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Vanishing set of inverse Jacobi multipliers and attractor/repeller sets

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ABSTRACT

In this paper, we study conditions under which the zero-set of the inverse Jacobi multiplier of a smooth vector field contains its attractor/repeller compact sets. The work generalizes previous results focusing on sink singularities, orbitally asymptotic limit cycles, and monodromic attractor graphics. Taking different flows on the torus and the sphere as canonical examples of attractor/repeller sets with different topologies, several examples are constructed illustrating the results presented.

The aim of this paper is to study under what restrictions the zero-set of the inverse Jacobi multiplier of a smooth vector field contains its compact attractor or repeller sets. The results presented generalize the previous ones by extending the topology of the invariant set. Different flows on the torus and the sphere are given as canonical examples of attractor/repeller sets with different genus. Also, other additional examples are constructed.

I. INTRODUCTION AND MAIN RESULTS

We consider C^1 autonomous ordinary differential equations $\dot{x} = f(x)$ defined in $\mathbb{R}^n$ where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and its associated vector field $\mathcal{X} = \sum_{i=1}^n f_i(x)\partial_{x_i}$ where $f(x) = (f_1(x), \dots, f_n(x))$. We denote by $\text{sing}(\mathcal{X})$ the set of singular points of $\mathcal{X}$ and by $\text{div}\mathcal{X} = \sum_{i=1}^n \partial f_i(x)/\partial x_i$ its divergence.

Let ϕ_t be the flow associated to $\mathcal{X}$ with ϕ_0 being the identity function. A set $\Gamma \subset \mathbb{R}^n$ is *invariant* if $\phi_t(\Gamma) = \Gamma$ for any t for which the flow is defined. We define the distance from a point $p \in \mathbb{R}^n$ to the set Γ as $d(p;\Gamma) = \inf\{d(p,q) : q \in \Gamma\}$, where $d(p,q)$ is the Euclidean distance between the points p and q. If $d(\phi_t(p);\Gamma) \to 0$ when $t \to \infty$ we will write $\phi_t(p) \to \Gamma$ as $t \to \infty$. The *basin of attraction* of a set Γ is the set $\{p \in \mathbb{R}^n : \phi_t(p) \to \Gamma\}$, that is, the set of initial conditions that approaches to Γ asymptotically under the forward flow. The basin of attraction is an open set.

There are several equivalent ways to define an attractor set. In Refs. 13 and 14, Milnor introduced a definition of attractor. A set $\Gamma \subset \mathbb{R}^n$ is an *attractor set* if it possesses a compact neighborhood $U \subset \mathbb{R}^n$ and there is a positive increasing time sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ with $t_n \to \infty$ such that $\Gamma = \cap_{n\in\mathbb{N}}\phi_{t_n}(U)$, where the nested sequence of sets $U \supsetneq \phi_{t_1}(U) \supsetneq \phi_{t_2}(U) \supsetneq \cdots$ holds, this U is called a *trapping neighborhood* of Γ. Their intersection Γ is always invariant, $\phi_t(\Gamma) = \Gamma$.

We recall that a non-compact set Γ such that $\phi_t(p) \to \Gamma$ as $t \to \infty$ for any point $p \in \mathbb{R}^n$ is not necessarily an invariant set. An example with $n = 2$ is given by the curve $\Gamma = \{(x,y) \in \mathbb{R}^2 : y = \exp(-x)\sin(x)\}$ and the flow $\phi_t(x,y) = (x+t, y\exp(-t))$ associated with the system $\dot{x} = 1, \dot{y} = -y$.

There is a simple sufficient condition to guarantee that a compact invariant set Γ is indeed an attractor. It is based on the Liouville divergence-flow relation

$$\frac{d}{dt}|\phi_t(U)|\Big|_{t=t_0} = \int_{\phi_{t_0}(U)} \text{div}(\mathcal{X})dx \qquad (1)$$

for any Lebesgue measurable subset $U \subset \mathbb{R}^n$, see the proof in Proposition 14 of the Appendix. Here, $|A|$ denotes the volume of