REGULARIZATION OF BROUWER TRANSLATION THEOREM.

ARMENGOL GASULL[†,‡], JORGE GROISMAN[♭] AND FRANCESC MAÑOSAS[†]

Abstract. The celebrated Brouwer translation theorem asserts that for a preserving orientation fixed point free homeomorphism of the plane, each point belongs to an invariant region where the dynamics is continuously conjugate to a translation. In this work we prove that if we start with a $\mathcal{C}^m, m \in \mathbb{N} \cup \{\infty\}$, diffeomorphism then the referred conjugacy has the same kind of regularity.

2000 MSC: Primary: 37C15. Secondary: 54H20.
Keywords: Brouwer translation theorem, smooth conjugation, homeomorphism, diffeomorphism

Data Availability statement

All data generated or analysed during this study are included in this published article.

Introduction and Main result

Let F be an orientation-preserving homeomorphism of $\mathbb{R}^2$ which is fixed point free. The *Brouwer's plane translation theorem* asserts that every $z \in \mathbb{R}^2$ is contained in a domain of translation $\mathcal{U}$ for F, that is an open connected subset of $\mathbb{R}^2$, whose boundary is $L \cup F(L)$ where L is the image of a proper embedding of $\mathbb{R}$ in $\mathbb{R}^2$, such that L separates $F(L)$ and $F^{-1}(L)$ (L is called a Brouwer line). Moreover, if $E = \bigcup_{n\in\mathbb{Z}} F^n(\overline{\mathcal{U}})$, where $\overline{\mathcal{U}} = U \cup L \cup F(L)$, there exists a homeomorphism $\Phi : E \to \mathbb{R}^2$ such that $(\Phi \circ F \circ \Phi^{-1})(z) = z + (1, 0)$.

This result was proved in 1912 by Brouwer ([1]) although many other researchers, like for instance Scherrer, Kerékjártó, Terasaka or Sperner, during the 20's and the 30's gave their own proofs trying to fix and clarify his approach. For more details, it is interesting to read Section 2 of Guillou's paper [5] dedicated to some historical remarks about this Brouwer's result. Nowadays the interested reader can also take a look for instance to any of the proofs given in the more recent works of Franks ([4]), Guillou ([5]), or Le Calvez and Sauzet ([7]).

From now on if L is a Brouwer line for a planar orientation preserving fixed point free map we denote by $E_L = \bigcup_{n\in\mathbb{Z}} F^n(\overline{\mathcal{U}})$ the F-invariant space associated to L.

In a few words, in this note we will prove that if F is a $\mathcal{C}^m$ diffeomorphism, $m \in \mathbb{N} \cup \{\infty\}$, then the above conjugacy Φ can be chosen of class $\mathcal{C}^m$.

Notice that from the above paragraphs, orientation preserving fixed point free diffeomorphisms have always Brouwer lines. Our result is:

Theorem A. *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a orientation preserving fixed point free $\mathcal{C}^m$ diffeomorphism, $m \in \mathbb{N}\cup\{\infty\}$. Let L be a Brouwer line. Then, there exists a $\mathcal{C}^m$ diffeomorphism $\Psi : E_L \to \mathbb{R}^2$ such that $(\Psi \circ F \circ \Psi^{-1})(z) = z + (1, 0)$.*

Note that there is no loss of derivatives in this result, that is we find a map Ψ which is as smooth as the original map F. Throughout the paper m takes any value in $\mathbb{N} \cup \{\infty\}$ and all diffeomorphisms are of class $\mathcal{C}^m, m \in \mathbb{N} \cup \{\infty\}$.

The study of the regularity of conjugacies between some dynamical systems is a classical subject of interest and goes back to Poincaré. For instance, nowadays this question is well understood for smooth diffeomorphisms of the circle, see [3] and its references. It is interesting to observe that for some dynamical systems results an agreement between the regularity of the dynamical system and the one of the conjugation happens while for others do not. Like in Theorem A, another example of the first situation we have the Kerékjártó's theorem of local linearization of periodic maps in a neighborhood of a fixed point ([2]). A well-known result of the second situation is given by the Grobman-Hartman theorem which states that a $\mathcal{C}^m$ diffeomorphism, with m a positive integer, can be locally $\mathcal{C}^0$ linearized near a hyperbolic fixed point but in general it can not be $\mathcal{C}^m$ linearized for $m > 0$ ([8]).

