Planar central configurations of some restricted $(4+1)$-body problems

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We start with the 13 central configurations of the restricted $(4+1)$–problem where the four primaries have equal masses and are located at the vertices of a square. Then we describe the evolution of these central configurations when some of the masses of the four primaries tend to zero and the remainder ones keep constant. More precisely, we consider the cases where one of the masses tends to zero, where either two adjacent or two opposite equal masses tend to zero simultaneously, and where three equal masses tend to zero simultaneously. Here simultaneously means that the masses which go to zero take the same value at any moment.

I. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The *n-body problem* is the problem of studying the motions of n punctual masses interacting between them under the Newtonian gravitation.

A configuration of the bodies of the n-body problem is called *central* when the acceleration of each body is proportional to the position vector of the body with respect to the center of mass.

The set of all planar central configurations is invariant under homotheties with respect to the center of mass and rotations. When we count the number of central configurations we mean the number of equivalence classes with respect the equivalence relations defined by these homotheties and rotations.

Central configurations are important in the analysis of the n-body problem for several reasons, here we only mentioned briefly some of them.

(1) They allow to compute all the homographic solutions of this problem (see[16]).

(2) Every motion starting or ending in a total collision is asymptotic to a central configuration (see[4,10]).

(3) Every parabolic motion of the n bodies is asymptotic to a central configuration (see[4,10]).

(4) They play a role in the study of the invariant sets obtained fixing the energy and the angular momentum (see[13,14]).

(5) They have been used for different missions in the solar system (see[6,7]).

The central configurations of the 2– and 3-body problem are known (see[9]), but the problem of finding the central configurations when $n>3$ is far to be solved. More precisely, for $n>3$ we only know the central configurations for some particular n-body problems where, in general, the configurations satisfy some geometrical properties, or some of the masses are equal.

The objective of this paper is to study some families of central configurations of the planar restricted $(4+1)$-body problem. More precisely, we start with the 13 central configurations of the restricted $(4+1)$-body problem with the four primaries having equal masses localized at the vertices of a square (see for instance[5,11]). Then we describe the evolution of the families of central configurations coming from the numerical continuation of these 13 central configurations when we decrease the mass of either one, or two, or three primaries with equal masses to zero. These families end at a central configuration of a restricted problem with either two, three or four infinitesimal masses. All numerical computations have been done using enough precision to ensure that all results provided here are accurate at least up to twelve decimal places.

We define the following restricted 5-body problems.

- The *restricted square* $(4+1)$*-body problem* with four equal masses at the vertices of a square and a fifth infinitesimal mass.
- The *restricted isosceles trapezoidal* $(4+1)$*-body problem* with two pairs of adjacent equal masses at the vertices of an isosceles trapezoid and a fifth infinitesimal mass.
- The *restricted kite* $(4+1)$*-body problem* with the four masses at the vertices of a kite and with either a pair of opposite equal masses or three equal masses and a fifth infinitesimal mass.
- The *restricted* $(2+3)$*-body problem* with two primaries having equal masses and three infinitesimal masses.
- The *restricted equilateral triangular* $(3+2)$*-body problem* with three primaries having equal masses at the vertices of an equilateral triangle and two infinitesimal masses.
- The *restricted* $((1+3)+1)$*-body problem* with the primary at the origin, three equal infinitesimal masses at

a central configuration of the restricted $(1+3)$-body problem and a fifth infinitesimal mass equal to zero. We note that our restricted $((1+3)+1)$-body problem is a particular case of the general restricted $(1+4)$-body problem (see Section III for more details).

The paper is structured as follows. In Section II we give the equations of central configurations of the general 5-body problem. In Section III we give the 13 central configurations of the restricted square $(4+1)$-body problem and we give the central configurations of the restricted $(3+2)$, $(2+3)$ and $((1+3)+1)$-body problems which appear as the limit cases of the continued families as either one, two or three infinitesimal masses tend to zero.

In Section IV we describe the evolution of the families of central configurations emanating from the restricted square $(4+1)$-body problem when two adjacent equal masses of the square tend simultaneously to zero along the family of the restricted isosceles trapezoidal $(4+1)$-body problem central configurations. These families end at central configurations of the restricted $(2+3)$-body problem. The obtained results are summarized in Figure 6.

In Subsection V A we describe the evolution of the families of central configurations emanating from the restricted square $(4+1)$-body problem when two opposite equal masses of the square tend simultaneously to zero along the family of the restricted kite $(4+1)$-body problem central configurations with two pairs of equal masses. These families also end at central configurations of the restricted $(2+3)$-body problem. The obtained results are summarized in Figure 9.

