LOWER BOUNDS FOR THE NUMBER OF LIMIT CYCLES IN A GENERALIZED RAYLEIGH-LIÉNARD OSCILLATOR

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Abstract. In this paper a generalized Rayleigh-Liénard oscillator is consider and lower bounds for the number of limit cycles bifurcating from weak focus equilibria and saddle connections are provided. By assuming some open conditions on the parameters of the considered system the existence of up to twelve limit cycles is provided. More precisely, the approach consists in perform suitable changes in the sign of some specific parameters and apply Poincaré-Bendixson Theorem for assure the existence of limit cycles. In particular, the algorithm for obtaining the limit cycles through the referred approach is explicitly exhibited. The main techniques applied in this study are the Lyapunov constants and the Melnikov method. The obtained results contemplate the simultaneity of limit cycles of small amplitude and medium amplitude, the former emerging from a weak focus equilibrium and the latter from homoclinic or heteroclinic saddle connections.

1. Introduction

1.1. **Historical facts and equations of Rayleigh and Liénard.** Ordinary differential equations (ODEs) have been largely studied in mathematics since the invention of Calculus back in 17th century. Since that the theory have proved to be very accurate to model real problems from mechanics movements and chemical reactions to social and financial sciences. The interest by ODEs gained even more attraction after the remarkable work of Poincaré entitled *Mémoire sur les courbes définies par une équation différentielle*, see [17]. This paper, dated 1882, is consider one of the starting points of the so called qualitative theory of ODEs. In particular, Poincaré formally introduced the concept of limit cycle, an isolated periodic orbit inside the set of all periodic orbits of an ODE, and exhibited an *ad hoc* example of ODE presenting a limit cycle without any connection to some concrete problem. However, the first reported case of a limit cycle surging from a real model ODE was probably provided by Rayleigh in 1877 in his study on the oscillations of a violin string, see [18]. Posteriorly in 1908 another example of limit cycle emerged from a series of works of Poincaré addressing wireless telegraphy, although the most recognized example of a limit cycle is due to Van der Pol on the electrical circuits in 1927, see [21].

The goal of this paper is to study the existence and the number of limit cycles in a system which is a generalization of the Rayleigh and Van der Pol systems. The equation proposed by Rayleigh which is nowadays known as *Rayleigh equation* is

$$\ddot{x} + ax + \varepsilon(c_3 + c_4\dot{x}^2)\dot{x} = 0 \tag{1}$$

where ε is a small parameter. We also point to [13] for some historical facts about Rayleigh work.

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For the sake of applications of non-linear systems it is usually interesting to assume that the unperturbed part of (1) has a potential of the form $V(x) = ax^2 + bx^4$ so the total energy is $\dot{x}^2 + V(x)$. This is achieved by adding the term $2bx^3$ to the last equation obtaining then the equation

$$\ddot{x} + ax + 2bx^3 + \varepsilon(c_3 + c_4\dot{x}^2)\dot{x} = 0, \tag{2}$$

which is the *generalized Rayleigh system*. By replacing the term $c_4\dot{x}^2$ in the Rayleigh equations by $c_2x^2 + c_1x^4$ we obtain the famous generalized Van der Pol or Liénard equation. Therefore, by combining generalized Rayleigh and Liénard equations we obtain

$$\ddot{x} + ax + 2bx^3 + \varepsilon(c_3 + c_2x^2 + c_1x^4 + c_4\dot{x}^2)\dot{x} = 0. \tag{3}$$

Last equation is common referred in the literature as *mixed generalized Rayleigh-Liénard equation*. In the general case, one can study the problem

$$\ddot{x} + ax + 2bx^3 = \varepsilon f(x, \dot{x}),$$

see for instance the work of Guckenheimer and Holmes in [10].

In this paper we study the limit cycles for the case $f(x, \dot{x}) = (c_3 + c_2x^2 + c_1x^4 + c_4\dot{x}^2 + c_5x^6 + c_6\dot{x}^4)\dot{x}$ so that the mixed generalized Rayleigh-Liénard equation becomes a particular case of the equation we deal with. The equation we study is then

$$\ddot{x} + ax + 2bx^3 - \varepsilon(c_3 + c_2x^2 + c_1x^4 + c_4\dot{x}^2 + c_5x^6 + c_6\dot{x}^4)\dot{x} = 0, \tag{4}$$

that is equivalent to the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -ax - 2bx^3 + \varepsilon Q(x, y). \end{aligned} \tag{5}$$

where $Q(x, y) = (c_3 + c_2x^2 + c_1x^4 + c_4y^2 + c_5x^6 + c_6y^4)y$. Our purpose is to obtain a lower bound for the number of limit cycles of (5).

The problem of finding limit cycles involves several methods and approaches, some of them are briefly summarized in what follows. In [2] the authors study equation (3) by using the harmonic balance and Krylov-Bogoliubov methods to obtain up to two limit cycles. System (3) was also studied by Lynch in [14] by using Lyapunov constants obtaining three limit cycles. Then, the same authors of [2] using harmonic balance method and elliptic functions presented an example with seven limit cycles for the case $a < 0$ and $b > 0$, see [3]. In [22] the authors prove that system (3) can have eight limit cycles, improving the lower bound obtained in [3]. The authors use both Lyapunov constants as well as Melnikov method to get the results for the case $ab < 0$. It also worth to mention the recent work [5] on which the authors study the global dynamics from a particular case of system (2), namely, assuming $\varepsilon = c_4 = 2b = 1$. The main result of that paper provides conditions on a and c_3 so that system (2) presents pitchfork, Hopf, homoclinic and double limit cycle bifurcations, which would be a very difficult - if possible - task considering arbitrary parameters or a more general system (3).

The study of lower and upper bounds for the number of limit cycles is widely considered in the literature. An usual approach is to consider suitable perturbations on structural unstable systems and then to look at the bifurcations taking place. In this case, the bounds for the numbers of limit cycles that can bifurcate provide an idea about the co-dimension of the problem. In the local scenario, these bifurcations usually occur at degenerated Hopf bifurcations equilibria and they can be studied through the Lyapunov constants, we mention the work of Torregrosa and coauthors, see for instance [8] and [12]. The non-local case can be studied through some techniques, an effective one being the Melnikov function which can be applied in different ways. One of them is to consider global bifurcations from non hyperbolic limit cycles or saddle connections, see for instance [22] and [19]. Another way to apply Melnikov method is through the analysis of critical and regular level curves associated to some Hamiltonian function. This is a more general approach because it contemplates both local and global scenarios.

In particular, the study of Melnikov functions emerging from perturbations of Hamiltonian systems have been well established in the work of Petrov in a series of papers published some decades ago, see for instance [15], [16] and references therein. In those papers Petrov provides some upper bounds for the number of limit cycles bifurcating from distinct configurations of Hamiltonian functions in terms of the degree of the perturbations. In particular, Petrov use Tchebyshev spaces to associate the number of zeros of the Melnikov function to the number of zeros of some linear combinations involving elliptic integrals, and they represent a fine advance in the study of limit cycles. Nevertheless, a sharp bound is obtained in [7] where the authors use an approach similar to Petrov's to provide a new upper bound for the number of limit cycles.

1.2. **The main result of the paper.** In this paper we study system (5) which generalizes equation (3). We consider the case where a and b have opposite sign and apply both Lyapunov and Melnikov methods as the authors did in [22]. The approach consists in to consider suitable perturbations on the coefficients of the system to produce changes in the sign of the Lyapunov constants (derivatives of the Poincaré map) and Melnikov function. The limit cycles essentially bifurcate from weak focus equilibria and from heteroclinic and homoclinic loops containing those weak focus. We highlight that the lower bounds provided in this paper agree with Petrov's statements mentioned before.

We distinguish between the limit cycles that bifurcate from the weak focus equilibria from the ones that bifurcate from saddle connections, denoting them as small amplitude limit cycles, or medium amplitude limit cycles, respectively. Therefore we say that system (5) presents a configuration (i, j) of limit cycles if there exists i limit cycles of small amplitude and j limit cycles of medium amplitude.

The main contributions of this paper can be summarized as follows: first, we consider a quite general system without assuming any hypotheses on the parameters except the condition $ab < 0$. So we are able to obtain from one to twelve limit cycles in a region of the phase portrait containing the equilibria. On the other hand, we explicitly exhibit the algorithm to obtain those limit cycles and we provide the conditions for the realization of those number of limit cycles.

This important part of the process for obtaining the limit cycles is in general omitted in the literature. This is the case in [22], where the authors only pointed out the approach which *a priori* does not guarantee the realization of the limit cycles as claimed. We stress, however, that although paper [22] contain some minor miscalculations, assuming $c_5 = c_6 = 0$ our results points to an upper bound of eight limit cycles, which is the quantity obtained in that paper. Besides then, we emphasize that the procedure considered in the present paper, based in the Lyapunov constants and Melnikov method, allow us to consider the simultaneous existence of limit cycles bifurcating from the equilibrium of center type (small amplitude) and from the homoclinic/heteroclinic connection (medium amplitude) that represents a critical point of the Melnikov function. The main result of this paper is the following.

Theorem 1. *Consider system (5). Then there exists suitable values of parameters realizing the following configurations of limit cycles:*

(a) (s, m) *if* $a > 0$ *and* $b < 0$*, where* $m \in \{0,1,2,3\}$*,* $s \in \{0,1,2,3,4,5\}$ *and* $s + m \leq 5$*.*

(b) $(2s, 3m + k)$ *if* $a < 0$ *and* $b > 0$*, where* $m \in \{0,1,2\}$*,* $k \in \{0,1,2\}$*,* $s \in \{0,1,2,3,4\}$ *and* $2s + 3m + k \leq 12$*.*

The rest paper is organized as follows: In Section 2 the canonical forms that we consider in this paper are presented. In Section 3 we present the general facts about Lyapunov constants and Melnikov method. The Lyapunov constants and the Melnikov functions related with the generalized Rayleigh-Liénard system are given in Sections 4 and 5, respectively. Finally, in Section 6 we state and prove some auxiliary results and Theorem 1.

