EFFECTIVENESS OF THE BENDIXSON–DULAC THEOREM

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Abstract. We illustrate with several new applications the power and elegance of the Bendixson–Dulac theorem to obtain upper bounds of the number of limit cycles for several families of planar vector fields. In some cases we propose to use a function related with the curvature of the orbits of the vector field as a Dulac function. We get some general results for Liénard type equations and for rigid planar systems. We also present a remarkable phenomenon: for each integer $m \geq 2$, we provide a simple 1-parametric differential system for which we prove that it has limit cycles only for the values of the parameter in a subset of an interval of length smaller that $3\sqrt{2}(3/m)^{m/2}$ that decreases exponentially when m grows. One of the strengths of the results presented in this work is that although they are obtained with simple calculations, that can be easily checked by hand, they improve and extend previous studies. Another one is that, for certain systems, it is possible to reduce the question of the number of limit cycles to the study of the shape of a planar curve and the sign of an associated function in one or two variables.

1. Introduction

Despite all the efforts dedicated to solve the second part of the Hilbert's 16th problem, it is yet a very difficult task to obtain criteria that give explicit upper bounds for many concrete families of planar smooth vector fields. Although there is no any universal approach, the aim of this paper is to present several families of planar systems for which the Bendixson–Dulac theorem allows to get, in a fast and elegant way, an upper bound of their number of limit cycles. We will avoid results based on cumbersome computations.

The families that we will consider include extensions of Liénard systems and rigid systems. As we will see, we obtain new results and we also present simple proofs of some recent results. They give explicit upper bounds for several families of planar vector fields. These upper bounds are also sharpened when we deal with more particular systems, obtaining results of at most two, one, or none limit cycles.

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Our main results for Liénard type systems are contained in Section 3. They are given in Theorem 3.1, that deals with a version of Wilson Liénard systems which always have an algebraic limit cycle, in Theorem 3.4 that studies a family recently introduced in [26], in Theorem 3.9 that extends the classical theorem of Massera, and in Theorem 3.7. In fact, this last result includes the remarkable phenomenon highlighted in the abstract: the family

$$\begin{cases} \dot{x} = y - \lambda |y|^m (x^3 - x), \\ \dot{y} = -x, \end{cases}$$

introduced in [26], has for $m \geq 2$, limit cycles only for some values of λ contained in the interval of length $3\sqrt{2}(3/m)^{m/2}$, centered at the origin. Notice that for m big it is extremely thin. This interval decreases exponentially with m.
Our main result for general rigid systems is given in Theorem 4.1 of Section 4. It is applied to recover, in a simple way, known results for rigid cubic systems and to a family containing non polynomial vector fields.

It seems to us that not all the mathematical community that works on these topics is aware of the capability of the Bendixson–Dulac approach. The goal of this work is double: we try to change this perception and we also present several new results and easy proofs of some known results. For instance, in most textbooks, the proof of the uniqueness and hyperbolicity of the limit cycle for the classical van der Pol equation needs some work. By using this approach there are extremely simple proofs, see Corollary 3.5 and Remark 3.6.

The today known as Bendixson–Dulac theorem was first formulated by Ivar Bendixson in 1901 ([1]), and later developed by Henri Dulac in 1937 ([12]). He improved Bendixson's result by introducing a new parameterization of the time, via the today called Dulac functions. This result appears, under different versions, in most differential equations textbooks. One of the pioneers to try to go further with this approach was Yamato ([29]). Afterward, one of its main defenders was Cherkas, who used and developed it, see for instance [4, 5]. The authors of this work also often apply and try to extend this method, see [15, 16, 17]. More examples about its applicability can be seen in the survey [18].

In this paper we will use the version of the Bendixson–Dulac theorem that we state below, after introducing some notations and definitions. For completeness, in Section 2 we present a proof based on the version of Bendixson–Dulac theorem for multiply connected regions that is proved for instance in [4, 15, 23].

Given an open connected subset $\mathcal{U} \subset \mathbb{R}^2$, with finitely many holes, we will denote by $\ell = \ell(\mathcal{U})$ this number of holes, that is, the number of bounded components of $\mathbb{R}^2 \setminus \mathcal{U}$. Notice that if $\mathcal{U}$ is simply connected then $\ell(\mathcal{U}) = 0$. We also set $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$.

For a continuous function $f : \mathbb{R}^n \to \mathbb{R}$, not changing sign and vanishing on a null measure set, we will denote by $\operatorname{sign}(f)$ the sign of f at any of its point where it is not zero. Moreover, given an equilibrium point or a periodic orbit, when we say that its stability is given by the sign of f we mean that the object is an attractor (resp. a repeller) whenever $\operatorname{sign}(f) < 0$ (resp. $\operatorname{sign}(f) > 0$).

Definition 1.1. *Given a function $V : \mathbb{R}^2 \to \mathbb{R}$ of class $\mathcal{C}^1$ we will say that it is* admissible *if:*

(i) The vector ∇V vanishes on $\{V(x,y) = 0\}$ at finitely many points.
(ii) The set $\{V(x,y) = 0\}$ has finitely many connected components.
(iii) The set $\mathbb{R}^2 \setminus \{V(x,y) = 0\}$ has j connected components, $\mathcal{U}_i, i = 1, 2, \ldots j$, and for all of them $\ell(\mathcal{U}_i) < \infty$.

Associated to V, we define the non negative integer number

$$L(V) := \sum_{i=1}^{j} \ell(\mathcal{U}_i).$$

Theorem 1.2 (Bendixson–Dulac theorem)**.** *Consider a $\mathcal{C}^1$ planar differential system*

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \tag{1}$$

and denote by $X = (P, Q)$ its associated vector field. Let $V : \mathbb{R}^2 \to \mathbb{R}$ be an admissible function such that there exists $s \in \mathbb{R}^+$ for which the function

$$M_s := \frac{\partial V}{\partial x} P + \frac{\partial V}{\partial y} Q - s \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) V \tag{2}$$

does not change sign and vanishes only on a null measure set. Define

$$L_X(V) := N + L(V),$$

where N is the number of periodic orbits of X contained in the set $\mathcal{V} = \{V(x,y) = 0\}$.

Then, the differential system (1) *has at most $L_X(V)$ periodic orbits, which are limit cycles. Moreover, each limit cycle not contained in $\mathcal{V}$ is hyperbolic, it is contained in one of the connected components $\mathcal{U}_i$ of $\mathbb{R}^2 \setminus \mathcal{V}$ and, for each $i = 1, 2, \ldots, j$, there are at most $\ell(\mathcal{U}_i)$ limit cycles in the component $\mathcal{U}_i$. The stability of each of these limit cycles is given by the sign of $-VM_s$ on the region $\mathcal{U}_i$.*

Remark 1.3. *The function M_s, when $s \le 0$, can also be used to control te number of limit cycles of* (1)*, see* [4, 18]*. In particular, notice that $M_0 = \dot{V}$ and it can be readily seen that, when $s < 0$, the theorem also works, giving that $L_X(V) = N$. In this work, we do not use this range of values of s. In fact, in most of our applications we will use $s = 1$, although the values $s = 2$ and $s = 1/3$, also will appear.*

Observe also that, somehow, this version of the Bendixson–Dulac theorem relates the second part of the Hilbert's 16th problem, which deals with the number of limit cycles ([22]), with the first part, that deals with the number and distribution of the ovals of a planar algebraic curve ([27]).

Similarly of what happens when one tries to use Lyapunov functions, the main difficulty in the above theorem for its practical use is the choice of the function V and of the positive real number s. In other words, the choice of a suitable Dulac function. As we will see, the function that gives the curvature of the orbits of (1) is sometimes a good candidate for V.