In Subsection V B we describe the evolution of the families of central configurations emanating from the restricted square $(4+1)$-body problem when one of the masses of the square tends to 0 along the family of the restricted kite $(4+1)$-body problem central configurations with three big equal masses. These families end at central configurations of the restricted equilateral triangular $(3+2)$-body problem. The obtained results are summarized in Figure 12.

Finally in Subsection V C we describe the evolution of the families of central configurations emanating from the restricted square $(4+1)$-body problem when three equal masses of the square tend simultaneously to 0 along the family of the restricted kite $(4+1)$-body problem central configurations with three small equal masses. These families end at central configurations of the restricted $((1+3)+1)$-body problem. The obtained results are summarized in Figure 15.

Some preliminary numerical results on the planar central configurations of the restricted $4+1$-body problem can be found in[12].

II. EQUATIONS OF THE CENTRAL CONFIGURATIONS OF THE 5-BODY PROBLEM IN THE PLANE

Let (x_i, y_i) for $i = 1,\dots,N$ be the position of the punctual mass m_i of the i–th body. Then the center of masses (c_1, c_2) of the N-body problem is defined by

$$(c_1, c_2) = \frac{1}{M}\left(\sum_{i=1}^{N} m_i x_i, \sum_{i=1}^{N} m_i y_i\right),$$

where $M = \sum_{i=1}^{N} m_i$. Since in a central configuration the acceleration of each body is proportional with a constant λ to the position vector of the body with respect to the center of mass, a configuration

$$\{(x_1,y_1),(x_2,y_2),(x_3,y_3),\dots,(x_N,y_N)\}$$

of the N bodies is central if it satisfies the equations

$$e_{x_i} = 0, \qquad e_{y_i} = 0, \tag{1}$$

for $i = 1,\dots,N$, where $r_{ij} = \sqrt{(x_i-x_j)^2+(y_i-y_j)^2}$ and

$$e_{x_i} = \sum_{j=1, j\neq i}^{N} m_j \frac{x_i - x_j}{r_{ij}^3} - \lambda (x_i - c_1),$$

$$e_{y_i} = \sum_{j=1, j\neq i}^{N} m_j \frac{y_i - y_j}{r_{ij}^3} - \lambda (y_i - c_2).$$

We note that we have the next two relations

$$\sum_{i=1}^{N} m_i e_{x_i} = 0, \qquad \sum_{i=1}^{N} m_i e_{y_i} = 0.$$

Therefore for the 5–body problem ($N = 5$) the ten equations (1) can be reduced to the following eight equations

$$e_j = e_{x_{j+1}} - e_{x_1} = 0, \qquad e_{j+4} = e_{y_{j+1}} - e_{y_1} = 0, \tag{2}$$

for $j = 1,2,3,4$.

III. CENTRAL CONFIGURATIONS OF THE LIMITING PROBLEMS

Recall that the restricted $(L+N)$-body problem is the limit case of the $(L+N)$-body problem having L given masses and N small masses when these small masses tend to zero. Thus, $\mathbf{q}$ is a central configuration of the restricted $(L+N)$-body problem if $q = \lim_{\varepsilon\to 0}\mathbf{q}(\varepsilon)$ where $\mathbf{q}(\varepsilon)$ is a central configuration of the planar $(L+N)$-body problem with masses m_i for $i = 1,\dots,L$ and $m_{j+L} = \varepsilon\mu_j$ for $j = 1,\dots,N$. One can easily see, by taking the terms of order 0 in ε in the equations of central configurations of the $(L+N)$-body problem, that in the central configurations of the restricted $(L+N)$-body problem with $L > 1$ the L large masses must be in a central configuration of the L-body problem and the possible positions of each one of the infinitesimal masses are the possible positions of the infinitesimal mass in a central configuration of the restricted $(L+1)$-body problem. Thus in this case the positions of the infinitesimal masses do not depend on the values of μ_i. When $L = 1$ the terms of order 0 in ε in the equations are not enough to determine the position of the N infinitesimal masses and we must use the terms of order 1. This causes that the positions of the infinitesimal masses in a central configuration of the restricted $(1+N)$-body problem depend on μ_i (see for more details[3]).

A. Restricted square $(4+1)$-body problem central configurations

First we describe the central configurations of the *restricted square* $(4+1)$*-body problem* with four equal masses at the vertices of a square. By taking conveniently the units of mass and length it is not restrictive to consider four equal masses $m_1 = m_2 = m_3 = m_4 = 1$ at the vertices of the square with coordinates $(x_1,y_1) = (-1,-1)$, $(x_2,y_2) = (1,-1)$, $(x_3,y_3) = (1,1)$ and $(x_4,y_4) = (-1,1)$ and a fifth infinitesimal mass $m_5 = 0$ at (x_5,y_5).