2. Canonical forms for the generalized mixed Rayleigh-Liénard oscillator

In the following we apply a linear change of coordinates to equation (4), providing a simpler expression for the system that models a generalized mixed Rayleigh-Liénard oscillator.

Lemma 1. *Consider system (5). The following statements holds.*

(a) *If* $a > 0$ *and* $b < 0$ *then system (5) is topologically equivalent to system*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \frac{2b}{a^2}x^3 + \varepsilon Q_1(x,y), \end{aligned} \tag{6}$$

where $Q_1(x,y) = (d_3 + d_2x^2 + d_1x^4 + d_4y^2 + d_5x^6 + d_6y^4)y$*,* $d_1 = c_1/\sqrt[5]{a}$*,* $d_2 = c_2/\sqrt[3]{a}, /, d_3 = c_3/\sqrt{a}$*,* $d_4 = c_4/\sqrt{a}$*,* $d_5 = c_5/\sqrt[7]{a}$ *and* $d_6 = c_6/\sqrt{a}$*.*

(b) *If* $a < 0$ *and* $b > 0$ *then system (5) is topologically equivalent to system*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + \frac{3\sqrt{b}}{2a}x^2 - \frac{b}{2a^2}x^3 + \varepsilon Q_2(x,y), \end{aligned} \tag{7}$$

where $Q_2(x,y) = (e_3 + e_7x + e_2x^2 + e_8x^3 + e_1x^4 + e_9x^5 + e_5x^6 + e_4y^2 + e_6y^4)y, e_1 = \dfrac{2c_1b - 15ac_5}{8\sqrt{2}(-a)^{5/2}b}$*,* $e_2 = \dfrac{15a^2c_5 - 12ac_1b + 4c_2b^2}{8\sqrt{2}(-a)^{3/2}b^2}$*,* $e_3 = -\dfrac{a^3c_5 - 2a^2c_1b + 4ac_2b^2 - 8c_3b^3}{8\sqrt{2}\sqrt{-a}b^3}, e_4 = \dfrac{c_4}{\sqrt{-2a}}$*,* $e_5 = \dfrac{c_5}{8\sqrt{2}(-a)^{7/2}}$ *and* $e_6 = \dfrac{c_6}{\sqrt{-2a}}$*. Besides then the parameters* e_7, e_8 *and* e_9 *are linearly dependent with respect to the parameters* $e_1, \ldots, e_6$ *and satisfies* $e_7 = -\dfrac{2a\left(48a^4e_5 - 4a^2be_1 + b^2e_2\right)}{b^{5/2}}, e_8 = \dfrac{40a^3e_5 - 4abe_1}{b^{3/2}}$ *and* $e_9 = -\dfrac{6ae_5}{\sqrt{b}}$*.*

Proof. Consider system (5). We initially consider the case $a > 0$ and $b < 0$. Applying the change of coordinates $\widetilde{x} = \sqrt{a}x, \widetilde{y} = y$ and $\widetilde{t} = \sqrt{a}t$ we obtain system (6), where we have removed the tilde in the expression of the system.

For the case $a < 0$ and $b > 0$, we initially translate the point $p_1 = \left(\sqrt{\frac{-a}{2b}}, 0\right)$ to the origin by the change of coordinates $\widetilde{x} = x - \sqrt{\frac{-a}{2b}}$ and $\widetilde{y} = y$. So system (5) becomes

$$
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= x\left(3\sqrt{-2ab}x + 2a - 2bx^2\right) + \varepsilon Q(x, y),
\end{aligned}
\tag{8}
$$

where $Q(x,y) = \frac{y}{8b^3}\Big(8\sqrt{-2ab}abx\left(c_1 + 5c_5x^2\right) - 6\sqrt{-2ab}a^2c_5x - a^3c_5 + 2a^2b\left(c_1 + 15c_5x^2\right) - 8\sqrt{-2ab}b^2x\left(2c_1x^2 + c_2 + 3c_5x^4\right) - 4ab^2(6c_1x^2 + c_2 + 15c_5x^4) + 8b^3(c_1x^4 + c_2x^2 + c_3 + c_4y^2 + c_5x^6 + c_6y^4)\Big)$ and as in the previous case, we remove the tildes in the expression of the system.

After we consider the rescaling $\widetilde{\widetilde{x}} = \sqrt{-2a}\widetilde{x}, \widetilde{\widetilde{y}} = \widetilde{y}$ and $\widetilde{t} = \sqrt{-2a}t$. Applying this change of coordinates to system (8), we get system (7), where we remove the tildes in the expression of the system. This completes the proof of the lemma. □

3. Main techniques to analyze small and medium amplitude limit cycles

To analyze the types and the number of limit cycles that can bifurcate near to the perturbed center and from the homoclinic/heteroclinic connections we make use of two main techniques, namely, Lyapunov constants and Melnikov functions. We apply these two powerful techniques separately and posteriorly we combine them to obtain simultaneity of small and medium amplitude limit cycles.

In this way, exhibiting the Lyapunov constants with the local basis and improving the Bautin's algorithm we are able to analyze the number of small amplitude limit cycles, that bifurcates from the perturbed center when system (5) presents a homoclinic connection ($a < 0, b > 0$) or a heteroclinic one ($a > 0, b < 0$).

Next, we present the Melnikov functions that characterize the existence of the homoclinic/heteroclinic connections, as well as the stability of these connections. According to this approach of changing the stability of the connections and breaking them ultimately, we are able to provide conditions for the existence of limit cycles that bifurcates from the homoclinic/heteroclinic connections, which we call medium amplitude limit cycles. All these results are given in terms of the parameters of the systems. Finally, we analyze the simultaneous existence of the bifurcations of small and medium limit cycles, near to the perturbed center and the homoclinic/heteroclinic connection, respectively. In the following we present these techniques.

3.1. **Lyapunov constants.**

In the following we briefly present the approach to deal with the limit cycles emerging from changing signs in the Lyapunov constants. It can be founded for instance in [1, 4, 6]. Consider differential systems

$$
\dot{x} = \lambda x - y + P(x, y), \qquad \dot{y} = x + \lambda y + Q(x, y),
\tag{9}
$$

where P and Q are polynomials without constant and linear terms. Then the origin of system (9) is a *weak focus* if $\lambda = 0$. The limit cycles that bifurcate from a weak focus are called *small amplitude limit cycles.*

When $\lambda = 0$ we denote a local Lyapunov function by V, defined in a neighborhood of the origin. Then the origin is a weak focus it is stable or unstable if $\dot{V} < 0$ or $\dot{V} > 0$, respectively, where

$\dot{V}$ denotes the rate of changes of V along the trajectories of (9). The expression of V can be constructed, see [4] and [9], and it is of the form

$$\dot{V} = \eta_2(x^2+y^2) + \eta_4(x^2+y^2)^2 + \cdots + \eta_{2k}(x^2+y^2)^{2k} + \ldots,$$

where η_{2k} is a polynomial in the coefficients of the polynomials P and Q. We define η_{2k} the k-th *focal value*. In the following we get that the weak focus is stable, unstable if the first non-zero focal value is negative, positive, respectively, see [4].

When the origin is a weak focus it is a center if and only if $\eta_{2k} = 0$ for all k. Moreover the stability of the origin is determined by the sign of the first non–zero focal value. As η_{2k} is relevant only when $\eta_{2l} = 0$ for $l < k$, we put $\eta_2 = \eta_4 = \ldots = \eta_{2k-2} = 0$ in the expression for η_{2k}. Instead of working with the η_{2k} following to many authors we prefer to work with

$$V_{2k+1} = 2\pi\eta_{2k},$$

and call it the n–th *Lyapunov constant*.

For $\varepsilon > 0$ and small consider the interval $J = \{(x,0) : 0 \le x < \varepsilon\}$ and the *Poincaré return map* $x \mapsto h(x)$ defined from $J \to \{(x,0) : 0 \le x\}$. It assigns to x the abscissa $h(x)$ of the point where the orbit of the differential system (9) starting at the point $(x,0) \in J$ first returns to the positive half-axis $\{(x,0) : 0 \le x\}$. Then the *displacement function* is defined as $x \mapsto d(x)$ from $J \to \mathbb{R}$ by $d(x) = h(x) - x$. Therefore the orbit of system (9) through the point $(x,0) \ne (0,0)$ is periodic if and only if x is a zero of the displacement function.

Clearly the Lyapunov constants are related with the coefficients of the displacement function, because the origin of system (9) is a center if and only if the displacement function is identically zero, if and only if the Lyapunov constants $V_{2k+1} = 0$ for $k \ge 1$. In fact it is known that

$$d(x) = \lambda x \left[a_0 + \sum_{j=1}^{\infty} a_j^1 x^j \right] + \sum_{k\ge1} V_{2k+1} x^{2k+1} \left[1 + \sum_{j=1}^{\infty} a_j^{2k+1} x^k \right],$$

for $|x| < \varepsilon$, where

$$a_0 = \frac{e^{2\pi\lambda} - 1}{\lambda} = 2\pi + O(\lambda),$$

and the a_j^{2k+1} for $k = 1, 2, \ldots$ are analytic functions in λ and in the coefficients of the polynomials P and Q. For more details in the displacement function see [20].

Since P and Q are polynomials, by the Hilbert basis theorem there is a constant m such that $V_{2k+1} = 0$ for all $k \ge 1$ if and only if $V_{2k+1} = 0$ if $k = 1, \ldots, m$. Therefore it is necessary to compute only a finite number of the Lyapunov constants, though with few exceptions for any given case it is unknown a priori how many are required.