Moreover, the most difficult condition to be checked is that M_s does not change sign. Hence, several approaches try to arrive to functions for which this question can be more easily studied. For instance, one of these situations is when it is a function of only one variable or the product of two functions of one variable, see again [18] for some examples. Another one is when, from some point of view, we can look to M_s as a quadratic polynomial.

Finally, notice that, given V and X, the computation of the number N in $L_X(V)$ is usually not difficult, while $L(V)$ depends on the topology of the set $\{V(x,y)=0\}$, see Section 2.2. When V is polynomial in one of its variables, to get an upper bound of $L(V)$ is an affordable task.

2. Preliminary results

For the sake of notation, from now on, in this paper we will denote the partial derivatives as subscripts. Hence, for instance, for $F=F(x,y)$, $F_x=\frac{\partial F}{\partial x}$, or $F_{x,y}=\frac{\partial^2 F}{\partial x \partial y}$.

2.1. **Proof of Theorem 1.2.**

We recall a version of the Bendixson–Dulac theorem for multiply connected regions, see [4, 15, 23].

Theorem 2.1. *Consider a $\mathcal{C}^1$ planar differential system*

$$\dot{x}=P(x,y), \quad \dot{y}=Q(x,y), \tag{3}$$

defined on $\mathcal{U}\subset\mathbb{R}^2$, an open subset such that $\mathbb{R}^2\setminus\mathcal{U}$ has ℓ bounded components, and denote by $X=(P,Q)$ its associated vector field. Let $B:\mathcal{U}\to\mathbb{R}^+\cup\{0\}$ be a $\mathcal{C}^1$ function such that

$$\operatorname{div}(BX)=(BP)_x+(BQ)_y$$

does not change sign and vanishes only on a null measure set. Then the system (3) has at most ℓ limit cycles in $\mathcal{U}$. Moreover, all of them are hyperbolic and their stability is given by the sign of $\operatorname{div}(BX)$.

To prove Theorem 1.2 first one has to show that the periodic orbits of (1) are either contained in $\mathcal{V}$ or do not cut this set. This fact follows because $M_s\big|_{\mathcal{V}}=\nabla V\cdot X=\dot{V}$ does not change sign. Hence system (1) can have some periodic orbits contained in $\mathcal{V}$, say that it has N, and all the others that are strictly contained in each of the connected components of $\mathbb{R}^2 \setminus \mathcal{V}$. Fix one of these connected components, say $\mathcal{U}_i$. To control the number of periodic orbits in this set we will apply Theorem 2.1 with $B = |V|^{-1/s}$ and $\mathcal{U} = \mathcal{U}_i$. Some computations give that

$$\operatorname{div}\left(|V|^{-1/s}X\right) = -\frac{1}{s}\operatorname{sign}(V)|V|^{-1/s-1}M_s \tag{4}$$

and, by hypothesis, this function does not change sign on $\mathcal{U}_i$. As a consequence, the maximum number of periodic orbits in $\mathcal{U}_i$ is $\ell(\mathcal{U}_i)$, as we wanted to prove. Moreover, by using (4) and again Theorem 2.1 we get that all of them are hyperbolic and their stability is given by the sign of $-VM_s$.

Remark 2.2. *Notice that in Theorem 1.2 nothing is said about the hyperbolicity of the limit cycles contained in $\mathcal{V}$. As we will see in Corollary 3.2, they can be hyperbolic or not.*

2.2. **About the practical calculation of $L(V)$.** Given an admissible function V, the computation of $L(V)$ relies on the study of the topology of each of the connected components $\mathcal{U}_i$, of $\mathbb{R}^2\setminus\mathcal{V}$, where $\mathcal{V} = \{V(x,y) = 0\}$. Then $L(V)$ is the sum of all the quantities $\ell(\mathcal{U}_i)$, where these values are the number of bounded components of $\mathbb{R}^2 \setminus \mathcal{U}_i$. In fact, it also holds that the fundamental group of $\mathcal{U}_i$ is $\pi_1(\mathcal{U}_i) = \mathbb{Z} * \overset{\ell)}{\cdots} * \mathbb{Z}$, where $\ell = \ell(\mathcal{U}_i)$. In all concrete situations appearing in this work there is a more direct way for obtaining $L(V)$. This number is simply the number of bounded connected components of $\mathcal{V}$.

2.3. **Curvature of the orbits.** It is know that the function

$$K^{\perp} := Q^2P_x + P^2Q_y - PQ(P_y + Q_x),$$

that is the numerator of the curvature of the orbits of the vector field $X^{\perp} = (-Q, P)$, orthogonal to the vector field $X = (P, Q)$, associated to the system (1), can be used to know the stability of the periodic orbits of (1) and other dynamical features of its phase portrait, see [6, 10, 14] or [30, p. 29]. For instance, Diliberto in 1950 proved that a limit cycle is hyperbolic and stable (resp. unstable) if and only if

$$\int_0^l K^{\perp}(\gamma(s))\,\mathrm{d}s < 0 \quad (\text{resp. } > 0),$$

where $\gamma(s)$ is its parameterization by the arc length and l is its length.

In this work we will see that the function

$$K := Q^2P_y - P^2Q_x + PQ(P_x - Q_y), \tag{5}$$

proportional to the numerator of the curvature of the orbits of X is, in several cases, a good candidate for a suitable choice of V in Theorem 1.2. Notice that $K = Q\dot{P} - P\dot{Q} = Q(P_xP + P_yQ) - P(Q_xP + Q_yQ)$. As far as we know, this is the first time that this function K is used to control the number of limit cycles of planar differential systems. We prove:

Theorem 2.3. *Consider planar system* (1) *of class* $\mathcal{C}^2$. *Assume that the function*

$$D := P^2Q\big(P_{xx} - 2Q_{xy}\big) + PQ^2\big(2P_{xy} - Q_{yy}\big) + Q^3P_{yy} - P^3Q_{xx}$$

does not change sign and vanishes on a null measure set. Then the system (1) *has at most* $L_X(V)$ *limit cycles, where* $V = K$ *is given in* (5) *and* $L_X(V)$ *is defined in Theorem 1.2.*

Proof. By taking $V = K$, as in equation (5), and $s = 1$, the function M_1 given in Theorem 1.2 is $M_1 = D$ and the theorem follows. □

We will apply this result at the end of Section 3 for Liénard systems and in Section 4 to rigid systems.

3. Liénard type systems

We present several applications of the Bendixson–Dulac theorem to two families related with Liénard systems.

3.1. **Liénard systems with an explicit solution.** We study a family of Liénard type equations introduced recently in [21] that includes the Wilson family of Liénard equations ([28]), which gave the first example of such equations having an algebraic limit cycle. More concretely, we consider systems

$$\begin{cases} \dot{x} = y - (x^2-1)B(x), \\ \dot{y} = -x(1+yB(x)), \end{cases} \tag{6}$$

where B is a $\mathcal{C}^1$ function. They have the invariant algebraic curve $C(x,y) = x^2+y^2-1 = 0$, because $C_xP + C_yQ = -2xBC$, where $X = (P,Q)$ denotes the vector field associated to (6). Hence, when this system has not equilibrium points on the curve, it is a periodic orbit. Moreover, depending on the choice of the function B, it can be a limit cycle.

The following result allows to extend, and to prove in an easier way, the recent results about the maximum number of limit cycles of the above system when $B(x) = x^3 - bx$ given in [3, 21].