To show that if F is of class $\mathcal{C}^m$ then it is possible to $\mathcal{C}^m$ conjugate $F|_E$, with a translation we proceed in several steps. A first simple reduction comes by using the celebrated Riemann mapping theorem. Recall that it affirms that any non-empty simply connected open subset $\mathcal{W}$ of the complex number plane $\mathbb{C}$, different of the whole plane, can be biholomorphycally mapped to the open unit disk. Moreover, recall that by Brouwer Translation Theorem there exists a homeomorphism $\Phi : E_L \to \mathbb{R}^2$. As a consequence E_L is an open simply connected domain. Then, in case that $\mathcal{W} = E_L$ is not the whole plane, we can finally transform $\mathcal{C}^\infty$ diffeomorphically the unit disc into $\mathbb{R}^2$. In short it is not restrictive to assume that F has a Brouwer line L such that

$E_L = \mathbb{R}^2$. Therefore we will prove Theorem A only in the case that $E_L = \mathbb{R}^2$. The general case follows from the previous observation.

Next, we will prove that the $\mathcal{C}^m$ diffeomorphism F, on $\mathbb{R}^2$, is successively $\mathcal{C}^m$ conjugated, via some $\mathcal{C}^m$ conjugations Ψ_1, Ψ_2, Ψ_3, with some $\mathcal{C}^m$ diffeomorphisms of $\mathbb{R}^2$, say G_1, G_2 and G_3, respectively, obtaining finally

$$G_3(0,y) = (1,y) \quad \text{and} \quad \bigcup_{n\in\mathbb{Z}} G_3^n(\overline{\mathcal{U}}) = \mathbb{R}^2,$$

where here $\overline{\mathcal{U}} = \{(x,y) : \ 0 \le x \le 1\}$. More concretely, we prove:

- The diffeomorphism Ψ_1 can be taken such that $\Psi_1(L)$ is the straight line $\{0\} \times \mathbb{R}$ and it is a Brouwer line for G_1.
- The diffeomorphism Ψ_2 can be constructed such that the straight line $\{0\} \times \mathbb{R}$ is a Brouwer line and $G_2(0,y) = (1, p(y))$ for some increasing diffeomorphism p of $\mathbb{R}$.
- The diffeomorphism Ψ_3 can be taken such that the straight line $\{0\} \times \mathbb{R}$ is a Brouwer line and $G_3(0,y) = (1,y)$.

The next three sections of the paper are devoted to prove each of the steps of the proof we described above. The final section 4 contains the proof of the main theorem.

Next, we say some words about how we will prove it. By the above reductions, we can start with a diffeomorphism $G_3 = F$ that has $L = \{0\}\times\mathbb{R}$ as a Brouwer line, satisfying $F(0,y) = (1,y)$ and $\bigcup_{n\in\mathbb{Z}} F^n(\overline{\mathcal{U}}) = \mathbb{R}^2$, where $\overline{\mathcal{U}} = \{(x,y) \, : \, 0 \le x \le 1\}$. We want to prove the existence of a $\mathcal{C}^m$ diffeomorphism Ψ, defined on the whole plane, such that

$$\Psi(F(x)) = T(\Psi(x)), \quad \text{where} \quad T(x,y) = (x+1, y). \tag{1}$$

From this equality it is apparent that once Ψ is defined on $\overline{\mathcal{U}}$, then Ψ restricted to $F(\overline{\mathcal{U}})$ is determined by Ψ on $\overline{\mathcal{U}}$ because of condition (1). Hence, the main point is to be able to construct a suitable seed on $\overline{\mathcal{U}}$ such that its extension to $\overline{\mathcal{U}} \cup F(\overline{\mathcal{U}})$ is of class $\mathcal{C}^m$ on the common boundary $\{1\} \times \mathbb{R}$. As we will see, one of the main tools in our construction, and also in other parts of this paper, will be the sometimes called *Smoothing theorem*, due to Hirsch ([6]). We will recall it in Theorem 2.2. In a few words, it allows to modify piecewise smooth homeomorphism in order to produce diffeomorphisms.

The question that we tackle in Theorem A, but for analytical diffeomorhisms, seems also natural and interesting. Unfortunately, it is not covered at all by our approach.

1. First step: construction of G_1.

To prove this step first we show that it is possible to choose a $\mathcal{C}^m$ Brouwer line, and then, by conjugacy, we will transform it into the straight line $\{0\} \times \mathbb{R}$. Next lemma shows how to do this first choice.