Definition 1. *We say that the origin is a weak focus of order k of system* (9) *if $\eta_2 = \cdots = \eta_{2k} = 0$ and $\eta_{2k+2} \ne 0$.*

Remark 1. *By the previous definition, the origin is a weak focus of order k of system* (9) *if $V_3 = \cdots = V_{2k-1} = 0$ and $V_{2k+1} \ne 0$.*

In [4] the authors stated that if a system presents a weak focus at the origin of order k at most k small amplitude limit cycles can bifurcate from the origin under perturbation of the system. In the following we describe briefly how to provide a convenient perturbation of the original system presenting a weak focus of order k at the origin to produce k small amplitude limit cycles.

The expressions η_{2k} provides the Lyapunov constants V_{2k-1}. We assume that $V_1 = 1$, $V_3 = \cdots = V_{2k-1} = 0$ and $V_{2k+1} \neq 0$. Without loss of generality we assume that $V_{2k+1} < 0$. Therefore the origin is stable. Consider Γ_1 be a level curve of V which is sufficiently close to the origin. So the flow of system (9) is inward across it.

Now consider a suitable perturbation S_1 of system (9) such that the Lyapunov constants satisfy $V_3 = \cdots = V_{2k-3} = 0$ and $V_{2k-1} > 0$. Therefore, the origin now is unstable for S_1. As S_1 is sufficiently close to system (9) then the flow remains inward across Γ_1. Consider W_1 the Lyapunov function of S_1, take Γ_2 a level curve of W_1, inside of the region limited by Γ_1 and sufficiently close to the origin in such a way that the flow of S_1 is outward across of Γ_2. Therefore by the Poincaré-Bendixson Theorem we conclude that there exists a limit cycle of S_1 between Γ_1 and Γ_2. In the following working in a similar way we consider a convenient perturbation S_2 of S_1 with analogous properties.

In this way at most k limit cycles can be produced by convenient perturbations of system (9). Note that the Lyapunov constants must satisfy:

$$|V_{2i-1}| \ll |V_{2i+1}| \quad \text{and} \quad V_{2i-1} \cdot V_{2i+1} < 0,$$

for $i = 3, \ldots, k$. If all the constants V_{2k+1} are zero then the origin is a center.

3.2. **Melnikov method.** In order to present the Melnikov method, note that system (5) with $\varepsilon = 0$ is a Hamiltonian system with the Hamiltonian function

$$H(x, y) = \frac{1}{2}(y^2 + ax^2 + bx^4).$$

If $ab > 0$ then the origin is the unique equilibrium point and if $ab < 0$ there exists three equilibrium points: $O = (0, 0)$, $p_1 = \left(\sqrt{\frac{-a}{2b}}, 0\right)$ and $p_2 = \left(-\sqrt{\frac{-a}{2b}}, 0\right)$. In this way if $a > 0$ and $b < 0$, then $O = (0, 0)$ is a center point and p_1, p_2 are saddle points, and if $a < 0$ and $b > 0$ then $O = (0, 0)$ is saddle point and p_1, p_2 are center points. As system (5) with $\varepsilon = 0$ is Hamiltonian, then for the case $a > 0$ and $b < 0$ we have a heteroclinic loop between the two saddle points p_1 and p_2. On the other hand in the case $a < 0$ and $b > 0$ the system presents a homoclinic loop. It is important to note that system (5) is invariant under the transformation $(x, y) \mapsto (-x, -y)$, so its phase portrait is symmetric with respect to the origin, see Figure 1.

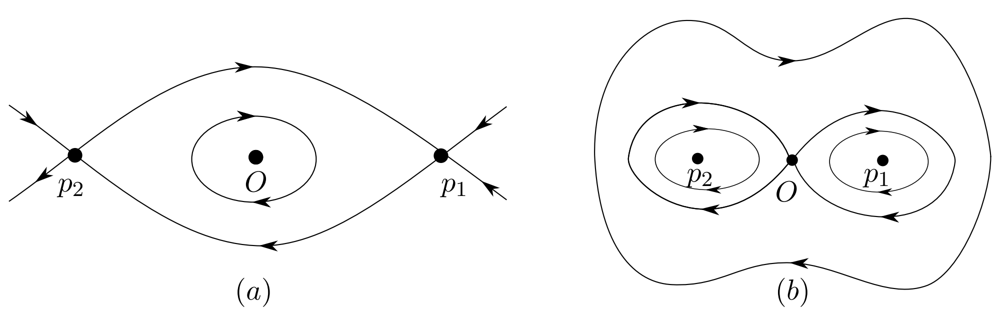

FIGURE 1. The topological structure of system (5) with $\varepsilon = 0$. In (a) we get $a > 0, b < 0$, and in (b) we have $a < 0, b > 0$.

We note that for $\varepsilon \neq 0$ system (5) is no longer Hamiltonian so if ε is sufficiently small generically the saddle connections are broken and the center structure is destroyed. Therefore in both scenarios we can eventually have the birth of limit cycles emerging from those equilibria or loops.

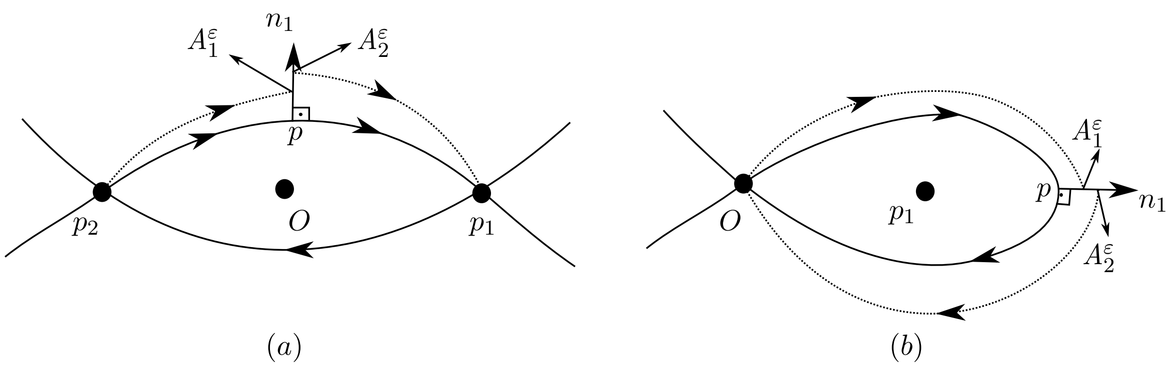

FIGURE 2. In figures (a) and (b) are presented the geometric approach of the construction of Melnikov functions of systems (6) and (7), respectively.

More precisely, consider A_1^ε the unstable manifold of p_2 and A_2^ε the stable manifold of p_1, see Figure 2. Let A^ε be the heteroclinic, homoclinic loop of systems (6), (7), resp., and $p \in A^\varepsilon$. Consider

$$n_1 = \frac{(H_x(p), H_y(p))}{||(H_x(p), H_y(p))||},$$

then Melnikov function is

$$d(\varepsilon, M_1) = -\langle n_1, \overrightarrow{A_1^\varepsilon A_2^\varepsilon} \rangle,$$

see Figure 2, and for more details about the homoclinic and heteroclinic loops see [19].

We notice that system (6) has the same equilibrium points than system (5) but in this case the Hamiltonian function associated to its unperturbed part is given by

$$H_1(x, y) = ax^2 + \frac{b}{2}x^4 + \frac{1}{2}y^2.$$

Since the orbits of system (6) with $\varepsilon = 0$ lie on the h-levels of function H_1, we see that for $h = -a^2/8b$ it has a heteroclinic loop A formed by the saddle equilibrium points and two orbits A_1 and A_2 connecting them, that is, $A = \{p_1\} \cup A_1 \cup \{p_2\} \cup A_2$. From the expression of H_1 we obtain the analytical expression of these arcs:

$$A_{1,2} : y = y(x) = \mp\sqrt{\frac{-b}{a}}x^2 \pm \sqrt{\frac{a}{-4b}},$$

where $|x| \leq \sqrt{-\frac{a}{2b}}$. In this case we note that A surrounds the center equilibrium $O = (0, 0)$.

On the other hand, the points $O = (0, 0)$, $q_1 = \left(\frac{a}{\sqrt{b}}, 0\right)$ and $q_2 = \left(\frac{2a}{\sqrt{b}}, 0\right)$ are equilibrium points for system (7) being

$$H_2(x, y) = \frac{x^2}{2} - \frac{\sqrt{b}}{2a}x^3 + \frac{b}{8a^2}x^4 + \frac{1}{2}y^2$$

the Hamiltonian function of the system for $\varepsilon = 0$. Doing a translation of the original system for obtaining system (7), q_1 is a saddle when $\varepsilon = 0$, the equilibria O and q_2 are centers. The level $h = a^2/8b$ of Hamiltonian function H_2 contains two homoclinic loops L_l and L_r and the saddle point centered at q_1. We call $L = L_l \cup \{q_1\} \cup L_r$. In this case L_l and L_r are determined by the following relations:

$$L_l : y = \operatorname{sign}(y)(-a+\sqrt{b}x)\sqrt{\frac{a^2+2a\sqrt{b}x-bx^2}{a^2b}},$$

if $\frac{a(1+\sqrt{2})}{\sqrt{b}} \le x \le \frac{a}{\sqrt{b}}$ and

$$L_r : y = \operatorname{sign}(y)(a-\sqrt{b}x)\sqrt{\frac{a^2+2a\sqrt{b}x-bx^2}{a^2b}},$$

if $\frac{a}{\sqrt{b}} \le x \le \frac{a(1-\sqrt{2})}{\sqrt{b}}$. Observe that the homoclinic loop L_i surrounds the equilibria O and q_2. Now we denoted by $Q_2(x,y)$ the perturbed part of system (7), that is, $Q_2(x,y) = (e_3+e_7x+e_2x^2+e_8x^3+e_1x^4+e_9x^5+e_4y^2+e_5x^6+e_6y^4)y$.