For the restricted square $(4+1)$-body problem central configurations, we know from the works[5,11] that the infinitesimal mass must be placed at an axis of symmetry of the square and that there exists exactly 13 possible positions for the infinitesimal mass. Now we compute the coordinates $s^k = (x_5^k, y_5^k)$ for $k = 1,\dots,13$ of the possible positions for the infinitesimal mass under the above assumptions on the positions and the masses. We substitute the assumptions into (1) with $N = 5$, then by isolating λ from equation $e_{x_1} = 0$ and substituting it into the remaining equations we get that equations $e_{x_i} = 0$ and $e_{y_i} = 0$ are trivially satisfied for $i = 1,2,3,4$ and that s^k for $k = 1,\dots,13$ are the solutions of equations $e_{x_5} = 0$ and $e_{y_5} = 0$ which are equivalent to

$$\frac{x_5+1}{((1+x_5)^2+(1+y_5)^2)^{3/2}} + \frac{x_5-1}{((1-x_5)^2+(1+y_5)^2)^{3/2}} + \frac{x_5-1}{((1-x_5)^2+(1-y_5)^2)^{3/2}} + \frac{x_5+1}{((1+x_5)^2+(1-y_5)^2)^{3/2}} - \left(\frac{1}{4}+\frac{\sqrt{2}}{16}\right)x_5 = 0,$$

$$\frac{y_5+1}{((1+x_5)^2+(1+y_5)^2)^{3/2}} + \frac{y_5-1}{((1+x_5)^2+(1-y_5)^2)^{3/2}} + \frac{y_5-1}{((1-x_5)^2+(1-y_5)^2)^{3/2}} + \frac{y_5+1}{((1-x_5)^2+(1+y_5)^2)^{3/2}} - \left(\frac{1}{4}+\frac{\sqrt{2}}{16}\right)y_5 = 0,$$

respectively. The implicit graphs of $e_{x_5} = 0$ and $e_{y_5} = 0$ are the red and the blue curve respectively in Figure 1. By solving numerically equations $e_{x_5} = 0$, $e_{y_5} = 0$ we get the 13 solutions $s^k = (x_5^k, y_5^k)$ for $k = 1,\dots,13$ given by (see Figure 2)

$$\begin{aligned}
s^1 &= (0,\ 0),\\
s^2 &= (0.986244975201..,\ 0),\\
s^3 &= (2.266147813663..,\ 0),\\
s^4 &= (0,\ 0.986244975201..),\\
s^5 &= (0,\ 2.266147813663..),\\
s^6 &= (1.880455410280..,\ 1.880455410280..),\\
s^7 &= (0,\ -0.986244975201..),\\
s^8 &= (0,\ -2.266147813663..),\\
s^9 &= (1.880455410280..,\ -1.880455410280..),\\
s^{10} &= (-0.986244975201..,\ 0),\\
s^{11} &= (-2.266147813663..,\ 0),\\
s^{12} &= (-1.880455410280..,\ -1.880455410280..),\\
s^{13} &= (-1.880455410280..,\ 1.880455410280..).
\end{aligned} \tag{3}$$

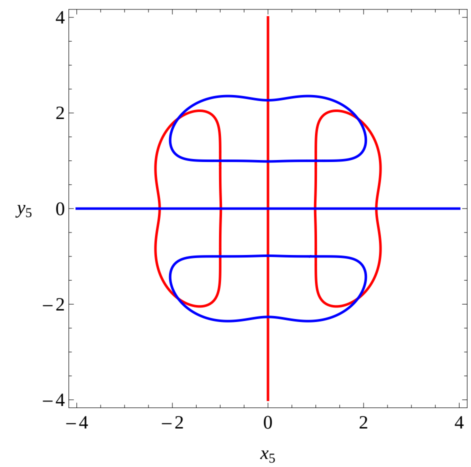


FIG. 1. Implicit graphs of $e_{x_5} = 0$ in red and $e_{y_5} = 0$ in blue. The intersection points of these two implicit graphs correspond to the possible positions of the infinitesimal mass for the restricted square (4+1)-body problem

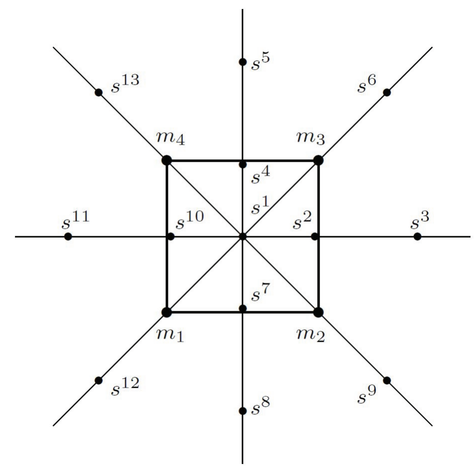


FIG. 2. Possible positions of the infinitesimal mass for the restricted square $(4+1)$-body problem