One of the key points to prove next result is that a proper embedding of $\mathbb{R}$ in $\mathbb{R}^2$ gives rise to a Jordan curve when the plane is compactified to $\mathbb{S}^2$, by adding one point.

Lemma 1.1. *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a preserving orientation fixed point free $\mathcal{C}^m$ diffeomorphism. Let L be a Brouwer line. Then there exists a $\mathcal{C}^\infty$ Brouwer line $S \subset E_L$ satisfying that $E_L = E_S$.*

Proof. As before we denote by $\mathcal{U}$ the connected component of $\mathbb{R}^2 \setminus (\{L\} \cup \{F(L)\})$ whose boundary is $L \cup F(L)$. Our first proposal is to construct a $\mathcal{C}^\infty$ curve that is a proper embedding of $\mathbb{R}$ in $\mathbb{R}^2$ that separates L and $F(L)$. To do this, let σ be a simple arc that joints a point in L with a point in $F(L)$ through $\mathcal{U}$. That is, σ is the image of a continuous map and injective map $\varphi : [0, 1] \longrightarrow \mathbb{R}^2$ such that $\varphi(0) \in L$, $\varphi(1) \in F(L)$ and $\varphi(t) \in \mathcal{U}$ for all $t \in (0, 1)$. Clearly σ separates $\mathcal{U}$ in two unbounded, open and simply connected components such that each one has one boundary component. We denote by $\mathcal{U}^+$ and $\mathcal{U}^-$ these open components.

Now we choose $z \in \sigma$ not in $L \cup F(L)$ and we will construct a $\mathcal{C}^\infty$ simple curve $\varphi_1 : [0, \infty) :\longrightarrow \mathcal{U}^+$ satisfying $\varphi_1(0) = z$ and $\lim_{t\to\infty} \|\varphi_1(t)\| = \infty$. To do this we consider $\Psi : \mathcal{U}^+ \longrightarrow A$ a holomorphic homeomorphism, given by the Riemann mapping theorem, between $\mathcal{U}^+$ and the unit ball A and we choose a sequence $x_n \in \mathcal{U}^+$ satisfying that the sequences $\|x_n\|$ and $\|\Psi(x_n)\|$ are strictly increasing and $\lim \|x_n\| = \infty$. Note that in this case we get $\lim \|\Psi(x_n)\| = 1$. Now we consider a $\mathcal{C}^\infty$ injective map $\alpha : [1, \infty) \longrightarrow A$ such that $\alpha(i) = \Psi(x_i)$ for all $i \in \mathbb{N}$ and $\|\alpha(t)\| \in [\|\Psi(x_i)\|, \|\Psi(x_{i+1})\|]$ if $t \in (i, i + 1)$. Then $\Psi^{-1} \circ \alpha$ gives a $\mathcal{C}^\infty$ arc in $\mathcal{U}^+$ beginning at x_1 such that $\lim_{t\to\infty} \|(\Psi^{-1} \circ \alpha)(t)\| = \infty$. Now joining this arc with a $\mathcal{C}^\infty$ arc in $\mathcal{U}^+$ that begins at z and ends at x_1 we obtain the desired $\mathcal{C}^\infty$ simple curve.

Using a symmetric construction we can obtain a $\mathcal{C}^\infty$ simple curve $\varphi_2 : (-\infty, 0) :\longrightarrow \mathcal{U}^-$ satisfying $\varphi_2(0) = z$ and $\lim_{t\to-\infty} \|\varphi_2(t)\| = \infty$. Lastly joining φ_1 and φ_2 and modifying both in a little neighborhood of 0 we obtain $\varphi : \mathbb{R} \longrightarrow \mathcal{U}$ a proper $\mathcal{C}^\infty$ embedding of $\mathbb{R}$ in $\mathbb{R}^2$. Set $S = \varphi(\mathbb{R})$. Now we consider $\mathbb{S}^2$ the compactification of $\mathbb{R}^2$ by adding one point and we obtain that $\widetilde{S}, \widetilde{L}$ and $\widetilde{F(L)}$, the corresponding compactifications of $L, F(L), S$, are Jordan curves in $\mathbb{S}^2$. Furthermore it is clear that $\widetilde{L}$ and $\widetilde{F(L)}$ belong to different connected components of $\mathbb{S}^2 \setminus \widetilde{S}$. Thus S separates L and $F(L)$ and this fact implies that S separates $F(S)$ and $F^{-1}(S)$. So S is a $\mathcal{C}^\infty$ Brouwer line. Let $\mathcal{V}$ be the open set that has boundary $S \cup F(S)$. Clearly $\overline{\mathcal{V}} \subset \overline{\mathcal{U}} \cup F(\overline{\mathcal{U}})$ and $\overline{\mathcal{U}} \subset \overline{\mathcal{V}} \cup F^{-1}(\overline{\mathcal{V}})$. Therefore $\bigcup_{n\in\mathbb{Z}} F^n(\overline{\mathcal{V}}) = \bigcup_{n\in\mathbb{Z}} F^n(\overline{\mathcal{U}})$. This ends the proof of the Lemma.