Theorem 3.1. *Consider the system* (6) *with* $B(x) = x\int_0^x W(t)/t\,\mathrm{d}t - bx$, *where* W *is any function that does not change sign, vanishes at isolated points, and such that* B *is of class* $\mathcal{C}^1$. *Then this system has at most* $L+N$ *limit cycles, where* L *is the number of bounded connected components of the set* $\mathcal{B} = \{x \in \mathbb{R} : (B(x)+2x)(B(x)-2x) \geq 0\}$ *plus one, and* $N \in \{0,1\}$. *In fact* $N = 1$ *when* $\mathcal{C} = \{x^2+y^2-1=0\}$ *is free of equilibrium points of the system, and then this set is one of the limit cycles, and* $N = 0$ *otherwise. Moreover, all the limit cycles* *but $\mathcal{C}$ are hyperbolic and their stability is given by the sign of VW in the connected component of $\mathbb{R}^2 \setminus \{V(x,y)=0\}$ where they lie, with*

$$V = (1-x^2-y^2)\left(x^2+y^2+B(x)y\right). \tag{7}$$

Proof. Consider the function V given in (7) and $s=1$ in Theorem 1.2. Then,

$$M_1(x,y) = x(x^2+y^2-1)^2\big(B(x)-xB'(x)\big) = -x^2(x^2+y^2-1)^2W(x).$$

Hence, thanks to the imposed conditions on W, we can apply Theorem 1.2. Moreover, since $\mathcal{C}$ is invariant, and contained in $\mathcal{V}=\{V(x,y)=0\}$, we have that $N\in\{0,1\}$ and the number of limit cycles of the system is bounded by $L(V)+N$. To get $L(V)$ we study the bounded connected components of $\mathcal{V}$, see Section 2.2. Notice that these components are formed by the oval $\mathcal{C}$ together with the bounded connected components of $x^2+y^2+B(x)y=0$. Since this curve also writes as

$$y = \frac{-B(x)\pm\sqrt{(B(x)+2x)(B(x)-2x)}}{2},$$

it is clear that these components are obtained by joining the curves plus and minus defined for x on each of the bounded connected components of $\mathcal{B}$. Hence $L(V)$ is at most L and the theorem follows. $\square$

The following corollary gives an easier and different proof of all the results about the maximum number of limit cycles of (6) when $B(x)=x^3-bx$,

$$\begin{cases} \dot{x} = y-(x^2-1)(x^3-bx), \\ \dot{y} = -x\big(1+y(x^3-bx)\big), \end{cases} \tag{8}$$

obtained in [3, 21]. It also solves in the best possible way the some times called *Coppel's problem* for polynomial systems, which in his own words (when restricted to quadratic systems) says: "Ideally one might hope to characterize the phase portraits of quadratic systems by means of algebraic inequalities on the coefficients," see [9]. The relevant values describing the bifurcations of the limit cycles of this system are $\underline{b}$ and b^*, see again [21]. We will show below that the value $\underline{b}\approx -1.44$ is algebraic. It is the negative root of the polynomial

$$4b^6-12b^5-4b^4+28b^3+56b^2-72b-229=0, \tag{9}$$

which is invariant under the change of variables $b\to 1-b$. The quantity $b(1-b)$ satisfies a third degree polynomial equation and then it is possible to express all the roots in terms of radicals but we prefer to omit their explicit expressions because they are rather complicated. The value $b^*\approx 0.747$ is the only zero of the function $Z:(\underline{b},1-\underline{b})\to\mathbb{R}$,

$$Z(b) = \int_0^1 \frac{8(b-3x^2)\sqrt{1-x^2}}{x^8-(2b+1)x^6+(b+2)bx^4-b^2x^2+1}\,\mathrm{d}x.$$

The sign of this function gives the stability of $\mathcal{C} = \{x^2 + y^2 - 1 = 0\}$ when $\mathcal{C}$ is a limit cycle. Most probably b^* is a non-algebraic number. The function Z was obtained in [21] from the integral of the divergence of the system on the algebraic limit cycle after some algebraic manipulations. In fact, the discriminant with respect to x of the denominator of the integrand gives the polynomial of the left-hand side of (9) that determines $\underline{b}$.

Corollary 3.2. *System* (8) *has at most two limit cycles, taking into account their multiplicities. More concretely:*

(i) It has no limit cycle for $b \le \underline{b}$.
(ii) $\mathcal{C}$ is its only limit cycle, and it is hyperbolic and attractor when $b \in (\underline{b}, 0]$.
(iii) It has two limit cycles, one hyperbolic, repeller and surrounded by $\mathcal{C}$, and $\mathcal{C}$ itself, which is hyperbolic and attractor, when $b \in (0, b^*)$.
(iv) $\mathcal{C}$ is its only limit cycle, and it is double and semi-stable when $b = b^*$.
(v) It has two limit cycles, one hyperbolic, attractor and surrounding $\mathcal{C}$, and $\mathcal{C}$ itself, which is hyperbolic and repeller, when $b \in (b^*, 1 - \underline{b})$.
(vi) It has one limit cycle surrounding $\mathcal{C}$ that is hyperbolic and attractor, when $b \ge 1 - \underline{b}$.

Proof. First, we will prove the most difficult part: the maximum number of limit cycles of the system is three. This will essentially be a direct consequence of Theorem 3.1. All the other results about this system can be obtained from the standard techniques of the qualitative theory of planar differential systems.

When $b \le 0$ the only limit cycle is $\mathcal{C}$ because in polar coordinates, $\dot r = r(r^2 - 1)(b - r^2 \cos^2\theta)\cos^2\theta$ does not vanish outside $\mathcal{C}$.

When $b \ge 3/2$, the maximum number of limit cycles is two. To prove this we apply Theorem 1.2 with $V(x, y) = x^2 + y^2 - 1$ and $s = 1/3$. Then

$$M_{1/3}(x, y) = \frac{1}{3}(x^2 + y^2 - 1)\big((2b - 3)x^2 + b\big) \ge 0.$$

Since for these values of b, $M_{1/3}$ does not vanish outside $\mathcal{C}$ the maximum number of limit cycles is two, one being $\mathcal{C}$ and at most another one can exist, and in this case it must surround $\mathcal{C}$.

Finally, consider the values of $b \in (0, 3/2)$. In fact, we can consider $b \in [0, 2]$. We apply Theorem 3.1 with $W(x) = 2x^2 \ge 0$. We get that $B(x) = x^3 - bx$, and

$$\begin{aligned}\mathcal{B} &= \{x \in \mathbb{R} : x^2(x^2 - 2 - b)(x^2 + 2 - b) \ge 0\} \\ &= \big(-\infty, -\sqrt{2 + b}\,\big] \cup \{0\} \cup \big[\sqrt{2 + b}\,, \infty\big).\end{aligned}$$

Hence, the number of bounded connected components of $\mathcal{B}$ is one and, as a consequence, $L(V) = 2$ and the maximum number of limit cycles is three. Also, from the proof we know that if the three limit cycles exist, one is $\mathcal{C}$, there is at most another one, say γ, surrounded by $\mathcal{C}$, and a third one Γ, surrounding $\mathcal{C}$.

