GLOBAL NILPOTENT REVERSIBLE CENTERS WITH CUBIC NONLINEARITIES SYMMETRIC WITH RESPECT TO THE x-AXIS

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Abstract. Let $P_3(x, y)$ and $Q_3(x, y)$ be polynomials of degree three without constant or linear terms. We characterize the global centers of all polynomial differential systems of the form $\dot{x} = y + P_3(x, y)$, $\dot{y} = Q_3(x, y)$ that are reversible and invariant with respect to the x-axis.

1. Introduction and statement of the main results

A planar polynomial differential system of degree three having a nilpotent center at the origin can be written as

$$\begin{aligned} x' &= y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ y' &= b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3. \end{aligned} \tag{1}$$

We consider systems (1) that are invariant under the symmetry $(x, y, t) \mapsto (x, -y, -t)$. Imposing that systems (1) are invariant under such symmetry we get that $a_{20} = a_{30} = a_{02} = a_{12} = b_{11} = b_{21} = b_{03} = 0$ and they become

$$\begin{aligned} x' &= y(1 + a_{11}x + a_{21}x^2 + a_{03}y^2), \\ y' &= b_{20}x^2 + b_{30}x^3 + b_{02}y^2 + b_{12}xy^2. \end{aligned} \tag{2}$$

Note that $(0, 0)$ is a nilpotent singular point. To be isolated we need that the second equation in (2) is not identically zero (which yields $b_{20}^2 + b_{30}^2 + b_{02}^2 + b_{12}^2 > 0$) and that both equations in (2) do not have the common factor y (which gives $b_{20}^2 + b_{30}^2 > 0$). We can prove that if $b_{20}^2 + b_{30}^2 > 0$, then the two equations in (2) cannot have a common factor of the form $ax + by$ with $a \neq 0$ or of the form $ax^2 + bxy + cy^2 + dx + ey$ with $a^2 + b^2 + c^2 > 0$. In short, the singular point $(0, 0)$ is isolated if and only if $b_{20}^2 + b_{30}^2 > 0$.

Now we apply [3, Theorem 3.5] to ensure that the singular point is a linear nilpotent center. Since system (3) is reversible, such a linear nilpotent center will be indeed a center. We compute the functions F and G defined in [3, Theorem 3.5] and we get

$$F(x) = b_{20}x^2 + b_{30}x^3 \quad \text{and} \quad G(x) = 0.$$

So the origin is a nilpotent center if and only $b_{20} = 0$ and $b_{30} < 0$. Note that under these conditions the origin is an isolated singular point.

Assume that $b_{20} = 0$ and $b_{30} = -\alpha^2$ with $\alpha \neq 0$. Then system (2) becomes

$$\begin{aligned} x' &= y(1 + a_{11}x + a_{21}x^2 + a_{03}y^2), \\ y' &= -\alpha^2x^3 + b_{02}y^2 + b_{12}xy^2. \end{aligned} \tag{3}$$

We characterize the planar polynomial differential systems (3) having a global center at the origin, called from now on global nilpotent centers. We recall that a center is a singular point filled up

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