4. Preliminary to obtain small limit cycles using Lyapunov constants

In this section we present the Lyapunov constants for each system.

Lemma 2. *Consider system (6) with $d_3 = 0$. The first Lyapunov constant is $V_1 = 1$, then*

$$V_3 = -\frac{1}{4}\pi\varepsilon(d_2+3d_4). \tag{10}$$

If $V_3 = 0$, then

$$V_5 = -\frac{\pi\varepsilon(a^2d_1+5a^2d_6+6bd_4)}{8a^2}. \tag{11}$$

If $V_3 = V_5 = 0$, then

$$V_7 = -\frac{\pi\varepsilon\left(-6a^2d_4^3\varepsilon^2+5a^2d_5+80bd_6\right)}{64a^2}. \tag{12}$$

If $V_3 = V_5 = V_7 = 0$, then

$$V_9 = \frac{3\pi\varepsilon\left(15a^4d_4^2d_6\varepsilon^2+12a^2bd_4^3\varepsilon^2-35b^2d_6\right)}{160a^4} \tag{13}$$

If $V_3 = V_5 = V_7 = V_9 = 0$, then

$$V_{11} = -\frac{3\pi d_4^3\varepsilon^3}{3200a^4\left(3a^4d_4^2\varepsilon^2-7b^2\right)^2}\left(315a^{12}d_4^6\varepsilon^6+2644a^8b^2d_4^4\varepsilon^4-9065a^4b^4d_4^2\varepsilon^2-7350b^6\right). \tag{14}$$

Proof. Consider the polynomial system

$$\begin{aligned}\dot{x} &= y+p(x,y),\\ \dot{y} &= -x+q(x,y).\end{aligned}$$

Applying the change of coordinates $x = r\cos\theta$ and $y = \sin\theta$, we can write

$$\frac{dr}{d\theta} = \sum_{i\geq 1} v_i(\theta) r^i, \tag{15}$$

where $v_i(\theta)$ are trigonometric polynomials in the variables $\cos\theta$ and $\sin\theta$. Denoting by $r(\theta, r_0)$ the solution of (15) satisfying $r(0, r_0) = r_0$, then in a neighborhood of $r = 0$ we obtain

$$r(\theta, r_0) = u_1(\theta) r_0 + \sum_{i\geq 2} u_i(\theta) r_0^i,$$

with $u_i(0) = 0$ for all $i \geq 2$, then the Poincaré return map is

$$\Pi(r_0) = r(2\pi, r_0) = u_1(2\pi) r_0 + \sum_{i\geq 2} u_i(2\pi) r_0^i.$$

As the Poincaré map is analytic the condition $\Pi(r_0) \equiv r_0$ is equivalent to the fact that system (6) presents a center at the origin. Note that $\Pi(r_0) \equiv r_0$ if and only if $u_i(2\pi) = 0$ for all $i \geq 2$. As stated in [1] and by direct computations we obtain that $u_{2k}(2\pi) = 0$ for all k a positive integer.

At this moment we are able to obtain the Lyapunov constants for system (6). Consider system (6) and assuming that $d_3 = 0$, then following the steps described previously, we get that the first Poincaré constant

$$u_1(2\pi) = 1.$$

and then we obtain

$$u_3(2\pi) = -\frac{1}{4}\pi\varepsilon(d_2 + 3d_4).$$

Assuming that $d_2 = -3d_4$ we obtain

$$u_5(2\pi) = -\frac{\pi\varepsilon(a^2 d_1 + 5a^2 d_6 + 6bd_4)}{8a^2}.$$

If $d_2 = -3d_4$ and $d_1 = -\dfrac{6bd_4}{a^2} - 5d_6$, then

$$u_7(2\pi) = -\frac{\pi\varepsilon\left(-6a^2 d_4^3\varepsilon^2 + 5a^2 d_5 + 80bd_6\right)}{64a^2}.$$

If $d_2 = -3d_4, d_1 = -\dfrac{6bd_4}{a^2} - 5d_6$ and $d_5 = \dfrac{6d_4^3\varepsilon^2}{5} - \dfrac{16bd_6}{a^2}$, then

$$u_9(2\pi) = \frac{3\pi\varepsilon\left(15a^4 d_4^2 d_6\varepsilon^2 + 12a^2 b d_4^3\varepsilon^2 - 35b^2 d_6\right)}{160a^4}.$$

If $d_2 = -3d_4, d_1 = -\dfrac{6bd_4}{a^2} - 5d_6, d_5 = \dfrac{6d_4^3\varepsilon^2}{5} - \dfrac{16bd_6}{a^2}$ and $d_6 = \dfrac{12a^2 b d_4^3\varepsilon^2}{5\left(7b^2 - 3a^4 d_4^2\varepsilon^2\right)}$ then

$$u_{11}(2\pi) = -\frac{3\pi d_4^3\varepsilon^3}{3200a^4\left(3a^4 d_4^2\varepsilon^2 - 7b^2\right)^2}\left(315a^{12} d_4^6\varepsilon^6 + 2644a^8 b^2 d_4^4\varepsilon^4 - 9065a^4 b^4 d_4^2\varepsilon^2 - 7350b^6\right).$$

As ε is a small parameter then $u_{11}(2\pi) = 0$ if and only if $d_4 = 0$. So if besides $d_3 = 0$ we assume $d_2 = -3d_4$, $d_1 = -\dfrac{6bd_4}{a^2} - 5d_6$, $d_5 = \dfrac{6d_4^3\varepsilon^2}{5} - \dfrac{16bd_6}{a^2}$, $d_6 = \dfrac{12a^2 b d_4^3\varepsilon^2}{5\left(7b^2 - 3a^4 d_4^2\varepsilon^2\right)}$ and $d_4 = 0$ then we obtain that all the others coefficients d_i are null and therefore system (6) is a Hamiltonian system, and therefore the origin is a center. In what follows we denote the Lyapunov constant $u_i(2\pi)$ by V_i. □

Lemma 3. *Consider system (7) with $e_3 = 0$ and $e_2 = \dfrac{4a^2be_1 - 48a^4e_5}{b^2}$. The first Lyapunov constant is $V_1 = 1$, then*

$$V_3 = -\frac{\pi\varepsilon\left(-48a^4e_5 + 4a^2be_1 + 3b^2e_4\right)}{4b^2}. \tag{16}$$

If $V_3 = 0$, then

$$V_5 = \frac{1}{8}\pi\varepsilon\left(\frac{3be_4}{a^2} + \frac{8a^2e_5}{b} - 5e_6\right). \tag{17}$$

If $V_3 = V_5 = 0$, then

$$V_7 = \frac{1}{32}\pi\varepsilon\left(\frac{24b^2e_4}{a^4} + 3e_4^3\varepsilon^2 + 44e_5\right) \tag{18}$$

If $V_3 = V_5 = V_7 = 0$, then

$$V_9 = \frac{9\pi e_4\varepsilon\left(2a^{12}e_4^6\varepsilon^6 + 7a^8b^2e_4^4\varepsilon^4 - 25a^4b^4e_4^2\varepsilon^2 + 35b^6\right)}{1760a^6b^3}. \tag{19}$$

Proof. Considering now system (7) with $e_3 = 0$ and $e_2 = \dfrac{4a^2be_1 - 48a^4e_5}{b^2}$. Proceeding in a similar way as in the proof of Lemma 2, the first Lyapunov constant is $u_1(2\pi) = 1$ and we conclude that the expression of $u_3(2\pi)$ is given in (16). Assuming that $e_1 = -\dfrac{3\left(b^2e_4 - 16a^4e_5\right)}{4a^2b}$ we conclude that the expression of $u_5(2\pi)$ is given in (17). If $e_1 = -\dfrac{3\left(b^2e_4 - 16a^4e_5\right)}{4a^2b}$ and $e_6 = \dfrac{8a^4e_5 + 3b^2e_4}{5a^2b}$ we conclude that the expression of $u_7(2\pi)$ is given in (18). If $e_1 = -\dfrac{3\left(b^2e_4 - 16a^4e_5\right)}{4a^2b}$, $e_6 = \dfrac{8a^4e_5 + 3b^2e_4}{5a^2b}$ and $e_5 = -\dfrac{3\left(a^4e_4^3\varepsilon^2 + 8b^2e_4\right)}{44a^4}$ we conclude that the expression of $u_9(2\pi)$ is given in (19).

Similarly we done in the previous case, $u_9(2\pi) = 0$ if and only if $e_4 = 0$. In this way, assuming $e_4 = 0$ and replacing this condition in the others parameters we conclude that all e_i's are null and therefore system (7) is a Hamiltonian system. So, the origin is a center. As we done in the previous case, we denote the Lyapunov constant $u_i(2\pi)$ by V_i. □

5. Preliminary to obtain medium limit cycles using Melnikov function

We now study the persistence and stability of the saddle connections of systems (6) and (7). We start analyzing on which (non generic) conditions the homoclinic and heteroclinic loops persist for these systems with $\varepsilon \neq 0$. This is provided by the first order Melnikov functions associated to each system. Once we have established conditions to persistence, we will deal with the stability of such loops.