To reduce this upper bound of three limit cycles by one it suffices to consider the stability of the origin, the infinity, the possible limit cycles and the invariant set $\mathcal{C}$. In fact we have that,

(A) The stability of the origin is given by the sign of b. Moreover, it is not difficult to see that when $b \in (\underline{b}, 1-\underline{b})$ the origin is the only equilibrium point of the system and that, otherwise, there are also other equilibrium points, but all of them are on $\mathcal{C}$.
(B) The set $\mathcal{C}$, which is always invariant by the flow, is a limit cycle if and only if $b \in (\underline{b}, \overline{b})$. Moreover it is hyperbolic and stable if $b \in (\underline{b}, b^*)$, hyperbolic and repeller if $b \in (b^*, \overline{b})$, and semi-stable and double when $b = b^*$. In fact, in this later case it is repeller from its interior and attractor from its exterior, see [21]. Moreover, it is also proved in that paper, that when $b \geq \overline{b}$, the set $\mathcal{C}$, that it is no more a periodic orbit, is also a repeller.
(C) The infinity is repeller for $b > 0$, see again [21].
(D) For $b \in (0, 2)$, whenever they exist, γ is hyperbolic and repeller and Γ is hyperbolic and attractor. This is a consequence of Theorem 3.1, because their respective stabilities are controlled by the sign of VW, that coincides with the sign of $1 - x^2 - y^2$, because

$$VW = 2x^2(1 - x^2 - y^2)(x^2 + y^2 + (x^3 - bx)y),$$

and for these values of b the limit cycles must lie in the region $\{x^2 + y^2 + (x^3 - bx)y > 0\}$ because it is the only connected component $\mathcal{U}$ of $\mathbb{R}^2 \setminus \mathcal{V}$ with $\ell(\mathcal{U}) \neq 0$.
(E) For $b \geq 3/2$, γ never exits and Γ is also hyperbolic and atractor, because as we have proved above by using Theorem 1.2, its stability is also given by the sign of

$$-V(x, y)M_{1/3}(x, y) = -\frac{1}{3}(x^2 + y^2 - 1)^2\big((2b - 3)x^2 + b\big) \leq 0.$$

For instance we will prove item (v). All the other cases follow similarly. First notice that by (B), $\mathcal{C}$ is a hyperbolic and repeller limit cycle. Recall that we already have proved that the system has at most one limit cycle surrounded by $\mathcal{C}$, and another one surrounding $\mathcal{C}$. Moreover, whenever they exist they are hyperbolic and their stabilities are given in (D). By (A) and (C), since the origin is attractor and the infinity is repeller, we get that there is no limit cycle surrounded by $\mathcal{C}$ and there is exactly one limit cycle, hyperbolic and stable, surrounding $\mathcal{C}$.

□

Remark 3.3. *System* (8) *can be transformed into the classical Liénard system*

$$\begin{cases} \dot{x} = y - bx + x^3 + \frac{4b}{3}x^3 - \frac{6}{5}x^5, \\ \dot{y} = -x + b^2x^3 - b(2+b)x^5 + (1+2b)x^7 - x^9. \end{cases} \tag{10}$$

By using Theorem 1.2 with $s = 1$ *and* $V(x,y) = A(x,y)B(x,y)$ *where*

$$\begin{aligned} A(x,y) =& -225 + 225x^2 + 25b^2x^6 - 30bx^8 + 9x^{10} \\ &+ (150bx^3 - 90x^5)y + 225y^2, \\ B(x,y) =& 225x^2 - 75b^2x^4 + 5b(24+5b)x^6 - 15(3+2b)x^8 + 9x^{10} \\ &+ (-225bx + 25(9+6b)x^3 - 90x^5)y + 225y^3, \end{aligned}$$

we get that $M_1 = 2x^4A^2(x,y) \geq 0$. *Hence, in these variables an upper bound of the number of limit cycles of system* (8) *can also be obtained. This example illustrates that although sometimes it is difficult to find a suitable* V *to apply Theorem 1.2, it seems to exist.*

3.2. **Some extended Liénard systems.** We consider planar differential equations of the form

$$\begin{cases} \dot{x} = y - |y|^m F(x), \\ \dot{y} = -G'(x)/2, \end{cases} \tag{11}$$

where F and G' are $\mathcal{C}^1$ functions satisfying $F(0) = 0$ and $G(x) = x^{2k} + o(x^{2k})$, $m \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$. Notice that when $G(x) = x^2$ and $m = 0$, they include the classical second order Liénard equations $\ddot{x} + F'(x)\dot{x} + x = 0$. The factor $|y|^m$ is added following the recent work [26], where this interesting system was studied for the first time. Notice that if instead of $y - |y|^m F(x)$ we consider the same system but with the first component equal to $y - y^m F(x)$, then, when m is odd, it would be invariant by the change of variables and time $(x, y, t) \to (x, -y, -t)$ and the origin would be a reversible center.

In all our study we skip the case $m = 1$, where the associated vector field is not of class $\mathcal{C}^1$. In any case, for $m = 1$, and on each of the regions $y > 0$ and $y < 0$, the vector field is integrable (it corresponds to a differential equation of separated variables) and by using the level curves of the corresponding first integrals, their phase portraits are easier to be studied. This approach is the one used in [26] for this case, when $G(x) = x^2$.

Theorem 3.4. *Consider the differential system* (11) *with* $m \neq 1$. *If the function* $H := (m-1)FG' + 2F'G$ *does not change sign and vanishes at isolated points, then the system has at most* J *limit cycles, all of them hyperbolic, where* J *is the number of zeroes of* G'. *In particular, if* G' *only vanishes at the origin the differential system has at most one limit cycle.*

Proof. We apply the Bendixson–Dulac theorem with $V(x,y) = G(x) + y^2 - y|y|^m F(x)$ and $s = 1$. Simple computations give that

$$M_1 = \frac{1}{2}|y|^m H(x).$$

Therefore, since M_1 satisfies the hypothesis of the Bendixson–Dulac theorem we have already proved that system (11) has at most $L_X(V)$ limit cycles. We claim that $L_X(V) \leq J$. Since the set $\mathcal{V} = \{V(x,y) = 0\}$ does not contain solutions of the differential system the claim will follow if we prove that $\mathcal{V}$ has at most J bounded connected components, see Section 2.2. Notice that each of these components can be an oval, an isolated point, or a more complicated set.

To prove this last assertion we first count the number of points of $\mathcal{V} \cap \{x = x_0\}$, taking into account their multiplicity, and we call it $K(x_0)$. When $m = 0$ it is clear that $K(x_0) \leq 2$, because $V(x_0, y) = 0$ is a quadratic equation in y. When $m \geq 2$, the equation $V(x_0, y) = 0$ splits into two trimonomial equations, one for $y \geq 0$ and another one for $y \leq 0$. By applying the Descarte's rule of signs to both equations, since the monomial y^2 appears in both, it can be seen that $K(x_0) \leq 3$.