B. Restricted equilateral triangular $(3+2)$-body problem

We consider the *restricted equilateral triangular* $(3+2)$*-body problem* with three masses $m_2 = m_3 = m_4 = 1$ at the vertices of an equilateral triangle and two infinitesimal masses $m_1 = m_5 = 0$. The possible positions of the infinitesimal masses are the positions for the infinitesimal in a central configuration of the *restricted equilateral triangular* $(3+1)$*-body problem* with three equal masses located at the vertices of an equilateral triangle and one infinitesimal mass.

These central configurations were studied first by Arenstorf in[1] and later on by Bang and Elmabsout in[2], and by Fernandes et al. in[5] also in the more general case of the $(n+1)$-body problem with n equal masses at the vertices of a regular n-gon. In all these papers the authors proved that for the central configurations of the restricted equilateral triangular $(3+1)$-body problem the infinitesimal mass must be on one of the three straight lines passing through the barycenter and a vertex of the triangle, and on each of these straight lines there are exactly four positions for the infinitesimal mass. But none of the authors provided the exact position for the infinitesimal mass in such central configurations.

We assume that the masses $m_2 = m_3 = m_4 = 1$ are at the vertices of the equilateral triangle T defined by $(x_3, y_3) = (1,0)$, $(x_4, y_4) = (T_a, T_b)$, and $(x_2, y_2) = (x_4, -y_4)$, with $T_b > 0$. This is not restrictive by taking conveniently the units of mass and length. The triangle T is the one that will appear later on when m tends to 0 and it is such that $(-1,0)$ is one of the possible positions for the infinitesimal mass in a central configuration of the corresponding restricted equilateral $(3+1)$–body problem. We also assume that the infinitesimal mass $m_1 = 0$ is at (x_1, y_1) and we compute the possible positions of m_1 on the straight line $y = 0$ in the following way. First we substitute the above assumptions on the positions and the masses into (1) with $N = 4$. By isolating λ from equation $e_{x_2} = 0$ and substituting it into the remaining equations we get a system equivalent to

$$
\begin{aligned}
&\frac{2T_a - 3x_1 + 1}{\left(T_a^2 - 2T_a + T_b^2 + 1\right)^{3/2}} + \frac{2(x_1 - T_a)}{\left((T_a - x_1)^2 + T_b^2\right)^{3/2}} + \\
&\frac{x_1 - 1}{\left((x_1 - 1)^2\right)^{3/2}} = 0, \\
&\frac{2T_b}{\left(T_a^2 - 2T_a + T_b^2 + 1\right)^{3/2}} - \frac{1}{4T_b^2} = 0
\end{aligned} \tag{4}
$$

Assuming now that $(-1,0)$ is one of the possible positions for the infinitesimal mass we get that $T_a = -0.145130124159..$ and $T_b = 0.661141185440..$ where T_a and T_b are the solutions of system (4) when $x_1 = -1$. The x coordinate of the remaining possible positions for the infinitesimal mass on the straight line $y = 0$ are given the solutions of the fist equation of (4) where T_a and T_b are the values computed above. We get the four positions $(x_1, y_1) = p^i = (p^i_x, p^i_y)$ for $i = 1, \ldots, 4$ with $p^i_y = 0$ and

$$
\begin{aligned}
p^1_x &= -1, & p^2_x &= -0.079390442398.., \\
p^3_x &= 0.236579917226.., & p^4_x &= 1.7968710056...
\end{aligned}
$$

Note that we also have the positions p^i rotated about the center of the equilateral triangle by an angle $2\pi/3$ which are denoted by p^{i*} and rotated by an angle $4\pi/3$ which are denoted by p^{i**} (see Figure 3).

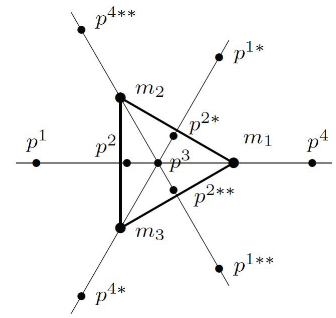


FIG. 3. Possible positions of the infinitesimal mass for the restricted square $(3+1)$-body problem

