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THE SECANT MAP APPLIED TO A REAL POLYNOMIAL WITH MULTIPLE ROOTS

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Abstract. We investigate the plane dynamical system given by the secant map applied to a polynomial p having at least one multiple root of multiplicity $d > 1$. We prove that the local dynamics around the fixed points related to the roots of p depend on the parity of d.

1. **Introduction and statement of the results.** The main goal of this paper is to investigate the dynamical system generated by the so called *secant map*, or *secant method* when considering it as a root finding algorithm, applied to the real monic polynomial of degree $k \geq 2$,

$$p(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0,\ a_k = 1,\ a_j \in \mathbb{R},\ j = 0, \dots k-1,$$

under the presence of real multiple roots. The secant map writes as

$$S(x,y) = \left(y, y - p(y)\frac{x-y}{p(x)-p(y)} \right). \tag{1}$$

We refer to [5] for a detailed discussion of the dynamics generated by S when all real roots of p are simple. As in [5] we consider $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ as a rational map (with *poles*). We note that S defines a rational map $S : \mathbb{C}^2 \to \mathbb{C}^2$. See [1] for a discussion on this context.

Let α be a root of p, and consider the set

$$\mathcal{A}(\alpha) = \{(x,y) \in \mathbb{R}^2 \,|\, S^n(x,y) \to (\alpha,\alpha), \quad \text{as} \quad n \to \infty\}. \tag{2}$$

Because S is a root finding algorithm, it is natural to investigate the structure and distribution of the sets $A(\alpha)$ for all roots of p. If α is a simple root, then S is regular (analytic) at (α,α), and $S(\alpha,\alpha) = (\alpha,\alpha)$. If α is a multiple root, then the map $S : R^2 \to R^2$ may (or may not) be continuous at (α,α), but it is not C^∞ smooth there.

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