We remark that, although in the introduction we had reduced the problem to the situation where $\bigcup_{n\in\mathbb{Z}} F^n(\overline{\mathcal{U}}) = \mathbb{R}^2$, in this proof we have not used at all this reduction. □

To end the construction of G_1 started in lemma 1.1 we need next result, that corresponds to Lemma 3.6 of [2]:

Proposition 1.1. *Let C be a closed, connected and non-compact one dimensional $\mathcal{C}^m$ submanifold of $\mathbb{R}^2$. Then there exists a $\mathcal{C}^m$ diffeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\varphi(C) = \{0\} \times \mathbb{R}$.*

Lemma 1.2. *Let F be a orientation preserving fixed point free $\mathcal{C}^m$ diffeomorphism. Let L be a $\mathcal{C}^\infty$ Brouwer line L and assume that $E_L = \mathbb{R}^2$. Then F is $\mathcal{C}^m$ conjugated via Ψ_1 to a $\mathcal{C}^m$ diffeomorphism G_1 of $\mathbb{R}^2$ and $\Psi_1(L) = \{0\} \times \mathbb{R}$ is a Brouwer line for G_1. Moreover $E_{\{0\}\times\mathbb{R}} = \mathbb{R}^2$.*

Proof. By lemma 1.1 we can choose a $\mathcal{C}^\infty$ Brouwer line L Then, by Proposition 1.1 there exists a $\mathcal{C}^\infty$ diffeomorphism $\Psi_1 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Psi_1(L) = \{0\} \times \mathbb{R}$. Set $G_1 = \Psi_1 \circ f \circ \Psi_1^{-1}$. Then G_1 is a $\mathcal{C}^m$ diffeomorphism, Ψ_1 conjugates F and G_1 and $\{0\} \times \mathbb{R}$ is a Brouwer line for G_1. Moreover $E_{\{0\}\times\mathbb{R}} = \Psi_1(E_L) = \Psi_1(\mathbb{R}^2) = \mathbb{R}^2$. □

2. Second step: construction of G_2.

We start this section with a couple of preliminary results. Theorem 3.11 in [6] asserts that two connected compact surfaces are $\mathbb{C}^\infty$ diffeomorphic if and only if they have the same Euler characteristic and the same number of boundary components. We will use the following specific non-compact version:

Theorem 2.1. *Two open connected subsets of $\mathbb{R}^2$ whose boundary consists of the same number of boundary components, where each one of them is a $\mathcal{C}^m$ proper embedding of $\mathbb{R}$ in $\mathbb{R}^2$, are $\mathcal{C}^m$ diffeomorphic.*

Next we recall the *Smoothing theorem*, that allows to modify a piecewise smooth homeomorphism in order to obtain a diffeomorphism. This result is proved in Hirsch book [9] in the $\mathcal{C}^\infty$ context. We will also use a $\mathcal{C}^m$ version with $m \in \mathbb{N}$ given in item (a) of [2, Thm. 3.2].

Theorem 2.2. *Let W_0 and W_1 be two manifolds without boundary of dimension n and assume that they can be decomposed in the form*

$W_i = M_i \cup N_i,\ i = 0, 1,$ where M_i and N_i are closed n-dimensional sub-manifolds satisfying

$$M_i \cap N_i = \partial M_i = \partial N_i = V_i.$$

In addition assume that $\Phi : W_0 \to W_1$ is a homeomorphism mapping $\mathcal{C}^m$ diffeomorphically M_0 onto M_1 and N_0 onto N_1. Then there exists a $\mathcal{C}^m$ diffeomorphism $\Psi : W_0 \to W_1$ such that

$$\Psi(M_0) = M_1,\ \Psi(N_0) = N_1 \text{ and } \Psi = \Phi \text{ on } V_0.$$

Moreover Ψ can be chosen in such a way that it coincides with Φ outside a given neighborhood of V_0.

Now we are ready to prove the goal of this section.