C. Restricted $(2+3)$-body problem central configurations

We consider the restricted $(2+3)$-body problem with two equal masses $m_1 = m_2 = 1$ located at $(x_1, y_1) = (-1,0)$ and $(x_2, y_2) = (1,0)$ and three infinitesimal masses $m_3 = m_4 = m_5 = 0$ located at (x_3, y_3), (x_4, y_4) and (x_5, y_5). This is not restrictive by taking conveniently the units of mass and length. The possible positions for each one of the infinitesimal masses in a central configuration of the restricted $(2+3)$-body problem coincide with the possible positions for the infinitesimal mass in a central configuration of the restricted $(2+1)$-body problem with two equal masses $m_1 = m_2 = 1$ located at $(x_1, y_1) = (-1,0)$ and $(x_2, y_2) = (1,0)$ and an infinitesimal mass located at (x, y). Under our assumptions, proceeding in a similar way than in the previous two restricted problems, we get that the equations of central configurations of the restricted $(2+1)$-body problem are

$$
\begin{aligned}
&\frac{1}{4} - \frac{1}{4}(x+1) + \frac{x-1}{\left((x-1)^2 + y^2\right)^{3/2}} + \frac{x+1}{\left((x+1)^2 + y^2\right)^{3/2}} = 0, \\
&y\left(-\frac{1}{4} + \frac{1}{\left((x-1)^2 + y^2\right)^{3/2}} + \frac{1}{\left((x+1)^2 + y^2\right)^{3/2}}\right) = 0.
\end{aligned}
$$

Its solutions are $q^k = (x^k, y^k)$ with

$$
\begin{aligned}
&q^1 = (0,0), \quad q^2 = (-\sqrt{\alpha}, 0), \quad q^3 = (\sqrt{\alpha}, 0), \\
&q^4 = (0, -\sqrt{3}), \quad q^5 = (0, \sqrt{3}),
\end{aligned}
$$

where $\alpha = 5.744709149227..$ is the unique real solution of the equation

$$x^5 - 4x^4 + 6x^3 - 68x^2 - 127x - 64 = 0,$$

(see Figure 4). For more details on this restricted problem see Szebehely[15], but take into account that there the two primaries have masses equal to $1/2$ and they are located at $(0,0)$ and $(1,0)$. Note that the central configurations of the restricted $(2+1)$–body problem with the infinitesimal mass at positions q^1, q^4 and q^5 correspond to the three Euler (collinear) central configurations and the ones with the infinitesimal mass at positions q^2 and q^3 correspond to the two Lagrange central configurations.

D. Restricted $((1+3)+1)$-body problem

By taking conveniently the units of mass and length we consider what we call the *restricted* $((1+3)+1)$*-body problem* which consists of four masses $m_1 = 1$ at $(-1,0)$, $m_2 = m_3 = m_4 = 0$ with m_3 at $(1,0)$ in a central configuration of the restricted $(1+3)$-body problem with three infinitesimal equal masses and a fifth infinitesimal mass $m_5 = 0$. It is a particular case of the restricted $(1+4)$-body problem where

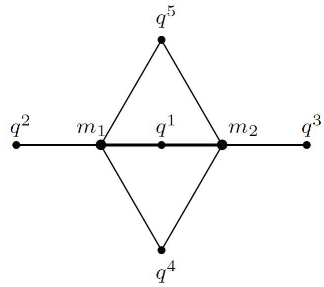


FIG. 4. Possible positions of the infinitesimal mass for the restricted square $(2+1)$-body problem

$\mu_1 = \mu_2 = \mu_3$ and $\mu_4 = 0$. In this sense this problem can be thought as a restricted problem of the restricted $(1+3)$-body problem with three equal infinitesimal masses. Is for that reason that we call it restricted $((1+3)+1)$-body problem.

Now we recall that the solutions of the central configurations $(1+3)$-body problem were studied in[3,8]. The $(1+3)$-body problem with three infinitesimal equal masses has three different classes of central configurations but here we only describe the one that will appear as limit when m tends to 0. In this central configuration m_1 is at $(-1,0)$, and the other three equal masses are on the circle of radius 2 centered at m_1 with m_3 at $(1,0)$, m_2 at $\tau^2 = (\tau_x^2, \tau_y^2) = (0.354757322483.., -1.471269043097..)$, and m_4 at $\tau^4 = (\tau_x^2, -\tau_y^2)$. The angles in counterclockwise starting at the positive x–axis of the masses m_2, m_3 and m_4 are $\alpha_2 = -0.8266029360..$, $\alpha_3 = 0$ and $\alpha_4 = -\alpha_2$.