5.1. **Persistence and stability of heteroclinic loop in system (6).** Denote by $Q_1(x, y)$ the perturbed part of system (6), that is, $Q_1(x, y) = (d_3 + d_2x^2 + d_1x^4 + d_4y^2 + d_5x^6 + d_6y^4)y$. For system (6) we have the following result:

Lemma 4. *System (6) has a heteroclinic loop A_ε close to A if and only if*

$$\begin{aligned} d_2 &= \varphi_1(d_1, d_3, d_4, d_5, d_6, \varepsilon) \\ &= -\frac{12d_4}{7} - \frac{5a^4 d_5}{84b^2} + \frac{a^2(99d_1 + 160d_6)}{462b} + \frac{10bd_3}{a^2} + O(\varepsilon). \end{aligned}$$

We write $A_\varepsilon = \{p_1\} \cup A_1^\varepsilon \cup \{p_2\} \cup A_2^\varepsilon$, where $A_{1,2}^\varepsilon$ denotes the orbits coming from $A_{1,2}$ for $\varepsilon \neq 0$. In what follows we prove Lemma 4 only for the arc A_1, because by symmetry A_1 is broken if and only if A_2 is broken.

Proof. In order to obtain the conditions for the persistence of the heteroclinic loop of system (6) with $\varepsilon = 0$ it is sufficient to study the zeros of the first order Melnikov function M_1 associated to the heteroclinic loop. As done in [19] this function depends on the parameters of the system and it is given by

$$M_1 = \int_{A_1} Q_1(x, y)\, dx,$$

where we highlight that $M_1 = M_1(a, b, d_1, \ldots, d_6)$. Using the expression of A_1 obtained in Subsection 3.2 we get

$$\begin{aligned} M_1 &= \int_{A_1} (d_3 + d_2x^2 + d_1x^4 + d_4y^2 + d_5x^6 + d_6y^4) y\, dx \\ &= 2\int_0^{\frac{a}{\sqrt{-2b}}} (d_3 + d_2x^2 + d_1x^4 + d_4y(x)^2 + d_5x^6 + d_6y(x)^4) y(x)\, dx \end{aligned}$$

where $y(x) = -\sqrt{\dfrac{-b}{a^2}}x^2 + \sqrt{\dfrac{a^2}{-4b}}$. Replacing M_1 into the expression of the integral we get

$$\int_0^{\frac{a}{\sqrt{-2b}}} \left[-\frac{a^2 + 2bx^2}{2\sqrt{-a^2 b}} (d_3 + d_2x^2 + d_1x^4 + d_5x^6 - \frac{d_4}{4a^2 b}(a^2 + 2bx^2)^2 + \frac{d_6}{16a^4b^2}(a^2 + 2bx^2)^4) \right] dx.$$

Integrating we obtain

$$M_1 = \frac{1}{13860\sqrt{2}b^4}(198a^6bd_1 - 924a^4b^2d_2 + 9240a^2b^3d_3 - -1584a^4b^2d_4 - 55a^8d_5 + 320a^6bd_6)$$

and finally solving this last equation with respect to the parameter d_2 we are done. □

Since system (5) is analytic the heteroclinic loop is isolated, so one can study its stability.

Lemma 5. *Assume that $d_2 = \varphi_1$. The heteroclinic loop A_ε is stable (respect. unstable) if the value*

$$div(p_2) = \frac{1}{231b^3}(-22a^6d_5 + a^4b(33d_1 - 40d_6) + 198a^2b^2d_4 - 924d_3b^3)\varepsilon.$$

in negative (respect. positive).

Proof. The proof is straightforward from the computation of the divergence at the saddle point p_2. □

If the condition of Lemma 4 occurs we have a simple loop. However if that divergence is zero in order to obtain the stability of the heteroclinic loop we must analyze higher orders of the return map, see [11]. In this direction we have the following result.

Lemma 6. *Assume that* $d_2 = \varphi_1$ *and* $div(p_2) = 0$, *that is,*

$$d_1 = \varphi_2(d_3, d_4, d_5, d_6) = \frac{40d_6}{33} + \frac{2a^2d_5}{3b} - \frac{6d_4b}{a^2} + \frac{28d_3b^2}{a^4}.$$

Then the heteroclinic loop A_ε *is stable (resp. unstable) if the value* d_5 *is*

$$\varphi_3(d_3, d_4, d_6, \varepsilon) = \varepsilon\frac{6b(-100a^4d_6 + 231a^2bd_4 + 308d_3b^2)}{11a^6}$$

is negative (resp. positive).

Proof. In [19] we see that the stability of A_ε is given by the sign of the expression

$$\varepsilon\int_{A_\varepsilon}\frac{\partial Q_1}{\partial y}dt = \varepsilon\left[\int_A \frac{\partial Q_1}{\partial y}dt + O(\varepsilon)\right]$$

Thus, since $\varepsilon \neq 0$ is sufficiently small it is sufficient to study the sign of the right hand side of the last equality.

Using the particular values of d_1 and d_2 we get

$$\begin{aligned}\frac{\partial Q_1}{\partial y} \quad &= \quad \frac{d_3(a^2 + 2bx^2)(a^2 + 14bx^2)}{a^4} + \frac{x^2(a^2 + 2bx^2)(80bd_6 + 11d_5(a^2 + 6bx^2))}{132b^2} \\ &+5d_6y^4 + 3d_4\left(-x^2 - \frac{2bx^4}{a^2} + y^2\right).\end{aligned}$$

Due to the symmetry and using the parametrization of arc A_1 obtained in Subsection 3.2 we get the expression for $\varepsilon\int_{A_1^\varepsilon}\frac{\partial Q_1}{\partial y}dt$ is

$$\varepsilon\frac{-11a^7d_5 - 600a^5bd_6 + 1386a^3b^2d_4 + 1848ab^3d_3}{3564\sqrt{-2b^7}}$$

This last equation vanishes for $d_5 = \varphi_3(d_3, d_4, d_6, \varepsilon)$, so we are done. □

In summary the function φ_1 controls the existence or not of the heteroclinic loop. If the loop remains for $\varepsilon \neq 0$, the stability of it is determined by the sign of φ_2 if the loop is simple, or by the sign of φ_2 if not. Now we perform the same computations for the case $a < 0$ and $b > 0$.

5.2. **Persistence and stability of homoclinic loop in system** (7)**.** Analogously to Subsection 5.1 we study the Melnikov function and the stability of the homoclinic loop when it persists for $\varepsilon \neq 0$.

Lemma 7. *System* (7) *has a homoclinic loop* L_ε *close to* L *if and only if*

$$e_2 = \phi_1(e_1, e_3, e_4, e_5, e_6, \varepsilon) = \frac{5be_3}{a^2} + \frac{6e_4}{7} - \frac{1417a^4e_5}{21b^2} + \frac{a^2(1287e_1 + 40e_6)}{231b} + O(\varepsilon).$$

Now we call $L_\varepsilon = L_l^\varepsilon \cup \{q_1\} \cup L_r^\varepsilon$, where in this case $L_{l,r}^\varepsilon$ denotes the orbit arcs coming from $L_{l,r}$ for $\varepsilon \neq 0$ sufficiently small.

Proof. The conditions for the persistence of the homoclinic loop is provided by the zeros of the first order Melnikov function M_2 associated to the homoclinic loop, that is

$$M_2 = \int_{L_r} Q_2(x, y)\,dx.$$

From the expression of the arc L_r we have M_2 into the form

$$\int_{\frac{a}{\sqrt{b}}}^{\frac{a-a\sqrt{2}}{\sqrt{b}}} \Big[\frac{1}{2}\sqrt{\frac{(a-\sqrt{b}x)^2(a^2+2a\sqrt{b}x-bx^2)}{a^2b}}\Big(e_3+e_2x^2+e_1x^4-\frac{6ae_5}{\sqrt{b}}x^5+e_5x^6$$

$$-\tfrac{2a(-4a^2be_1+b^2e_2+48a^4e_5)}{\sqrt{b^5}}x+\tfrac{-4abe_1+40a^3e_5}{\sqrt{b^3}}x^3+\tfrac{e_4(a-\sqrt{b}x)^2(a^2+2a\sqrt{b}x-bx^2)}{4a^2b}$$

$$+\tfrac{e_6(a-\sqrt{b}x)^2(a^2+2a\sqrt{b}x-bx^2)^2}{16a^2b}\Big)\Big]\,dx.$$

By integrating M_2 we get

$$M_2=\frac{\sqrt{2}a^2}{3465b^4}(1155b^3e_3+33a^2b^2(-7e_2+6e_4)-15587a^6e_5+a^4b(1287e_1+40e_6)).$$

Then solving this last equation with respect to the parameter e_2 the result follows. □

The proof of the next result is straightforward.

Lemma 8. *Assume that $e_2=\phi_1$. The homoclinic loop L_ε is stable (resp. unstable) if the value*

$$div(q_1)=-\frac{2}{231b^3}(462b^3e_3+99a^2b^2e_4-748a^6e_5+a^4b(66e_1+20e_6))\varepsilon$$

is negative (resp. positive).

Lemma 9. *Assume that $e_2=\phi_1$ and $div(q_1)=0$, that is,*

$$e_1=\phi_2(e_3,e_4,e_5,e_6)=\frac{-99b^2e_4+748a^4e_5-20a^2be_6}{66a^2b}.$$

Then the homoclinic loop L_ε is stable (resp. unstable) if the value

$$e_5=\phi_3(e_3,e_4,e_6,\varepsilon)=-\frac{3b(308b^2e_3-231a^2be_4-100a^4e_6)}{44a^6}$$

is negative (resp. positive).

Proof. As proceeded in the proof of Lemma 6 we must study the sign of $\varepsilon\int_{L_1}\frac{\partial Q_2}{\partial y}dt$. For fixed values of e_1 and e_2 according with Lemmas 8 and 9, we get

$$\frac{\partial Q_2}{\partial y}=\ \frac{x(a-\sqrt{b}x)^2(2a-\sqrt{b}x)(99b^2e_4+110a^4e_5+132a^3\sqrt{b}e_5x+2a^2b(10e_6-33e_5x^2))}{66a^2\sqrt{b^5}}$$

$$+\frac{e_3(a-\sqrt{b}x)^2(a^2+14a\sqrt{b}x-7bx^2)}{a^4}+3e_4y^2+5e_6y^4.$$

As before, using symmetry and the parametrization of L_1 obtained in Subsection 3.2 we obtain

$$\int_{L_1}\frac{\partial Q_2}{\partial y}dt=\frac{\sqrt{2}(924ab^3e_3-693a^3b^2e_4+44a^7e_5-300a^5be_6)}{3465\sqrt{b^7}}\varepsilon.$$

The last equation vanishes for $e_5=\phi_3(e_3,e_4,e_6,\varepsilon)$, so we are done. □

Function ϕ_1 has the same role than the function φ, but now it controls the existence and stability of the double homoclinic loop.