Lemma 2.1. *Let G_1 be the $\mathcal{C}^m$ diffeomorphism given in Lemma 1.2 that has the Brouwer line $\{0\} \times \mathbb{R}$. There exists a $\mathcal{C}^m$ diffeomorphism G_2 conjugated to G_1, such that $\{0\}\times\mathbb{R}$ is a Brouwer line and $G_2(0, y) = (1, p(y))$, for some $\mathcal{C}^m$ increasing diffeomorphism p of $\mathbb{R}$..*

Proof. Consider the manifolds (with boundary) A, B, A' and B' where A is the adherence of the region between $L = \{0\} \times \mathbb{R}$ and $G_1(L)$, the set B is the adherence of the connected component of $\mathbb{R}^2\setminus(L\cup\{G_1(L)\})$ whose boundary is exactly $\{G_1(L)\}$, and

$$A' = \{(x, y) : 0 \le x \le 1\}, \quad B' = \{(x, y) : x \ge 1\},$$

see Figure 1 for more details. By Theorem 2.1 there exist a $\mathcal{C}^m$ diffeomorphism Θ between A and A' and a $\mathcal{C}^m$ diffeomorphism Δ between B and B'.

Let r be the $\mathcal{C}^m$ diffeomorphism such that $\Theta \circ \Delta^{-1}(1, y) = (1, r(y))$. Let us consider now $R : B' \to B'$ defined by $R(x, y) = (x, r(y))$. Notice also that $\Theta(0, y) = (0, s(y))$ where s is another $\mathcal{C}^m$ diffeomorphism. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$\Phi(x, y) = \begin{cases} (x, s(y)), & \text{if } x \le 0; \\ \Theta(x, y), & \text{if } (x, y) \in A; \\ R \circ \Delta(x, y), & \text{if } (x, y) \in B. \end{cases}$$

Then Φ is a plane homeomorphism which is a $\mathcal{C}^m$ diffeomorphism in $\mathbb{R}^2 \setminus D$ where $D = L \cup \{G_1(L)\}$. Applying Theorem 2.2 to Φ we find a $\mathcal{C}^m$ diffeomorphism Ψ_2 in $\mathbb{R}^2$ such that $\Psi_2|_D = \Phi|_D$. Finally, $G_2 = \Psi_2 \circ G_1 \circ \Psi_2^{-1}$ is the desired $\mathcal{C}^m$ diffeomorphism. Note that by construction $G_2(0, y) = (1, p(y))$ for some diffeomorphism p that must be increasing because G_2 preserves orientation and sends points (x, y) with $x \in (0, 1)$ to points with first coordinate greater than one. □

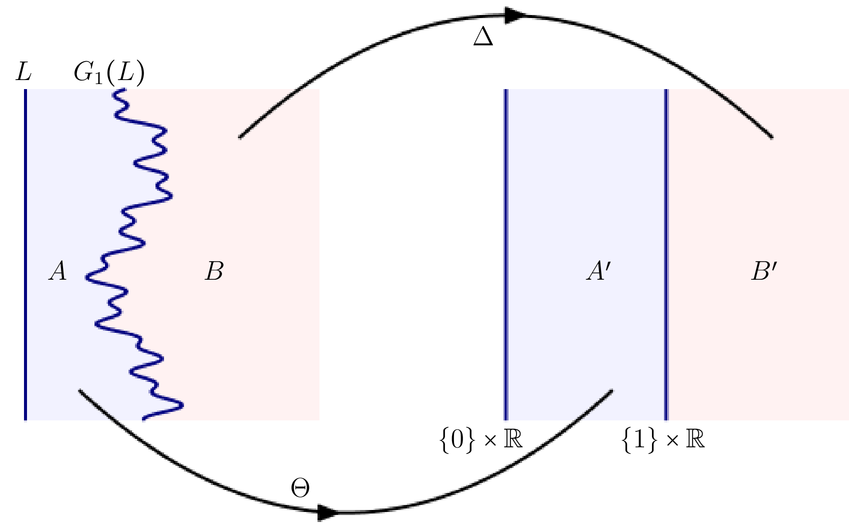

Figure 1. Diffeomorphisms Θ and Δ

3. Third step: construction of G_3.

Lemma 3.1. *Let G_2 be the $\mathcal{C}^m$ diffeomorphism given in lemma 2.1. There exists a $\mathcal{C}^m$ diffeomorphism G_3, conjugated to G_2, for which $\{0\}\times\mathbb{R}$ is a Brouwer line and $G_3(0,y)=(1,y)$.*