We need to study the central configurations of the restricted $((1+3)+1)$-body problem. To simplify the computations we assume that $m_1 = 1$ is located at the origin and the three infinitesimal equal masses are located on the circle of radius 1 centered at the origin with angles α_2, 0, and $-\alpha_2$. Using the results of[3] the equations of the central configurations for the infinitesimal mass m_5 located on the circle of radius 1 with angle α_1 are

$$\begin{aligned}
&-\sin(\alpha_1-\alpha_2)\left(1-\frac{1}{8\left|\sin\left(\frac{1}{2}(\alpha_2-\alpha_1)\right)\right|^3}\right)\\
&-\sin(\alpha_1+\alpha_2)\left(1-\frac{1}{8\left|\sin\left(\frac{1}{2}(\alpha_1+\alpha_2)\right)\right|^3}\right)\\
&-\sin\alpha_1\left(1-\frac{1}{8\left|\sin\frac{\alpha_1}{2}\right|^3}\right)=0.
\end{aligned}$$

Here all the angles are measured counterclockwise starting at the positive x-axis. The solutions of this equation are

$$\alpha_1^1 = -1.547748984048.., \quad \alpha_1^2 = -0.407577027360..,$$
$$\alpha_1^3 = -\alpha_1^2, \quad \alpha_1^4 = -\alpha_1^1, \quad \alpha_1^5 = \pi.$$

Now we consider the restricted $((1+3)+1)$-body problem with $m_1 = 1$ at $(-1,0)$, m_2 at (τ_x^2, τ_y^2), m_3 at $(1,0)$, and m_4 at (τ_x^4, τ_y^4), then the position of the central configurations for m_5 now is $t^k = (t_x^k, t_y^k) = (-1,0) + 2(\cos(\alpha_1^k), \sin(\alpha_1^k))$ for $k = 1,\dots,5$ (see Figure 5).

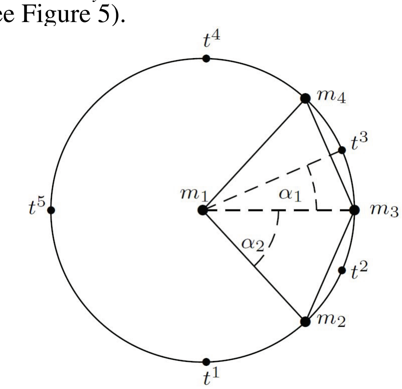


FIG. 5. Possible positions of the infinitesimal mass for the restricted square $((1+3)+1)$-body problem

IV. RESTRICTED ISOSCELES TRAPEZOIDAL $(4+1)$-BODY PROBLEM CENTRAL CONFIGURATIONS

We consider the *restricted isosceles trapezoidal* $(4+1)$-*body problem* having $m_1 = m_2 = 1$, $m_3 = m_4 = m$ located at the vertices of an isosceles trapezoid with coordinates $(x_1,y_1) = (-1,0)$, $(x_2,y_2) = (1,0)$, $(x_3,y_3) = (a,b)$ and $(x_4,y_4) = (-a,b)$ with $a,b > 0$. Here we have taken the unit of length so that the distance between m_1 and m_2 be two which is not restrictive.

Substituting these values into equations (2) with $m_5 = 0$ we get $e_5 = 0$ and $e_7 = e_6$. From $e_1 = 0$ we obtain

$$\lambda = \frac{1}{4} + m\left(-\frac{a-1}{((a-1)^2+b^2)^{3/2}} + \frac{a+1}{((a+1)^2+b^2)^{3/2}}\right).$$

Substituting λ in the remaining equations we get that $e_3 = -e_2$ and

$$\begin{aligned}
e_2 &= \frac{m-a^3}{4a^2} + \frac{(a-1)(am+1)}{((a-1)^2+b^2)^{3/2}} - \frac{(a+1)(am-1)}{((a+1)^2+b^2)^{3/2}} = 0,\\
e_4 &= -\frac{x_5}{4} + \frac{x_5-1}{((x_5-1)^2+y_5^2)^{3/2}} + \frac{x_5+1}{((x_5+1)^2+y_5^2)^{3/2}} + \frac{(a-1)mx_5}{((a-1)^2+b^2)^{3/2}} - \frac{(a+1)mx_5}{((a+1)^2+b^2)^{3/2}}\\
&\quad - \frac{m(a-x_5)}{((a-x_5)^2+(b-y_5)^2)^{3/2}} + \frac{m(a+x_5)}{((a+x_5)^2+(b-y_5)^2)^{3/2}} = 0,
\end{aligned} \tag{5}$$

$$e_6 = -\frac{b}{4} + \frac{b(am+1)}{((a-1)^2+b^2)^{3/2}} - \frac{b(am-1)}{((a+1)^2+b^2)^{3/2}} = 0,$$

$$e_8 = -\frac{y_5}{4} + \frac{y_5}{((x_5-1)^2+y_5^2)^{3/2}} + \frac{y_5}{((x_5+1)^2+y_5^2)^{3/2}} + \frac{m((a-1)y_5+b)}{((a-1)^2+b^2)^{3/2}} + \frac{m(b-(a+1)y_5)}{((a+1)^2+b^2)^{3/2}}$$
$$+ \frac{m(y_5-b)}{((a-x_5)^2+(b-y_5)^2)^{3/2}} + \frac{m(y_5-b)}{((a+x_5)^2+(b-y_5)^2)^{3/2}} = 0.$$