Notice that since on $\mathcal{V}$, $M_1 = \dot{V}$, each bounded connected component of $\mathcal{V}$ delimits some region either positively or negatively invariant, and as a consequence its interior must contain at least one equilibrium point (x^*, y^*) of the system. Observe also that even when the system has other equilibrium points on the line $\{x = x^*\}$, only one connected component of $\mathcal{V}$ can cut this line, because $K(x^*) \leq 3$. Hence the bounded connected components of the set $\mathcal{V}$ must cut the lines $\{x = x^*\}$, where $G'(x^*) = 0$, and at most one of them cuts each of the lines. As a consequence, $\mathcal{V}$ has at most J bounded connected components, and $L(V) \leq J$ as we wanted to prove. □

Theorem 3.4 can be applied to several differential systems (11). For m and G fixed, let W be a function that does not change sign, vanishes at isolated points, and such that the initial value problem for the linear differential equation

$$(m-1)F(x)G'(x) + 2F'(x)G(x) = W(x), \quad F(0) = 0, \tag{12}$$

has a regular solution F. Notice that (12) is singular at the zeroes of G and we impose that F must be smooth at these points. Then the correspondent differential system (11) is under the hypotheses of the theorem. By using this point of view, we have obtained several families of differential systems where it is easy to impose that their corresponding functions H do not change sign and, as a consequence, Theorem 3.4 can be applied. We will skip all the hypotheses that must be added to guarantee the desired property for H, and the other ones that the functions F and G must satisfy to fulfill all the other hypotheses of the theorem, because the reader can easily figure out them. These families are:

(i) When $F(x) = A^p(x)A'(x)B(x)$, $G(x) = cA^q(x)(A'(x))^2B^2(x)$, and $m = 0$. Then, it holds that
$$H(x) = (2p-q)cA^{p+q-1}(x)(A'(x))^4B^3(x).$$

(ii) When $m = 0$,
$$F(x) = A^{2p}(x)A'(x)B^{q+1}(x), \text{ and}$$
$$G(x) = cA^{4p}(x)(A'(x))^2B^q(x),$$
we get that
$$H(x) = (q+2)cA^{6p}(x)B^{2q}(x)(A'(x))^3B'(x)$$

(iii) When $m = 2k$, $G(x) = x^{2k}$, and
$$F(x) = \frac{1}{2}x^{k(1-2k)}\int_0^x y^{k(2k+1)}Z(y)\,\mathrm{d}y,$$
we obtain that $H(x) = x^{4k}Z(x)$.

(iv) When $m = 0$, $F(x) = a(x^3/3 - x)$ and
$$G(x) = x^2 - \Big(\frac{a^2}{8} + 6b\Big)x^4 + \Big(\frac{a^2}{48} + b\Big)x^6,$$
we get that
$$H(x) = \frac{a(16 - 3a^2 - 144b)}{12}x^4.$$

Now we will study in more detail some particular sub-cases of the above families and we will refine the upper bound for their number of limit cycles given in Theorem 3.4.

We start with an example contained in the family given in item (i). It corresponds to $p = 1$, $q = 0$, $c = 1/4$, $A(x) = x^3/3 - x^2/2$ and $B(x) = -2$, and writes as
$$\begin{cases} \dot{x} = y + \dfrac{1}{3}x^3(x-1)(2x-3), \\ \dot{y} = -x(x-1)(2x-1), \end{cases} \tag{13}$$
with $G(x) = x^2(1-x)^2$. We will prove that this system has at most one limit cycle, hyperbolic and stable. The existence of this limit cycle, that surrounds the three equilibrium points of (13), can be numerically confirmed.

By using Theorem 3.4 when $m = 0$ and with
$$V(x,y) = x^2(x-1)^2 + \frac{1}{3}x^3(x-1)(2x-3)y + y^2$$
we get that $H(x) = -4x^4(x-1)^4 < 0$ and the maximum number of limit cycles of the corresponding system is three, which is the number of zeroes of $G'(x) = 2x(x-1)(2x-1)$. This upper bound can be

reduced to two studying in more detail the set $\mathcal{V}$. This set is formed by two isolated critical points located at $(0,0)$ and $(1,0)$, and two disjoint curves going from infinity to infinity. The point $(0,0)$ is a weak focus and the point $(1,0)$ is a strong stable focus. The third critical point, located at $(1/2,-1/24)$, is a saddle point. By computing the first Lyapunov quantity associated to the weak focus at the origin we conclude that this point is repulsive. In fact, $\mathbb{R}^2 \setminus \mathcal{V}$ is formed by three open sets, two are simply connected and the third one has two holes (the two critical points located on the x axes). In short $L(V)=2$ and since $\mathcal{V}$ does not contain periodic orbits, the upper bound of two limit cycles follows from Theorem 1.2.

Finally, we prove that one is the actual upper bound for the number of limit cycles of (13). By Theorem 1.2 the stability of the limit cycles is given by the sign of $-V(x,y)M_1(x)$ that coincides with the sign of $-H(x)>0$. Hence all of them are repelling hyperbolic limit cycles. By using the Poincaré–Bendixson theorem it can be seen that the only situations compatible with these results are that either (13) has no limit cycle, or that it has exactly one, as we wanted to show.

The same tools allow to prove that

$$\begin{cases} \dot{x} = y - bx^3(x-a)(2x-3a), \\ \dot{y} = -x(x-a)(2x-a), \end{cases}$$

has at most one limit cycle. In fact, notice that when $b=0$ it can be easily integrated. It has two centers and a saddle point, and the separatrices of this saddle point form two homoclinic trajectories which, together with the critical point, have an eight shape. Numerically, the limit cycle seems to bifurcate for $b \approx 0$ from this double loop and whenever it exists, it surrounds the three equilibrium points of the system, a saddle, a strong focus and a weak focus, both with different stabilities.

Similar examples to system (13), with more equilibrium points surrounded by a limit cycle and for which Theorem 3.4 also works are not difficult to be constructed. For instance, if we take $m=0$, $F(x) = cx^3(1-x)(2-x)^3$ and $G(x)=x^2(1-x)^2(2-x)^2$, with $|c|<2$, we get a system with five critical points, two saddles and three foci, for which $H(x)=8cx^4(1-x)^4(2-x)^4$.

Also, because it contains the classical van der Pol differential equation, we particularize in detail a subfamily of the one given in item (ii). If we consider $p=q=0$, $c=1$, and $A'=C$ in (ii) we get the following result.

Corollary 3.5. *Consider the $\mathcal{C}^1$ differential system*

$$\begin{cases} \dot{x} = y - C(x)B(x), \\ \dot{y} = -C(x)C'(x), \end{cases}$$

with $C(0) = 0$ *and* $C'(x) \neq 0$ *for* $x \neq 0$. *If* $C(x)B'(x)$ *does not change sign and vanishes at isolated points, then this system has at most one limit cycle and when it exists it is hyperbolic.*

Notice that the van der Pol equation corresponds to $C(x) = x$ and $B(x) = \lambda(x^2/3-1)$. Then $C(x)B'(x) = 2\lambda x^2/2$, which does not change sign.

Proof of Corollary 3.5. For these particular cases of differential systems contained in the family (ii) we get that

$$H(x) = 2C^3(x)B'(x).$$

Hence, it does not change sign and vanishes at isolated points. Notice that C only vanishes at $x = 0$, because if $C(z) = 0$, by Rolle's theorem C' would vanish at a point between 0 and z. Hence, $G'(x) = 2C(x)C'(x) = 0$ only at $x = 0$, and the corollary follows. Observe also that in this case $V(x,y) = C^2(x) + y^2 - yC(x)B(x)$, and the set $\{V(x,y) = 0\}$ has only one bounded connected component, the origin, and then $L_X(V) = 1$. □

Remark 3.6. *For completeness we reproduce a second easy proof of the uniqueness and hyperbolicity of the limit cycle of the van der Pol equation attributed to Cherkas in* [7, p. 105]. *Write the equation as the system*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - \lambda(x^2-1)y. \end{cases}$$

By applying the Bendixson–Dulac theorem with $V = x^2+y^2-1$ *and* $s = 2$ *we get that* $M_2 = 2\lambda(x^2-1)^2$. *Clearly, the unit circle is not a periodic orbit of the system, and* $\{V(x,y) = 0\}$ *has two connected components, one bounded and simply connected and a second one, say* $\mathcal{U}$, *with* $\ell(\mathcal{U}) = 1$. *Hence* $L_X(V) = 1$ *and the result follows.*

When $b = 0$, the system introduced in item (iv) corresponds to the Wilson Liénard equation ([28]) and when $|a| < 2$ it has the algebraic limit cycle

$$y^2 - \frac{a}{6}x^3y + \frac{1}{144}(a^2x^6 + 144x^2 - 576) = 0.$$

Since this limit cycle is also hyperbolic we get that for $|b|$ small enough the limit cycle persists and our theorem applies to get an upper bound of the total number of limit cycles of the system when $b \neq 0$. We skip more details because the study of this system is quite similar to the one that we did for system (13).