Proof. We know that $G_2(0,y)=(1,p(y))$, for some $\mathcal{C}^m$ increasing diffeomorphism p. Let $\Phi:\mathbb{R}^2\to\mathbb{R}^2$ be defined as

$$\Phi(x,y)=\begin{cases}(x,y) & \text{if } x\leq 0;\\ (x,xp^{-1}(y)+(1-x)y) & \text{if } x\in[0,1];\\ (x,p^{-1}(y)) & \text{if } x\geq 1.\end{cases}$$

Clearly Φ is a homeomophism that restricted to $\mathbb{R}^2\setminus\big((\{0\}\times\mathbb{R})\cup(\{1\}\times\mathbb{R})\big)$ is a $\mathcal{C}^m$ diffeomorphism. Applying Theorem 2.2 to the function Φ, we obtain a $\mathcal{C}^m$ diffeomorphism Ψ_3 such that its restriction to $(\{0\}\times\mathbb{R})\cup(\{1\}\times\mathbb{R})$ agrees with Φ. Thus, conjugating G_2 by Ψ_3 we obtain G_3 with the property stated in this lemma. □

4. Proof of Theorem A.

Recall that from all the previous reductions we can start with a $\mathcal{C}^m$ plane diffeomorphism F satisfying $F(0,y)=(1,y)$ and $\bigcup_{n\in\mathbb{Z}}F^n(\overline{\mathcal{U}})=\mathbb{R}^2$, where $\overline{\mathcal{U}}=[0,1]\times\mathbb{R}$.

Recall also, that as we have already explained in the introduction, if a conjugation Ψ satisfying $\Psi\circ F\circ\Psi^{-1}=T$, where $T(x,y)=(x+1,y)$, is defined on $\overline{\mathcal{U}}$, automatically it is also defined on the whole plane. This is so, because the conjugation condition writes as $\Psi(F(x)) = T(\Psi(x))$. In short, Ψ restricted to $F(\overline{\mathcal{U}})$ is determined by the definition of Ψ in $\overline{\mathcal{U}}$.

Hence, we start with a $\mathcal{C}^m$ diffeomorphism Ψ only defined on $\overline{\mathcal{U}}$, and we firstly need to find conditions on it, that imply that its extension to $\overline{\mathcal{U}} \cup F(\overline{\mathcal{U}})$ is also of class $\mathcal{C}^m$ on the common boundary $\{1\} \times \mathbb{R}$.

With this aim, we introduce $G(x, y) = F(x, y) - (1, 0)$. We claim that if we define a $\mathcal{C}^m$ diffeomorphism $\Psi : \overline{\mathcal{U}} \to \overline{\mathcal{U}}$ satisfying:

- The map Ψ identically coincides with G in a neighborhood of $\{0\} \times \mathbb{R}$,
- The map Ψ is the identity in a neighborhood $\{1\} \times \mathbb{R}$,

then its extension to $\overline{\mathcal{U}} \cup F(\overline{\mathcal{U}})$ also is a $\mathcal{C}^m$ diffeomorphism. From it, clearly, this extension procedure can be continued until Ψ is defined on the whole plane because $\bigcup_{n\in\mathbb{Z}} F^n(\overline{\mathcal{U}}) = \mathbb{R}^2$.

Let us prove the claim. The conjugacy condition can be written as $\Psi = T \circ \Psi \circ F^{-1}$. We know that in $\overline{\mathcal{U}}$ and near $\{1\} \times \mathbb{R}$ the map Ψ is simply the identity. Therefore, to prove that its extension to $\overline{\mathcal{U}} \cup F(\overline{\mathcal{U}})$ gives rise to a $\mathcal{C}^m$ map it suffices to prove that it is also the identity near $\{1\} \times \mathbb{R}$ but in $F(\overline{\mathcal{U}})$. If $(x, y) \in F(\overline{\mathcal{U}})$ and close enough to $\{1\} \times \mathbb{R}$, $F^{-1}(x, y)$ is in $\overline{\mathcal{U}}$ and near $\{0\} \times \mathbb{R}$. Hence, for $(x, y) \in F(\overline{\mathcal{U}})$ and close enough to $\{1\} \times \mathbb{R}$,

$$\begin{aligned}\Psi(x, y) =& T \circ \Psi \circ F^{-1}(x, y)\\ =& \Psi(F^{-1}(x, y)) + (1, 0) = G(F^{-1}(x, y)) + (1, 0)\\ =& F(F^{-1}((x, y)) - (1, 0) + (1, 0) = (x, y),\end{aligned}$$

as we wanted to prove. Notice that we have used that Ψ coincides with G in a neighborhood of $\{0\} \times \mathbb{R}$.