6. Proof of the main result

In this section we prove the main result of the paper. Since Theorem 1 contemplates several cases for different values of s, m and k, we split the proof in some lemmas, see next subsection. Once the lemmas are stated and proved, the proof of the main theorem follows quite straightforwardly. However we emphasize that lemmas are also important by themselves because they explicitly exhibit the approach of choosing suitable perturbations of the parameters to obtain limit cycles, as described in Section 3. This is important because such calculations sometimes are neglected or exhibited for weak focus of low multiplicity as occurs in Hopf Theorem.

More precisely, we start considering the values in the parameters space for which the systems present a center. In the following we perform suitable perturbations of these parameters to obtain in each step a change in the sign of the Lyapunov constants. In other words, if for the values of the parameters $(\alpha_1^*, \ldots, \alpha_m^*)$ the system presents a center, then we start choosing a parameter, that we denote by α_m, that determinate the sign of V_{2k+1} (the last Lyapunov constant), i.e., if $\alpha_m > \alpha_m^*, (\alpha_m < \alpha_m^*)$ then $V_{2k+1} > 0$ ($V_{2k+1} < 0$), resp.. Fixing this value of α_m, in the following we consider a convenient perturbation of the next parameter, we denote by α_{m-1}^*, that vanishes the previous Lyapunov constant V_{2k-1}. We consider a convenient perturbation of α_{m-1}^*, denoted by $\alpha_{m-1}(\varepsilon)$ such that $|\alpha_{m-1}^* - \alpha_{m-1}(\varepsilon)| << \varepsilon$ and satisfies $V_{2k-1} \cdot V_{2k+1} < 0$ and $|V_{2k-1}| << |V_{2k+1}|$. We repeat this procedure until reach the first Lyapunov constant V_1, where we consider a convenient perturbation such that the equilibrium point be a unstable or stable focus at the system.

6.1. Technical lemmas for the heteroclinic connection occurring in system (6).

In this subsection we state and prove some lemmas where we explicitly apply Bautin's algorithm, see [1, 6], combined with Melnikov techniques. That is actually the very essence of the proof of Theorem 1. We start exhibiting Bautin's algorithm for obtaining five limit cycles which bifurcate from the center at the origin.

Lemma 10. *There exists at least five limit cycles bifurcating from the origin for system (6).*

Proof. In the proof of Lemma 2 under the hypothesis $d_3 = 0$ we obtained that: if the parameters d_i's satisfy $d_2 = -3d_4$, $d_1 = -\dfrac{6bd_4}{a^2} - 5d_6$, $d_5 = \dfrac{6d_4^3\varepsilon^2}{5} - \dfrac{16bd_6}{a^2}$, $d_6 = \dfrac{12a^2bd_4^3\varepsilon^2}{5\left(7b^2 - 3a^4d_4^2\varepsilon^2\right)}$ and $d_4 = 0$, then system (6) is Hamiltonian and therefore the origin is a center. Our objective here is to consider small perturbations in these values of parameters to obtain a variation of the signs of the Lyapunov constants and the stability of the origin.

First we consider the expression of V_{11} given in (14). Note that if $d_4 > 0, d_4 < 0$ then the origin is a weak repeller focus, weak attractor focus, resp.. Without loss of generality, we assume that $d_4 > 0$.

Consider now the expression of V_9 given in (13) and the perturbation of the parameter d_6 given by $d_6(\varepsilon) = d_{62}\varepsilon^2 + d_{63}\varepsilon^3$, with $d_{62} = \dfrac{12a^2d_4^3}{35b}$. Note that $|d_6 - d_6(\varepsilon)| = \left|\frac{36a^6d_4^5}{245b^3}\right|\varepsilon^4 + O\left(\varepsilon^6\right) << \varepsilon$. Then we obtain the new expression of

$$V_9 = -\frac{21\pi b^2 d_{63}\varepsilon^4}{32a^4} + \frac{27\pi a^2 d_4^5\varepsilon^5}{280b} + \frac{9}{32}\pi d_4^2 d_{63}\varepsilon^6.$$

So the sign of V_9 is given by the sign of d_{63}. Then $|V_9| \ll |V_{11}|$, and if $d_4 > 0$ and $d_{63} > 0$ then $V_9 \cdot V_{11} < 0$. In this way we obtain at least one limit cycle bifurcating from the origin under these assumptions.

Consider the expression of V_7 given in (12), replacing d_6 by $d_6(\varepsilon) = d_{62}\varepsilon^2 + d_{63}\varepsilon^3$ and d_5 by $d_5(\varepsilon) = d_{52}\varepsilon^2 + d_{53}\varepsilon^3 + d_{54}\varepsilon^4$ with $d_{62} = \dfrac{12a^2d_4^3}{35b}, d_{52} = -\dfrac{1}{7}\left(30d_4^3\right)$ and $d_{53} = -\dfrac{16bd_{63}}{a^2}$. We have $|d_5 - d_5(\varepsilon)| = |d_{54}|\varepsilon^4 << \varepsilon$. Therefore we obtain

$$V_7 = \frac{-5}{64}\pi d_{54}\varepsilon^5.$$

As in the previous case we obtain $|V_7| \ll |V_9| \ll |V_{11}|$, and if $d_4 > 0, d_{63} > 0$ and $d_{54} < 0$ then $V_9 \cdot V_{11} < 0$ and $V_7 \cdot V_9 < 0$. So we obtain that at least two limit cycles bifurcating from the origin.

Working similarly we consider the expression V_5 given in (11), replacing d_6 by $d_6(\varepsilon) = d_{62}\varepsilon^2 + d_{63}\varepsilon^3, d_5$ by $d_5(\varepsilon) = d_{52}\varepsilon^2 + d_{53}\varepsilon^3 + d_{54}\varepsilon^4$ and d_1 by $d_1(\varepsilon) = d_{10} + d_{11}\varepsilon + d_{12}\varepsilon^2 + d_{13}\varepsilon^3 + d_{14}\varepsilon^4 + d_{15}\varepsilon^5$ with $d_{62} = \dfrac{12a^2d_4^3}{35b}, d_{52} = -\dfrac{1}{7}\left(30d_4^3\right), d_{53} = -\dfrac{16bd_{63}}{a^2}, d_{10} = -\dfrac{6bd_4}{a^2}, d_{11} = 0, d_{12} = -\dfrac{12a^2d_4^3}{7b}, d_{13} = -5d_{63}$ and $d_{14} = 0$, we get $|d_1 - d_1(\varepsilon)| = |d_{15}|\varepsilon^5 << \varepsilon$ and

$$V_5 = -\frac{1}{8}\pi d_{15}\varepsilon^6.$$

So we obtain that $|V_5| \ll |V_7| \ll |V_9| \ll |V_{11}|$, and if $d_4 > 0, d_{63} > 0, d_{54} < 0$ and $d_{15} > 0$ then $V_9 \cdot V_{11} < 0, V_7 \cdot V_9 < 0$ and $V_5 \cdot V_7 < 0$. So we obtain that at least three limit cycles bifurcating from the origin.

Consider now the expression of V_3 given in (10), replacing d_6 by $d_6(\varepsilon) = d_{62}\varepsilon^2 + d_{63}\varepsilon^3, d_5$ by $d_5(\varepsilon) = d_{52}\varepsilon^2 + d_{53}\varepsilon^3 + d_{54}\varepsilon^4, d_1$ by $d_1(\varepsilon) = d_{10} + d_{11}\varepsilon + d_{12}\varepsilon^2 + d_{13}\varepsilon^3 + d_{14}\varepsilon^4 + d_{15}\varepsilon^5$ and d_2 by $d_2(\varepsilon) = d_{20} + d_{21}\varepsilon + d_{22}\varepsilon^2 + d_{23}\varepsilon^3 + d_{24}\varepsilon^4 + d_{25}\varepsilon^5 + d_{26}\varepsilon^6$ with $d_{62} = \dfrac{12a^2d_4^3}{35b}, d_{52} = -\dfrac{1}{7}\left(30d_4^3\right), d_{53} = -\dfrac{16bd_{63}}{a^2}, d_{10} = -\dfrac{6bd_4}{a^2}, d_{11} = 0, d_{12} = -\dfrac{12a^2d_4^3}{7b}, d_{13} = -5d_{63}, d_{14} = 0, d_{20} = -3d_4, d_{21} = 0, d_{22} = 0, d_{23} = 0, d_{24} = 0$ and $d_{25} = 0$, we obtain $|d_2 - d_2(\varepsilon)| = |d_{26}|\varepsilon^6 << \varepsilon$ and

$$V_3 = -\frac{1}{4}\pi d_{26}\varepsilon^7.$$

Therefore we get that $|V_3| \ll |V_5| \ll |V_7| \ll |V_9| \ll |V_{11}|$, and if $d_4 > 0, d_{63} > 0, d_{54} < 0, d_{15} > 0$ and $d_{26} < 0$ then $V_9 \cdot V_{11} < 0, V_7 \cdot V_9 < 0, V_5 \cdot V_7 < 0$ and $V_3 \cdot V_5 < 0$. So we obtain that at least four limit cycles bifurcating from the origin.