We end this section studying in more detail the particular family of differential systems of the form (11), introduced in [26],

$$\begin{cases} \dot{x} = y - \lambda|y|^m(x^3 - x), \\ \dot{y} = -x, \end{cases} \tag{14}$$

where $m \geq 2$ is an integer number and $\lambda \in \mathbb{R}$. Notice that the factor $x^3 - x$ in (14) can be changed by $c^2x^3 - x$, with another value of λ, obtaining the same phase portrait. This is so, because by doing the change of variables $(x, y) \to (cx, cy)$, with $c > 0$, the first equation writes as $\dot{x} = y - c^m\lambda|y|^m(c^2x^3 - x)$ and the second one remains invariant. We do not take the factor as $x^3/3 - x$, which corresponds to the van der the Pol equation when $m = 0$, simply to keep the notation of [26]. We prove:

Theorem 3.7. *Consider the differential system* (14) *with* $m \in \mathbb{N}$, *and* $m \geq 2$. *Then, the following holds:*

(i) For $|\lambda| \neq 0$ *small enough it has at least one limit cycle.*

(ii) For $|\lambda| \geq \dfrac{3}{\sqrt{2}}\left(\dfrac{3}{m}\right)^{m/2}$ *it has no limit cycle.*

Proof. Notice that the case $\lambda = 0$ corresponds to a linear center, and the phase portrait when $\lambda < 0$ can be easily obtained from the one with $\lambda > 0$, simply by doing the change of variables and time $(x, y, t) \to (x, -y, -t)$. Then, it suffices to make the proof for the case $\lambda > 0$.

(i) Given any $\mathcal{C}^1$ perturbed Hamiltonian systems,

$$\begin{cases} \dot{x} = \dfrac{\partial H(x,y)}{\partial y} + \varepsilon R(x,y), \\ \dot{y} = -\dfrac{\partial H(x,y)}{\partial x} + \varepsilon S(x,y), \end{cases} \tag{15}$$

where ε is a small parameter, its associated Melnikov–Poincaré–Pontryagin function is

$$M(h) = \int_{\gamma(h)} S(x,y)\,\mathrm{d}x - R(x,y)\,\mathrm{d}y,$$

where the curves $\gamma(h)$, for $h \in (h_0, h_1)$, form a continuum of ovals contained in $\{H(x,y) = h\}$. It is known that each simple zero $\bar{h} \in (h_0, h_1)$ of M gives rise to a limit cycle of (15) that tends, when $\epsilon \to 0$, to $\gamma(\bar{h})$, see for instance [8, 11].

Consider the differential system (14) with $\lambda = \varepsilon$. By applying the above result with $H(x,y) = x^2 + y^2 = h = r^2$, with $r \in (0, \infty)$, and taking the parameterization of the level sets as $x = r\cos\theta$, $y = r\sin\theta$, we get that

$$\begin{aligned} M(r^2) &= \int_{x^2+y^2=r^2} |y|^m(x^3 - x)\,\mathrm{d}y \\ &= \int_0^{2\pi} r^m|\sin\theta|^m(r^4\cos^4\theta - r^2\cos^2\theta)\,\mathrm{d}\theta \\ &= \frac{\sqrt{\pi}}{2}\frac{\Gamma\big((m+1)/2\big)}{\Gamma\big((m+6)/2\big)} r^{m+2}\big(3r^2 - (m+4)\big), \end{aligned}$$

where Γ is the Euler Gamma function. Hence, for each m, this function has a simple positive zero $r = \sqrt{(m+4)/3}$, that gives rise to the desired limit cycle.

(ii) We will apply Theorem 1.2 with $s = 1/3$ and

$$V(x,y) = \exp\left(\frac{\lambda^2 y^{2m}}{9m}\right)\left(3 + \lambda xy|y|^{m-2}\right).$$

Some calculations give that

$$M_{1/3} = -\frac{1}{9}\exp\left(\frac{\lambda^2 y^{2m}}{9m}\right)\lambda x^2|y|^{m-2}\left(2\lambda^2 y^{2m} - 27y^2 + 9(m-1)\right).$$

We need that $M_{1/3}$ does not change sign. Hence, writing $y^2 = z$ we want that

$$z^m - \frac{27}{2\lambda^2}z + \frac{9(m-1)}{2\lambda^2} \geq 0 \quad \text{for} \quad z \geq 0. \tag{16}$$

Let us prove, that given a real polynomial $P(z) = z^m + bz + c$, with $m \geq 2$, it holds that $P(z) \geq 0$ for all $z \geq 0$ if and only if

$$b \geq -m\Big(\frac{c}{m-1}\Big)^{(m-1)/m}. \tag{17}$$

Since $P(0) = c$, an obvious first condition is that $c \geq 0$. When $b \geq 0$ the result is trivial. When $b < 0$, since $p'(z) = mz^{m-1} + b$, the function P has a minimum at $z = z_0 = (-b/m)^{1/(m-1)}$. By imposing that $P(z_0) \geq 0$, (17) follows after some straightforward computations.

Condition (17) applied to the polynomial (16) gives that

$$-\frac{27}{2\lambda^2} \geq -m\left(\frac{9}{2\lambda^2}\right)^{(m-1)/m}.$$

After some manipulations we get that this inequality is equivalent to the one given in the statement.

Hence, we are under the hypotheses of Theorem 1.2. Moreover, since $\{V(x,y) = 0\}$ does not contain ovals, and all the connected components of $\mathbb{R}^2 \setminus \{V(x,y) = 0\}$ are simply connected, we have that $L_X(V) = 0$ and the system has no limit cycle, as we wanted to prove. □

Similar computations that the ones of item (i) are done in the Appendix of [26] for the case $m = 2$.

Remark 3.8. *The result of item (ii) of Theorem 3.7 shows that for any $m \geq 2$ there exits a value $\lambda = \lambda^*(m)$ such that for $|\lambda| \geq \lambda^*(m)$ system* (14) *has no limit cycle. Moreover, it gives an upper bound of this value.*

For $m = 2$ it is not a sharp bound, because in [26] *the authors study numerically the system and they find that $\lambda^*(2) \in (1.474, 1.475)$, while our bound is $9\sqrt{2}/4 \approx 3.182$. Nevertheless, by using it, for m big enough, we prove that the limit cycles only exist for λ in an extremely*

thin interval of length $3\sqrt{2}\Big(\dfrac{3}{m}\Big)^{m/2}$ *that decreases exponentially when* m *grows.*
It is the first time that the authors see a proof of the existence of this type of exponentially small intervals for the presence of limit cycles.

3.3. **About Massera's theorem.** Consider the classical Liénard equation

$$\begin{cases} \dot{x} = y - F(x), \\ \dot{y} = -x, \end{cases} \tag{18}$$

with F a class $\mathcal{C}^2$ function satisfying $F(0) = 0$. We prove, in a very simple way, the following extension of the classical Massera's theorem ([24, 25]), where the hyperbolicity of the limit cycle is also guaranteed. Other authors had already proved this hyperbolicity, see for instance [19].