To construct the seed Ψ on $\overline{\mathcal{U}}$, satisfying the above two properties, we will proceed by defining it on several pieces of $\overline{\mathcal{U}}$ and then $\mathcal{C}^m$ joining them by applying several times Theorem 2.2.

Let us start our construction. Let V be a neighborhood of $\{0\} \times \mathbb{R}$ of the form $V = \{(x, y) : \ 0 \le x \le \sigma(y)\}$, where σ is a $\mathcal{C}^\infty$ function, and such that $G(V) \subset [0, 1/2) \times \mathbb{R}$, see Figure 2.(a). This can be done, simply by the continuity of G, because $G(0, y) = (0, y)$. In that figure $\Sigma = \{(\sigma(y), y)\,,\ y \in \mathbb{R}\}$.

Let $k \in \mathbb{R}$ be such that $1/2 < k < 1$. Set $A = \{(x, y) : \ \sigma(y) \le x \le k\}$ which is $\mathcal{C}^m$ diffeomorphic to $[0, 1] \times \mathbb{R}$ (through a $\mathcal{C}^m$ diffeomorphism Δ) which moreover we can assume that satisfies $\Delta(\sigma(y), y) = (0, y)$. Let B be the subset of $\overline{\mathcal{U}}$ whose boundary is the curve $G(\Sigma)$ and the straight line $x = k$ which as above is also $\mathcal{C}^m$ diffeomorphic to $[0, 1] \times \mathbb{R}$

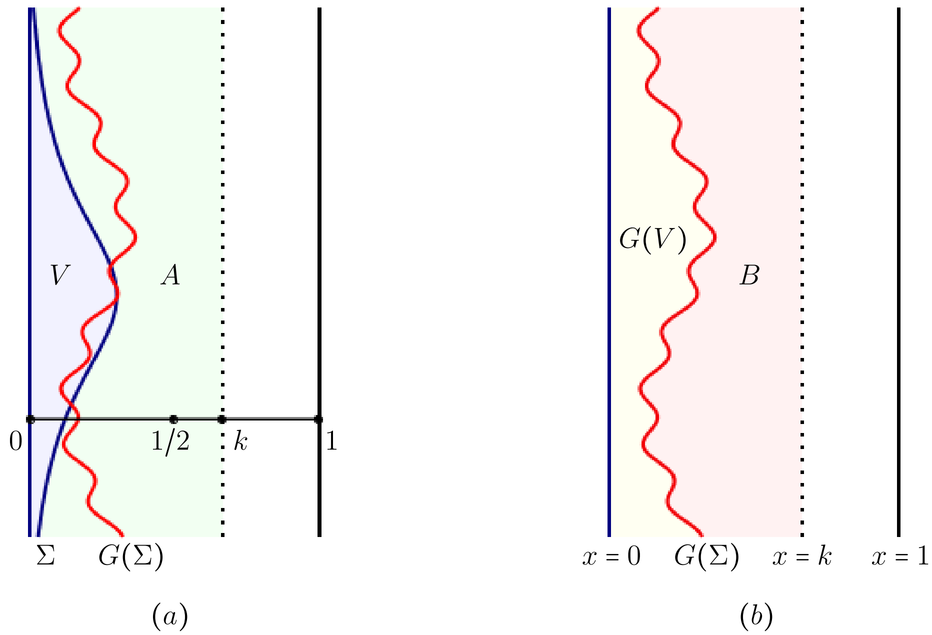

FIGURE 2. Subsets of the strip $[0,1] \times \mathbb{R}$ introduced to define the diffeomorphisms Θ and Δ in the proof of Theorem A

(through a $\mathcal{C}^m$ diffeomorphism Θ) that satisfies $\Theta(G(\sigma(y), y)) = (0, y)$, see Figure 2.(b).

