\theta)=-1/g(\theta)$.

Proof. Note that for (6) $c=\lambda(n-1)2\pi$. Hence in order to finish the proof of this proposition it suffices to show that the possible periodic solutions of (6) that Lemma 6 gives do not produce periodic orbits of system (3) except in the cases in which the origin is a center and in cases (b2) and (c2). Consider Case (a). In this case $f\equiv\lambda g$. So

$$\begin{aligned} d &= -\int_0^{2\pi}((n-1)\lambda g+g')\,e^{\lambda(n-1)\theta}\,d\theta \\ &= -g(\theta)\,e^{\lambda(n-1)\theta}\Big|_0^{2\pi}=g(0)(1-e^{\lambda(n-1)2\pi}). \end{aligned}$$

Hence, by Lemma 6, the initial condition (different from zero) that gives us a possible limit cycle when $\lambda\neq 0$ is

$$x(0)=\frac{1-e^{\lambda(n-1)2\pi}}{g(0)(1-e^{\lambda(n-1)2\pi})}=\frac{1}{g(0)}.$$

Consequently, from (R6), case (a) follows. Case (b) follows, from Lemma 6, in a way similar to case (a). In case (c), again from Lemma 6, and with calculations similar to those in case (a), we have that the initial conditions that could give periodic orbits of (3) are $x(0)=\pm 1/g(0)$. So from (R6), the proposition is proved. □

PROPOSITION 8. (a) *If* $A(\theta)\not\equiv 0$, $A(\theta)$ *does not change sign and* n *is even, system* (3) *has at most one limit cycle surrounding the origin. Furthermore, it can exist only if* $c\cdot\operatorname{sign}(A(\theta))<0$.

(b) *Assume* $A\not\equiv 0$, $A(\theta)\geqq 0$, *and that* n *is odd*; *then the following hold.*

(1) *If* $g(\theta)\equiv 0$ *then the number of limit cycles of system* (3) *surrounding the origin is at most the number appearing in the first row of Table* 1, *according to the signs of* c *and* d. *Furthermore, the limit cycles turn in the sense* $\dot\theta>0$.

(2) *If* $g(\theta)>0$ *for all* $\theta\in[0,2\pi]$, *then the number of limit cycles of system* (3) *surrounding the origin is at most the number appearing in the first row of Table* 1 *minus* 1, *according to the signs of* c *and* d. *Furthermore, they turn in the sense* $\dot\theta>0$.

(3) *If* $g(\theta)<0$ *for all* $\theta\in[0,2\pi]$ *system* (3) *has at most one limit cycle surrounding the origin. It can exist only if* $c<0$ *and then it turns in the sense* $\dot\theta>0$ *or if* $c>0$ *and* $d<0$ *and then it turns in the sense* $\dot\theta<0$.

The value c *is equal to* $\lambda(n-1)2\pi$ *and* d *is given in Proposition* 7. *If* $A(\theta)\leqq 0$ *we have similar results by reversing the inequalities for* c *and* d.

Proof. (a) The proof follows from Table 1 and results (R1), (R2), (R3), and (R8).

(b) The proof follows from Table 1 and results from (R1) to (R7). See also Fig. 1. Note that the case $A(\theta) \leqq 0$ can be obtained from case $A(\theta) \geqq 0$ by using Lemma 3. □

Most results of the two above propositions are already proved in [3].

PROPOSITION 9. *Assume that $B(\theta) \not\equiv 0$ and $B(\theta) \geqq 0$ (hence n is odd). Then the following hold.*

(a) *If $g(\theta) \equiv 0$ then the number of limit cycles of system* (3) *surrounding the origin is at most the number appearing in the first row of Table* 2, *according to the sign of c. Furthermore, the limit cycles turn in the sense $\dot{\theta} > 0$.*

(b) *If $g(\theta) > 0$ for all $\theta \in [0, 2\pi]$, then the number of limit cycles of system* (3) *surrounding the origin is at most the number appearing in the first row of Table* 2 *minus* 1, *according to the sign of c. Furthermore, they turn in the sense $\dot{\theta} > 0$.*

(c) *If $g(\theta) < 0$ for all $\theta \in [0, 2\pi]$ system* (3) *has at most one limit cycle surrounding the origin. It can exist only if $c < 0$ and then it turns in the sense $\dot{\theta} > 0$ or if $c > 0$ and $d < 0$ and then it turns in the sense $\dot{\theta} < 0$.*

The value c is equal to $\lambda(n-1)2\pi$. If $B(\theta) \leqq 0$ we have similar results by reversing the inequalities for c.

The proof of this proposition follows in a way similar to the proof of Proposition 8.

From Propositions 7–9 we obtain Theorem C.

Remark. Theorems A and B can also be applied to more general differential equations (not necessarily polynomial). It is enough that we can find a system of differential equations such that there exists a change of variables (usually polar coordinates) that transforms it into (2). So, for instance, we can apply Theorems A and B to a subclass of planar vector fields $X(v) = Cv + h(v)Dv$, studied in [9], where C and D are 2×2 matrices, h a smooth homogeneous function, when the functions $A(\theta)$ or $B(\theta)$ associated with this differential equation do not change sign.

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