Theorem 3.9. *Consider the differential system* (18)*. If the function* $xF''(x)$ *does not change sign and vanishes at isolated points, then it has at most one limit cycle and when it exists it is hyperbolic.*

Proof. We will apply Theorem 1.2 with V given by the function K, defined in (5), associated to the curvature of the system, and $s = 1$. By using the results of Theorem 2.3 when $P = y - F(x)$ and $Q = -x$, we obtain that

$$V = K = x^2 + y^2 + F^2 - 2yF + x\big(y - F\big)F'$$

and $M_1 = (y - F)^2 xF''$. Hence M_1 satisfies the hypothesis of Theorem 1.2. To end the proof we have to show that $L_X(V) \leq 1$. Since the set $\mathcal{V} = \{V(x, y) = 0\}$ does not contain orbits of the system, it suffices to prove that $\mathcal{V}$ has at most one bounded connected component, see Section 2.2. Clearly the points of $\mathcal{V}$ lie on the two curves

$$y = F(x) - \frac{1}{2}xF'(x) \pm \frac{1}{2}\sqrt{x^2\big((F'(x))^2 - 4\big)}.$$

Therefore the bounded connected components of $\mathcal{V}$ are given by $x = 0$ and the bounded subsets of $\mathbb{R}$, where $(F'(x))^2 - 4 \geq 0$. These components are either positively or negatively invariante by the flow of the system because $M_1\big|_{\mathcal{V}} = \dot{V}$ does not change sign. Hence they must surround some of the equilibrium points of the system. Since the origin is the only equilibrium point, there is at most one of these components. Hence, $L_X(V) \leq 1$, as we wanted to prove. $\square$

We want to emphasize the surprising simplicity of the proof of this classical theorem with the methods employed in this work.

4. Rigid systems

These systems write as

$$\begin{cases} \dot{x} = -y + xF(x,y), \\ \dot{y} = \quad x + yF(x,y), \end{cases} \tag{19}$$

where F is an arbitrary smooth function. This name is due to the fact that in the usual polar coordinates (r,θ) it holds that $\dot{\theta} = 1$ and, therefore, their flow rotates around the origin with constant angular velocity, as a rigid rotation. Despite their simplicity and the fact that they have the origin as the unique equilibrium point, the control of the number of limit cycles of these systems is far to be completely known. They were introduced by Conti in [2] and studied by several authors. We prove the following result for them:

Theorem 4.1. *Let X be the vector field associated to* (19)*. If F is of class $\mathcal{C}^2$ and it holds that*

$$H := F_{xx}F_{yy} - F_{xy}^2 \geq 0, \tag{20}$$

and H vanishes on a null measure set, then (19) *has at most $L_X(V)$ limit cycles, where*

$$V = (x^2+y^2)\left(xFF_x + yFF_y + xF_y - yF_x - 1 - F^2\right) \tag{21}$$

and $L_X(V)$ is defined in Theorem 1.2.

Proof. We apply again Theorem 1.2 with $V = K$ and $s = 1$, where K is given in (5). We can use the results of Theorem 2.3 with $P = -y + xF$ and $Q = x + yF$. We get that V is as in (21) and

$$\begin{aligned} M_1 = D = (x^2+y^2)\Big(&\big(x^2F_{xx} + 2xyF_{xy} + y^2F_{yy}\big)F^2 \\ &+ 2\big((x^2-y^2)F_{xy} + xy(F_{yy} - F_{xx})\big)F \\ &+ \big(x^2F_{yy} - 2xyF_{xy} + y^2F_{xx}\big)\Big). \end{aligned}$$

To control the sign of M_1 we first remove the factor x^2+y^2. Notice that the discriminant of the remaining part, thinking it as a second degree polynomial in F, $AF^2 + BF + C$, is $B^2 - 4AC = -4(x^2+y^2)^2H \leq 0$. Moreover, looking to A and B as quadratic homogenous polynomials of the form $ax^2+bxy+cy^2$, we get that their corresponding discriminants coincide and are given by $b^2-4ac = -4H \leq 0$. Therefore, the condition (20) implies that M_1 does not change sign and vanishes only on a null measure set and hence our result follows. □

Notice that the upper bound for the number of limit cycles given in the above theorem essentially depends on the shape of the set $\{V(x,y) = 0\}$. To get the actual value of $L_X(V)$ for each case this set must be carefully studied. We present now a concrete application when F is a quadratic polynomial.

Corollary 4.2. *Consider the rigid cubic system* (19)*, with* $F = a + bx + cy + dx^2 + exy + hy^2$*. If* $4dh - e^2 > 0$ *this system has at most one limit cycle, and when it exists it is hyperbolic.*

This result is not new. It was proved in [20] by using a totally different approach: the authors transform the system into a periodic Abel differential equation and then they apply know results about these equations. In that work it is also proved that when $4dh - e^2 < 0$ there are systems with at least two limit cycles. Our proof is different and self-contained. Another proof, based on the study of the stability of the possible periodic orbits, is given in [13].

Proof. The function H of Theorem 4.1 is $H(x, y) \equiv 4dh - e^2 > 0$ and hence the system has at most $L_X(V)$ limit cycles. Here

$$V(x, y) = (x^2+y^2)\Big(-1-a^2+(c-ab)x-(ac+b)y+e(x^2-y^2)+2(h-d)xy + (bx + cy)(dx^2 + exy + hy^2) + (dx^2 + exy + hy^2)^2\Big).$$

It is easy to verify that the set $\{V(x, y) = 0\}$ does no contain orbits of the system.

As the origin is the unique finite critical point of the system and M_1 does not change sign, the bounded connected components of the set $\mathcal{V} = \{V(x, y) = 0\}$ must surround the equilibrium point. In principle, from the degree of $V(x, y)$ we can conclude that the maximum number of them is three, being the origin one of these components. But it is easy to show that there are at most two bounded connected components. This is so, because if we take $y = 0$ in the second factor of V we obtain a polynomial in x of degree four, where the coefficient of x^4 and the independent term are of opposite sign. Then it is not possible to have two positive roots and two negative roots at the same time. Therefore, the number of connected components in the set $\mathcal{V}$ is at most two, one of them being the origin.

In the case where the second bounded connected component exists it is not difficult to show that a limit cycle exterior to it cannot exist. The first step is to determine the stability of infinity. Writing the system in polar coordinates it is possible to show that, if $4dh - e^2 > 0$, the infinity is an attractor for $d > 0$ and it is repulsive for $d < 0$. Moreover, it can be seen by using that $M_1\big|_{\mathcal{V}} = \dot{V}$, that the flow associated to the system traverses this second bounded component forward for $d > 0$ and inward for $d < 0$. As between this bounded connected component of $\mathcal{V}$ and infinity only one limit cycle can exist, and if it exists it is hyperbolic, taking into account the stability of infinity we conclude that no limit cycle exists in this region. We conclude then that the system can have at most one limit cycle. As the origin is an attractor for $a < 0$ and it is repulsive for $a > 0$, the limit cycle appears via a Hopf bifurcation at the origin and must be located in the interior of the non trivial bounded connected component of the set $\mathcal{V}$. In conclusion, the limit cycle exits if and only if $ad<0$ and it is unique. $\square$

We end this work with a second corollary of Theorem 4.1 that also covers some non-polynomial rigid differential systems.