Notice that when $m = 1$, $a = 1$ and $b = 2$ the four primaries with masses $m_1 = m_2 = m_3 = m_4 = 1$ are located at a square with vertices $(x_1, y_1) = (-1, 0)$, $(x_2, y_2) = (1, 0)$, $(x_3, y_3) = (1, 2)$ and $(x_4, y_4) = (-1, 2)$. This square is the one in Section III displaced one unit above the y-axis. Therefore when $m = 1$, $a = 1$ and $b = 2$ the restricted isosceles trapezoidal $(4+1)$-body problem becomes the restricted square $(4+1)$-body problem and the possible positions of the infinitesimal mass $m_5 = 0$ that provide central configurations are $\tilde{s}^k$ where $\tilde{s}^k = s^k + (0, 1)$ for $k = 1, \ldots, 13$ with s^k given in (3). Let S^k for $k = 1, \ldots, 13$ denote the central configuration of the restricted square $(4+1)$-body problem with the infinitesimal mass $m_5 = 0$ located at the position $\tilde{s}^k$.

We want to continue numerically the central configurations of the restricted square $(4+1)$-body problem to the restricted isosceles trapezoidal $(4+1)$-body problem with the mass m playing the role of a parameter varying from $m = 1$ to $m = 0$.

When $m = 0$ the restricted isosceles trapezoidal $(4+1)$-body problem becomes the restricted $(2+3)$-body problem with two equal masses $m_1 = m_2 = 1$ located at $(x_1, y_1) = (-1, 0)$ and $(x_2, y_2) = (1, 0)$ and three infinitesimal masses $m_3 = m_4 = m_5 = 0$ located at (x_3, y_3), (x_4, y_4) and (x_5, y_5) studied in Subsection III C.

Now we make the continuation of central configurations from the restricted square $(4+1)$-body problem to the restricted $(2+3)$-body problem with masses $m_1 = m_2 = 1$ at $(-1, 0)$ and $(1, 0)$ and three infinitesimal masses $m_3 = m_4 = m_5 = 0$ through the family of the restricted isosceles trapezoidal $(4+1)$-body problem. Note that we only need to continue the central configurations of the restricted square $(4+1)$-body problem S_k with $k = 1, \ldots, 9$ (i.e. with $x \geq 0$). The continuation of S_{13} (respectively, S_{11}, S_{10}, and S_{12}) can be obtained from the continuation of S_6 (respectively, S_3, S_2, and S_9) by symmetry.

The equations of the central configurations of the restricted isosceles trapezoidal $(4+1)$-body problem are the four equations (IV) with the four unknowns a, b, x_5, y_5. So we want to find numerically the solutions of this set of four equations as m varies from 1 to 0. Let

$$J = \begin{pmatrix} \frac{\partial e_2}{\partial a} & \frac{\partial e_2}{\partial b} & \frac{\partial e_2}{\partial x_5} & \frac{\partial e_2}{\partial y_5} \\ \frac{\partial e_4}{\partial a} & \frac{\partial e_4}{\partial b} & \frac{\partial e_4}{\partial x_5} & \frac{\partial e_4}{\partial y_5} \\ \frac{\partial e_6}{\partial a} & \frac{\partial e_6}{\partial b} & \frac{\partial e_6}{\partial x_5} & \frac{\partial e_6}{\partial y_5} \\ \frac{\partial e_8}{\partial a} & \frac{\partial e_8}{\partial b} & \frac{\partial e_8}{\partial x_5} & \frac{\partial e_8}{\partial y_5} \end{pmatrix}. \tag{6}$$

A central configuration given by $\sigma^0 = (a^0, b^0, x_5^0, y_5^0)$ for a fixed value of $m = m^0$ is said to be *degenerate* if the rank of the matrix J is not maximal at m^0 and σ^0. From the Implicit Function Theorem we know that every non-degenerate central configuration can be continued to a unique family of central configurations when the parameter m varies. So the number of central configurations can only change if the degeneracy condition holds for some $m \in [0, 1)$.