Then $\Theta^{-1} \circ \Delta : A \to B$ is a $\mathcal{C}^m$ diffeomorphism that agrees with G restricted to the curve Σ. Thus, we are able to define $S : [0,k] \times \mathbb{R} \to [0,k] \times \mathbb{R}$ as

$$S(x,y) = \begin{cases} G(x,y), & \text{if } (x,y) \in V; \\ \Theta^{-1} \circ \Delta(x,y), & \text{if } (x,y) \in A, \end{cases}$$

which is a diffeomorphism in $[0,k] \times \mathbb{R} \setminus \Sigma$, although is a global homeomorphism. Applying Theorem 2.2 we obtain $\Phi_1 : [0,k] \times \mathbb{R} \to [0,k] \times \mathbb{R}$ which agrees with G in some neighborhood of $\{0\} \times \mathbb{R}$ and such that $\Phi_1(k,y) = (k, \ell(y))$ for some $\mathcal{C}^m$ diffeomorphism ℓ. Let $\delta > 0$ be such that $k < \delta < 1$ and consider $R : [k,1] \times \mathbb{R} \to [k,1] \times \mathbb{R}$ defined as

$$R(x,y) = \begin{cases} \left(x, \dfrac{x-k}{\delta-k} y + \dfrac{\delta - x}{\delta - k} \ell(y)\right), & \text{if } k \leq x \leq \delta; \\ (x,y), & \text{if } \delta \leq x \leq 1. \end{cases}$$

Notice that R is a $\mathcal{C}^m$ diffeomorphism on each of the above two pieces and a global homeomorphism on the strip. Applying once more Theorem 2.2 we obtain a $\mathcal{C}^m$ diffeomorphism defined on $[k,1]\times\mathbb{R}$, that we call Φ_2. Notice that Φ_2 agrees with Φ_1 on $x=k$ and is the identity in $[\delta,1]\times\mathbb{R}$. Thus, applying again Theorem 2.2 to $\Phi:[0,1]\times\mathbb{R}\to[0,1]\times\mathbb{R}$ defined as

$$\Phi(x,y)=\begin{cases}\Phi_1(x,y), & \text{if } 0\le x\le k;\\ \Phi_2(x,y), & \text{if } k\le x\le 1,\end{cases}$$

we obtain the desired $\mathcal{C}^m$ diffeomorphism Ψ defined on $\overline{\mathcal{U}}=[0,1]\times\mathbb{R}$ that is the searched seed. This is so, because near $x=0$ the diffeomorphism Φ coincides with Φ_1 which, in turn, coincides with G, and near $x=1$ it coincides with Φ_2 that in that region is the identity. As a consequence, the theorem follows.

Acknowledgements

The first author is partially supported by the project PID2019-104658GB-I00 grant) and Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M), and the third author by the project MTM2017-86795-C3-1-P, all of the Spanish Ministerio de Ciencia e Innovación. The first and third authors are also supported by the grant 2017-SGR-1617 from AGAUR, Generalitat de Catalunya. The second author is partially supported by CSIC group 618-Uruguay.

References

[1] L. E. J. Brouwer. *Proof of the plane translation theorem (in German).* Math. Ann. **72** (1912), 37–54.

[2] A. Cima, A. Gasull, F. Manosas and R. Ortega. *Smooth linearization of planar periodic maps.* Math. Proc. Cambridge Philos. Soc. **167** (2019), 295–320.

[3] R. de la Llave and N. P. Petrov. *Regularity of conjugacies between critical circle maps: an experimental study.* Experiment. Math. **11** (2002), 219–241.

[4] J. Franks. *A new proof of the Brouwer plane translation theorem.* Ergod. Th. and Dynamic. Sys. **12** (1991), 217–226.

[5] L. Guillou. *Brouwer's plane translation theorem and generalizations of the Poincaré-Birkhoff theorem (in French).* Topology **33** (1994), 331–351.

[6] M. W. Hirsch. Differential Topology. Springer-Verlag, New York, 1976.

[7] P. Le Calvez and A. Sauzet. *A dynamical proof of Brouwer's translation theorem (in French).* Expo. Math. **14** (1996), 277–287.

[8] W. Zhang, K. Lu and W. Zhang *Differentiability of the conjugacy in the Hartman-Grobman theorem.* Trans. Amer. Math. Soc. **369** (2017), 4995–5030.

[†]Dept. de Matemàtiques, Universitat Autònoma de Barcelona, Edifici C, 08193 Cerdanyola del Vallès (Barcelona), Spain.
Email address: gasull@mat.uab.cat, manyosas@mat.uab.cat

[‡] Centre de Recerca Matemàtica, Edifici Cc, Campus de Bellaterra, 08193 Cerdanyola del Vallès (Barcelona), Spain

[♭] IMERL, Facultad de Ingeniería, Universidad de la República, Montevideo Uruguay
Email address: jorgeg@fing.edu.uy