Corollary 4.3. *Consider the differential system* (19)*, with* $F(x,y)=f(x)+g(y)$*, where* $f(x)=\sum_{k=0}^{2n} f_k x^k$*, with* $f_k\ge 0, k\ge 2$ *and* $f_{2n}>0$*. We assume that* $f''(x)\ge 0$*,* g *is of class* $\mathcal{C}^2$*, with* $g''(y)\ge 0$ *and it vanishes only at isolated points. Then the system has at most two limit cycles. Moreover, if there exists* $R>0$ *such that for all* $(x,y)\in\mathbb{R}^2$ *with* $x^2+y^2\ge R^2$ *it holds that* $F(x,y)\ge c>0$*, then the corresponding differential system has at most one limit cycle, and when it exists, it is hyperbolic.*

Proof. For this case the function H given in Theorem 4.1 is $H(x,y)=f''(x)g''(y)\ge 0$ and it vanishes on a null measure set. Hence the system has at most $L_X(V)$ limit cycles, where V is the function given in (21). To study $L_X(V)$, notice first that it is not restrictive to assume that $g(0)=0$. Then, $V(x,0)=x^2W(x)$, where

$$W(x)=xf(x)f'(x)+g'(0)x-1-f^2(x).$$

Since when $f(x)=f_kx^k$ it holds that $xf'(x)-f(x)=(k-1)f_kx^k$, we get easily that $W(x)=\sum_{j=0}^{4n} w_kx^k$, where all $w_k\ge 0$ for $k\ge 2$ and $w_0=-1-f^2(0)<0$. Hence, by the Descarte's rule of signs the number of positive roots of W is 1. As a consequence, the set $\mathcal{V}=\{V(x,y)=0\}$ has at most one bounded connected component surrounding the origin, different from the origin itself. Recall that V has the factor x^2+y^2. Notice that the set $\mathcal{V}$, which is not invariant by the flow of X, can not contain other bounded connected components. This is so, because M_1 does not change sign, and $M_1\big|_{\mathcal{V}}=\dot{V}$. Therefore, these connected components must surround some equilibrium point of X, but the origin is the only one. As a consequence of the above reasoning $L_X(V)\le 2$, see Section 2.2.

Let us prove now that under the hypothesis on the growth of F the maximum number of limit cycles is 1. Notice that if $r=\sqrt{x^2+y^2}$, it holds that $\dot r=rF(r\cos\theta, r\sin\theta)\ge cr$. Hence, the infinity is an attractor, and we can use similar arguments that in the proof of Corollary 4.2 to show that when $L_X(V)=2$ the differential system has no limit cycle in the unbounded components of $\mathbb{R}^2\setminus\mathcal{V}$. Therefore, the existence of at most one limit cycle follows. $\square$

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References

[1] Bendixson, I. Sur les courbes définies par des équations différentielles. *Acta Math. 24*, (1901) 1–88.

[2] Conti, R. Uniformly isochronous centers of polynomial systems in $\mathbf{R}^2$. In *Differential equations, dynamical systems, and control science*, vol. 152 of *Lecture Notes in Pure and Appl. Math.* Dekker, New York, 1994, pp. 21–31.

[3] Chen, H., and Chen, H. Global dynamics of a Wilson polynomial Liénard equation. *Proc. Amer. Math. Soc. 148*, (2020) 4769–4780.

[4] Cherkas, L.A. Dulac function for polynomial autonomous systems on a plane. *Differ. Equ. 33*, (1997) 692–701.

[5] Cherkas, L.A., and Grin', A.A. On the Dulac function for the Kukles system. *Differ. Equ. 46*, (2010) 818–826.

[6] Chicone, C. Bifurcations of nonlinear oscillations and frequency entrainment near resonance. *SIAM J. Math. Anal. 23*, (1992) 1577–1608.

[7] Chicone, C. *Ordinary differential equations with applications.* In: Texts in Applied Mathematics, vol. 34, 2nd edn. Springer, New York (2006).

[8] Christopher, C., and Li, C. *Limit cycles of differential equations.* Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2007.

[9] Coppel, W. A. A survey of quadratic systems. *J. Differential Equations 2*, (1966) 293–304.

[10] Diliberto, S. P. On systems of ordinary differential equations. *Contributions to the Theory of Nonlinear Oscillations,* pp. 1–38. Annals of Mathematics Studies, no. 20. Princeton University Press, Princeton, N.J., 1950.

[11] Dumortier, F., Llibre, J., and Artés, J. C. *Qualitative theory of planar differential systems.* Universitext. Springer-Verlag, Berlin, 2006.

[12] Dulac, H. Récherches des cycles limites. *C.R. Acad. Sci. Paris Sér. I Math. 204*, (1937) 1703–1706.

[13] Freire, E., Gasull, A., and Guillamon, A. Limit cycles and Lie symmetries. *Bull. Sci. Math. 131*, (2007) 501–517.

[14] Garcia, R.A., Gasull, A., and Guillamon, A. Geometrical conditions for the stability of orbits in planar systems. *Math. Proc. Cambridge Philos. Soc. 120*, (1996) 499–519.

[15] Gasull, A., and Giacomini, H. A new criterion for controlling the number of limit cycles of some generalized Liénard equations. *J. Differ. Equations 185*, (2002) 54–73.

[16] Gasull, A., and Giacomini, H. Upper bounds for the number of limit cycles through linear differential equations. *Pacific J. Math. 226*, (2006) 277–296.

[17] Gasull, A., and Giacomini, H. Upper bounds for the number of limit cycles of some planar polynomial differential systems. *Discrete Contin. Dyn. Syst. 27*, (2010) 217-229.

[18] Gasull, A., and Giacomini, H. Some applications of the extended Bendixson–Dulac theorem. In *Progress and challenges in dynamical systems*, vol. 54 of *Springer Proc. Math. Stat.* Springer, Heidelberg, 2013, pp. 233–252.

[19] Gasull, A., Giacomini, H., and Llibre J. New criteria for the existence and non-existence of limit cycles in Liénard differential systems. *Dyn. Syst. 24*, (2009) 171–185.

[20] Gasull, A., Prohens, R., and Torregrosa, J. Limit cycles for rigid cubic systems. *J. Math. Anal. Appl. 303*, (2005) 391–404.

[21] Gasull, A., and Sabatini, M. Fixed and moving limit cycles for Liénard equations. *Ann. Mat. Pur. Appl. 198*, (2019) 1985–2006.

[22] Ilyashenko, Yu. Centennial history of Hilbert's 16th problem. *Bull. Am. Math. Soc. (N.S.) 39*, (2002) 301–354.

[23] Lloyd, N. G. A note on the number of limit cycles in certain two-dimensional systems. *J. London Math. Soc. (2) 20*, (1979), 277–286.

[24] Massera, J. L. Sur un théorème de G. Sansone sur l'équation di Liénard. *Boll. Un. Mat. Ital. 9*, (1954) 367–369.

[25] Sansone, G. Soluzioni periodiche dell'equazione di Liénard. Calcolo del periodo. *Rend. Sem. Mat. Univ. Politec. Torino 10*, (1951), 155–171.

[26] Villari, G, and Zanolin, F. On the qualitative behavior of a class of generalized Liénard planar systems. Preprint 2020.

[27] Wilson, G. Hilbert's sixteenth problem. *Topology 17*, (1978) 53–73.

[28] Wilson, J. C. Algebraic periodic solutions of $\ddot{x} + f(x)\dot{x} + g(x) = 0$. *Contrib. Differ. Equ. 3*, (1964) 1–20.

[29] Yamato, K. An effective method of counting the number of limit cycles. *Nagoya Math. J. 76*, (1979) 35–114.

[30] Ye, Yan Qian; et alt. Second edition. Translations of Mathematical Monographs, 66. American Mathematical Society, Providence, RI, 1986.

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