Finally, replacing d_6 by $d_6(\varepsilon) = d_{62}\varepsilon^2 + d_{63}\varepsilon^3, d_5$ by $d_5(\varepsilon) = d_{52}\varepsilon^2 + d_{53}\varepsilon^3 + d_{54}\varepsilon^4, d_1$ by $d_1(\varepsilon) = d_{10} + d_{11}\varepsilon + d_{12}\varepsilon^2 + d_{13}\varepsilon^3 + d_{14}\varepsilon^4 + d_{15}\varepsilon^5, d_2$ by $d_2(\varepsilon) = d_{20} + d_{21}\varepsilon + d_{22}\varepsilon^2 + d_{23}\varepsilon^3 + d_{24}\varepsilon^4 + d_{25}\varepsilon^5 + d_{26}\varepsilon^6$ and d_3 by $d_3(\varepsilon) = d_{37}\varepsilon^7$, with $d_{62} = \dfrac{12a^2d_4^3}{35b}, d_{52} = -\dfrac{1}{7}\left(30d_4^3\right), d_{53} = -\dfrac{16bd_{63}}{a^2}, d_{10} = -\dfrac{6bd_4}{a^2}, d_{11} = 0, d_{12} = -\dfrac{12a^2d_4^3}{7b}, d_{13} = -5d_{63}, d_{14} = 0, d_{20} = -3d_4, d_{21} = 0, d_{22} = 0, d_{23} = 0, d_{24} = 0$ and $d_{25} = 0$, we get that $|d_3 - d_3(\varepsilon)| = |d_{37}|\varepsilon^7 << \varepsilon$ and the origin is a stable focus if $d_{37} < 0$ and unstable focus if $d_{37} > 0$.

Hence we obtain that $|V_3| \ll |V_5| \ll |V_7| \ll |V_9| \ll |V_{11}|$, and if $d_4 > 0, d_{63} > 0, d_{54} < 0, d_{15} > 0, d_{26} < 0$ and $d_{37} > 0$ then $V_9 \cdot V_{11} < 0, V_7 \cdot V_9 < 0, V_5 \cdot V_7 < 0$ and $V_3 \cdot V_5 < 0$. In this way we obtain one more limit cycle bifurcating from the origin. Therefore we get at least five limit cycles bifurcating from the origin, concluding the proof. □

Remark 2. *One could argue about the existence of more than five limit cycles inside the heteroclinic loop. Although that situation could be realizable, usually the multiplicity of the weak focus determines an upper bound for the number of limit cycles inside the loop. For instance, in the previous lemma it is easy to see that the first limit cycle bifurcation from the origin, call γ, is unstable. Moreover we are able to compare this information with the persistence or not of the heteroclinic loop. Indeed by fixing the values of parameters for which we obtain five limit cycles, the expression of M_1 from Subsection 5.1 we get*

$$M_1 = \frac{a^8 d_4^3}{2310\sqrt{2}b^4}\varepsilon^2 + O(\varepsilon^3).$$

Since we are assuming $d_4 > 0$ we get that $M_1 > 0$. From Subsection 5.1 the orbit leaving p_1 goes away from γ, so by using the Poincaré-Bendixson Theorem a convenient annular region it follows that system (6) has either no limit cycles between γ and the saddle point p_1 or it appears in pairs. So we conjecture that five is an upper bound for the number of limit cycles on the region located between the weak focus and the saddle point.

Lemma 11. *There exist a suitable choice of parameters such that system (6) has simultaneously 3 limit cycles bifurcating from the heteroclinic loop and 2 limit cycles bifurcating from the origin.*

Proof. As we see in Subsection 5.1 system (6) has a heteroclinic loop A_1^ε if $d_2 = -\frac{12d_4}{7} - \frac{5a^4d_5}{84b^2} + \frac{a^2(99d_1+160d_6)}{462b} + \frac{10bd_3}{a^2} + O(\varepsilon)$. Moreover, if $d_1 = \frac{40d_6}{33} + \frac{2a^2d_5}{3b} - \frac{6d_4b}{a^2} + \frac{28d_3b^2}{a^4}$ then the stability of that loop is established by the sign of $d_5 = 6b(-100a^4d_6 + 231a^2bd_4 + 308d_3b^2)/11a^6$. The first part of the proof consists in changing the stability of the loop to get 2 limit cycles, and finally to destroy the loop in order to obtain one more limit cycle. We start assuming $d_5 < 0$ so A_1^ε is stable. Now we write $d_5(\varepsilon) = d_{50} + \varepsilon d_{51}$ with $d_{50} = (1848b^3d_3 + 1386a^2b^2d_4 - 600a^4bd_6)/11a^6$ so

$$M_1 = \int_{A_1} \frac{\partial Q_1}{\partial y} dt = -\frac{a^7 d_{51}}{315\sqrt{-2b^7}}\varepsilon^2. \tag{20}$$

Consequently since $a > 0$ the loop A_1^ε now is defined by the sign of d_{51}. Indeed, if $d_{51} < 0$ then the loop changes its stability from stable to unstable and by applying Poincaré-Bendixson Theorem a stable limit cycle emerges.

In order to obtain a new limit cycle we change again the stability of A_1^ε. For doing this, consider the expression of $div(p_2) = \frac{\partial Q_1}{\partial y}$ considering $d_1(\varepsilon)$ a convenient perturbation of the parameter d_1 given by $d_1(\varepsilon) = d_{10} + \varepsilon d_{11} + \varepsilon^2 d_{12}$ with $d_{11} = 2a^2d_{51}/3b$ and $d_{10} = (66b(70bd_3 + 39a^2d_4) - 1160a^4d_6)/33a^4$, we get

$$div(p_2) = \frac{a^2 d_{12}}{7b^2}\varepsilon^3. \tag{21}$$

Therefore by choosing $d_{12} < 0$ the heteroclinic loop changes stability from unstable to stable and again, by the Poincaré-Bendixson theorem, a limit cycle bifurcates from A_1^ε. More precisely, that limit cycles is unstable. Finally to obtain the third limit cycle we choose the parameters so that the loop A_1^ε is broken. We do that considering $d_2(\varepsilon) = d_{20} + \varepsilon d_{21} + \varepsilon^2 d_{22} + \varepsilon^3 d_{23}$ where now we choose

$d_{22} = 3a^2 d_{12}/14b$, $d_{21} = a^4 d_{51}/12b^2$ and $d_{20} = (1980a^2b^2d_3 + 495a^2bd_4 - 260a^4bd_6)/66a^2b$. Then we obtain

$$M_1 = -\frac{a^4 d_{23}}{15\sqrt{2}b^2}\varepsilon^3.$$

We see that assuming $d_{23} \neq 0$ the loop is broken. Moreover, from Section 5 and since the second limit cycle is unstable, by choosing $d_{23} > 0$ we get $M_1 < 0$. Consequently, by the Poincaré-Bendixson Theorem we obtain a third limit cycle which is stable. Therefore we obtain three limit cycles by using Melnikov method. Moreover, for ε positive and sufficiently small we have that the expression of equations (20) and (21) have opposite sign and $|div(p_2)| \ll |M_1|$. It means that the change of stability of A_1^ε are only local so we can apply the Poincaré-Bendixson Theorem to obtain the limit cycles.

The proof of the lemma follows by applying the same approach used in Lemma 10 having now fixed the values $d_5(\varepsilon) = d_{50} + \varepsilon d_{51}$, $d_1(\varepsilon) = d_{10} + \varepsilon d_{11} + \varepsilon^2 d_{12}$ and $d_2(\varepsilon) = d_{20} + \varepsilon d_{21} + \varepsilon^2 d_{22} + \varepsilon^3 d_{23}$. That is, it can be obtained two more limit cycles but now bifurcating from the origin of system (6) so we get 5 limit cycles for such a system. The simultaneity occurs because at each step the obtained limit cycles are hyperbolic so they remain by assuming perturbations of the parameters involving higher orders of ε. □

Lemma 12. *There exist a suitable choice of parameters such that system* (6) *has s limit cycles bifurcating from the origin and m limit cycles bifurcating from the heteroclinic loop with* $s \in \{0, 1, 3, 4, 5\}$, $m \in \{0, 1, 2\}$ *and* $s + m \leq 5$.

Proof. The proof of this Lemma 12 is straightforward by using the same construction of Lemmas 10 and 11. □

6.2. **Technical lemmas for the homoclinic connection occurring in system (7).** We now state similar results but concerning system (7). Again we obtain limit cycles bifurcating from the center and from the homoclinic loops considering convenient values of the parameters of system (7). In addition we also consider some symmetry in system (7) so that limit cycles may emerge in pairs.

Lemma 13. *There exists at least four limit cycles bifurcating from the origin of system* (7).

Proof. In the proof of Lemma 3 under the hypothesis $e_3 = 0$ and $e_2 = \dfrac{4a^2be_1 - 48a^4e_5}{b^2}$ we obtain that: if the parameters e_i satisfies the conditions $e_1 = -\dfrac{3\left(b^2e_4 - 16a^4e_5\right)}{4a^2b}$, $e_6 = \dfrac{8a^4e_5 + 3b^2e_4}{5a^2b}$, $e_5 = -\dfrac{3\left(a^4e_4^3\varepsilon^2 + 8b^2e_4\right)}{44a^4}$ and $e_4 = 0$ then the system (7) is Hamiltonian and therefore the origin is a center.

Similarly we did in the proof of Lemma 10, we will consider small perturbations in these values of parameters to obtain a variation of the sign of the Lyapunov constants and the stability of the origin.

The signal of V_9, given in (19), is given by the sign of e_4. Therefore, if $e_4 > 0, e_4 < 0$, resp., then the origin is a weak repeller focus, weak attractor focus, resp.. Without loss of generality we assume that $e_4 > 0$. Consider the expression of V_7 given in (18) and the perturbation of the parameter e_5

given by $e_5(\varepsilon) = e_{50} + e_{52}\varepsilon^2$, with

$$e_{50} = -\frac{6b^2 e_4}{11a^4},$$

we obtain $|e_5 - e_5(\varepsilon)| = \left|\left(-\frac{3e_4^3}{44} - e_{52}\right)\right|\varepsilon^2 << \varepsilon$ and

$$V_7 \;\; = \frac{1}{32}\pi\varepsilon^3\left(3e_4^3 + 44e_{52}\right).$$

So the sign of V_7 is given by the sign of $3e_4^3 + 44e_{52}$. Then $|V_7| \ll |V_9|$ and if $e_4 > 0$ and $e_{52} < -\frac{3}{44}e_4^3$ then $V_7 \cdot V_9 < 0$. Therefore we conclude that at least one limit cycle bifurcating from the origin with these assumptions.

Consider the expression of V_5 given in (17) and replacing e_6 by $e_6(\varepsilon) = e_{60} + e_{62}\varepsilon^2 + e_{63}\varepsilon^3$ and e_5 by $e_{50} + e_{52}\varepsilon^2$, with $e_{50} = -\frac{6b^2e_4}{11a^4}, e_{60} = -\frac{3be_4}{11a^2}$ and $e_{62} = \frac{8a^2e_{52}}{5b}$. We get $|e_6 - e_6(\varepsilon)| = |e_{63}|\varepsilon^3 << \varepsilon$ and

$$V_5 \;\; = -\frac{5}{8}\pi e_{63}\varepsilon^4.$$

Then $|V_5| \ll |V_7| \ll |V_9|$, and if $e_4 > 0, e_{52} < -\frac{3}{44}e_4^3$ and $e_{63} < 0$ then $V_7 \cdot V_9 < 0$ and $V_5 \cdot V_7 < 0$. So we obtain that at least two limit cycles bifurcating from the origin.

Consider the expression of V_3 given in (16), replacing e_6 by $e_6(\varepsilon) = e_{60} + e_{62}\varepsilon^2 + e_{63}\varepsilon^3, e_5$ by $e_5(\varepsilon) = e_{50} + e_{52}\varepsilon^2$ and e_1 by $e_1(\varepsilon) = e_{10} + e_{14}\varepsilon^4$, with $e_{50} = -\frac{6b^2e_4}{11a^4}, e_{60} = -\frac{3be_4}{11a^2}$ and $e_{62} = \frac{8a^2e_{52}}{5b}$ and $e_{10} = -\frac{321be_4}{44a^2}$ therefore $|e_1 - e_1(\varepsilon)| = |e_{14}|\varepsilon^4 << \varepsilon$ and

$$V_3 = -\frac{\pi a^2 e_{14}\varepsilon^5}{b}.$$

So we obtain that $|V_3| \ll |V_5| \ll |V_7| \ll |V_9|$ and if $e_4 > 0, e_{52} < -\frac{3}{44}e_4^3, e_{63} < 0$ and $e_{14} > 0$ then $V_7 \cdot V_9 < 0, V_5 \cdot V_7 < 0$ and $V_3 \cdot V_5 < 0$, then we obtain that at least three limit cycles bifurcating from the origin.

Finally, replacing e_6 by $e_6(\varepsilon) = e_{60} + e_{62}\varepsilon^2 + e_{63}\varepsilon^3, e_5$ by $e_5(\varepsilon) = e_{50} + e_{52}\varepsilon^2, e_1$ by $e_1(\varepsilon) = e_{10} + e_{14}\varepsilon^4$ and e_3 by $e_3(\varepsilon) = e_{36}\varepsilon^6$, with $e_{50} = -\frac{6b^2e_4}{11a^4}, e_{60} = -\frac{3be_4}{11a^2}$ and $e_{62} = \frac{8a^2e_{52}}{5b}$ and $e_{10} = -\frac{321be_4}{44a^2}$, we get that $|e_3 - e_3(\varepsilon)| = |e_{36}|\varepsilon^6 << \varepsilon$ and the origin is a stable focus if $e_{36} < 0$ and unstable focus if $e_{36} > 0$.

Hence we obtain that $|V_3| \ll |V_5| \ll |V_7| \ll |V_9| \ll |V_{11}|$, and if $e_4 > 0, e_{52} < -\frac{3}{44}e_4^3, e_{63} < 0, e_{14} > 0$ and $e_{36} > 0$ then $V_7 \cdot V_9 < 0, V_5 \cdot V_7 < 0, V_3 \cdot V_5 < 0$ with $V_3 < 0$ and the origin an unstable focus.

In this way we obtain one more limit cycle bifurcating from the origin. Therefore we get at least four limit cycles bifurcating from the origin, concluding the proof. ☐

Lemma 14. *There exist a suitable choice of parameters such that system (7) has 4 limit cycles bifurcating from the homoclinic loop and 2 limit cycles bifurcating from the origin.*

Proof. The proof of the lemma is similar to the proof of Lemma 11, so we only highlight some minor differences between them. Indeed, we perform replacements of the parameters e_5, e_1 and e_2 in terms of order 1, 2 and 3 in ε, analogously to what we have done for the parameters d_5, d_1 and d_2 in Lemma 11. Through the replacement of e_5 we change the stability of the homoclinic loop L_1^ε.

Applying the Poincaré-Bendixson Theorem in the convenient annular regions, one of them internal to L_1^ε and other one external to it, we obtain a limit cycle in each annulus. Proceeding in a complete analogous way we obtain two more limit cycles from the replacement of the parameter e_1. Therefore, we get four limit cycles. In order to obtain the fifth limit cycle, we destroy the loop of L_1^ε doing the referred replacement in the parameter e_2. In this case whatever is the sign of M_2 (see Section 5.1) we obtain a limit cycle, being *internal* if $M_2 < 0$ and *external* to it otherwise. In any case we obtain five limit cycles bifurcating from the homoclinic loop. To obtain the two limit cycle bifurcating from the weak focus we proceed as in Lemma 13. □

6.3. **Proof of Theorem 1.** As commented before, the proof of Theorem 1 consists in apply the lemmas from Subsection 6.1. We remark that in the proof of the lemmas we obtain limit cycles of opposite stability for the cases $a > 0$, $b < 0$ and $a > 0$, $b < 0$ assuming that $d_3 = 0$ and $e_3 = 0$, respectively. Then one more limit cycle bifurcates by suitable perturbations of d_3 and e_3, respectively, preserving the already obtained limit cycles.

- **Proof of statement** (a)**:** Assume that $a > 0$ and $b < 0$. The configurations $(s, 0)$ with $s \in \{0, 1, \dots, 5\}$ follows from Lemma 10. The case $s = 2$ and $m = 3$ follows from Lemma 11. The remaining configurations $(s, m), m \in \{1, 2, 3\}$ and $s + m \leq 5$ follows from Lemma 12. The proof of the case $a > 0$ and $b < 0$ is then completed since the configurations of limit cycles of systems (5) and (6) are the same.

Realization of the maximal ciclicity and simultaneity of limit cycles: We notice that the sharper lower bound of 5 limit cycles when $a > 0$ and $b < 0$ can be obtained by the configurations (s, m) given by $(5, 0)$, $(4, 1)$, $(3, 2)$ and $(2, 3)$. The simultaneity is achieved in four out of the five situations.

- **Proof of statement** (b)**:** Now assume that $a < 0$ and $b > 0$. We proceed in a similar way to the previous case but now the double homoclinic loop plays a role. In what concerns the configurations of limit cycle bifurcating from the weak focus, after a translation of p_1 to the origin, configurations $(2s, k)$ with $s \in \{0, 1, \dots, 4\}, k \in \{0, 1, 2\}$ follows from both Lemma 13 and by symmetry with respect to the $y-$axis, see Remark 3. Now we prove the other configurations of limit cycles. First, the configuration $(2s, 3m), m \in \{1, 2\}$ is obtained by Lemma 14 because every internal limit cycle bifurcating from the homoclinic loop appears pairwise due to the symmetry.

Therefore, each one of the first two steps of the proof of Lemma 14 generates three limit cycles, being one external and two internal, so we get $3m$ medium limit cycles with $m = 1, 2$. The $2s$ small limit cycles of the configuration $(2s, 3m)$ follows also from Lemma 14 and symmetry. Finally the configuration $(2s, 3m + k)$ is obtained from the previous cases and observing that the value of k is determined from the break of the homoclinic loop. That break could generate no limit cycles $(k = 0)$, one external medium limit cycle $(k = 1)$, or two internal medium limit cycles by using symmetry $(k = 2)$, see Lemma 14.

Realization of the maximal ciclicity and simultaneity of limit cycles: In this case the maximal ciclicity of 12 limit cycles in obtained in only one situation which corresponds to a configuration of simultaneity between small and medium limit cycles. Indeed, the configuration $2s + 3m + k = 12$ can be obtained with $s = m = k = 2$, represented by the pair $(2, 10)$. This is precisely the situation of Lemma 14 after applying the symmetry.

The cases $2s + 3m + k < 12$ are obtained similarly from the other lemmas from the current section. Finally we notice that the configurations of limit cycles after and before the translation of system (7) are clearly preserved. The same results can be obtained for system (5) with $a < 0$ and $b > 0$. So we are done.

Remark 3. *We finish remarking some aspects of Theorem 1 that must be clarified.*

(I) *It does not provide any information about configurations having more than three medium limit cycles. Indeed to obtain such configurations one should take into account higher orders of Melnikov function, which is not considered in this paper.*

(II) *The value k is determined by breaking the loop, then an extra limit cycle can emerge. When a limit cycle takes place, if such a loop is broken in such way that a limit cycle appears internally, then by the symmetry we have a second limit cycle and in this case $k = 2$. Otherwise, we get an externally limit cycle so in this case $k = 1$.*

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