Let $\sigma_k^0 = (1, 2, \tilde{s}_k)$ for $k = 1, \ldots, 9$ be the solution of (IV) with $m = 1$ corresponding to the central configuration of the restricted square $(4+1)$-body problem S^k. For each $k = 1, \ldots, 9$ we compute the values of the determinant of J evaluated at the solution $\sigma_k^0 = (1, 2, \tilde{s}_k)$ that we denote by $|J^k|$ and we get

$$\begin{aligned} |J^1| &= -0.482357853887.., \\ |J^2| = |J^7| &= 2.646929597501.., \\ |J^3| = |J^8| &= -0.136787789633.., \\ |J^4| &= 2.646929597501.., \\ |J^5| &= -0.136787789633.., \\ |J^6| = |J^9| &= 0.145282467634.. \end{aligned}$$

Since all these determinants are different from zero, the central configuration S_k is non–degenerate for all $k = 1, \ldots, 9$ and, from the Implicit Function Theorem, it can be continued to a unique family of central configurations with values of m sufficiently close to 1. We continue numerically these families of central configurations for m decreasing from 1 to 0 by using the following methodology. For each $k = 1, \ldots, 9$ we continue numerically the solution $\sigma_k^0 = (1, 2, \tilde{s}_k)$ of (IV) from $m = 1$ to either $m = 0$, or to a value m^* where the determinant $|J|$ evaluated at the corresponding solution becomes 0. The continuation method is based in the Newton's algorithm for finding zeroes of a vectorial function. We see that the determinant of J along the continued families is never zero. Therefore each central configuration S^k can be continued to a unique family of central configurations of the restricted isosceles trapezoidal $(4+1)$-body problem for $m \in [1, 0)$. The continued families tend to a central configuration of the restricted $(2+3)$-body problem with the two infinitesimal masses (m_3, m_4) colliding at q^5. Moreover, the infinitesimal mass m_5 is located at: q^5 for S^k with $k = 1, \ldots, 6$; q^1 for S^7; q^4 for S^8; and q^3 for S^9. The position of the five masses along the continued families is plotted in Figure 6. In Figure 7 we plot the values of a and b as functions of m along the continued families and in Figure 8 we do the same with the values x_5 and y_5. Notice that since the mass m_5 is zero, the position of the masses m_3 and m_4 as a function of m is the same for all the continued families.

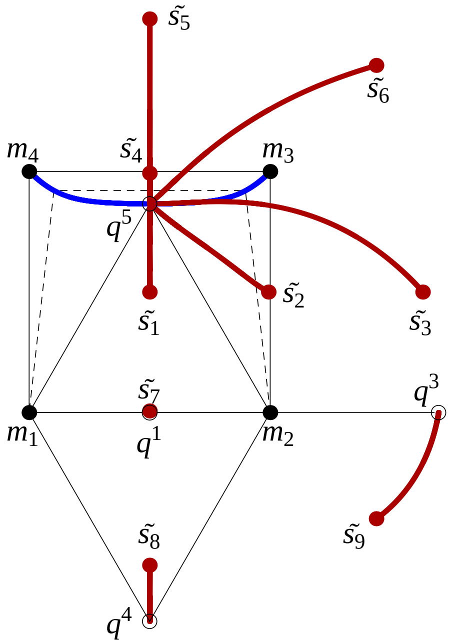


FIG. 6. Continuation from the restricted square $(4+1)$-body problem to the restricted $(2+3)$-body problem through the family of the restricted isosceles trapezoidal $(4+1)$-body problem.

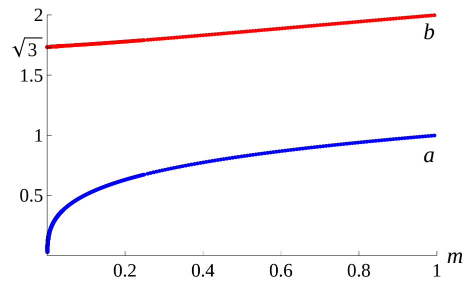


FIG. 7. Evolution of the values a and b along the family of central configurations that comes from the continuation from the restricted square $(4+1)$-body problem to the restricted $(2+3)$-body problem through the family of the restricted isosceles trapezoidal $(4+1)$-body problem.

V. RESTRICTED KITE $(4+1)$-BODY PROBLEM CENTRAL CONFIGURATIONS

We consider the *restricted kite* $(4+1)$*-body problem* having m_1, m_3 and $m_2 = m_4$ located at the vertices of a kite with coordinates $(x_1,y_1) = (-1,0)$, $(x_2,y_2) = (a,-b)$, $(x_3,y_3) = (1,0)$ and $(x_4,y_4) = (a,b)$ with $b > 0$. Notice that here we have taken the unit of length so that the distance between m_1 and m_3 be 2 which is not restrictive.

Substituting these values into equations (2) with $m_5 = 0$ we get $e_6 = 0$, $e_3 = e_1$ and $e_7 = -e_5$. From $e_1 = 0$ we obtain

$$\lambda = \frac{2m_2+m_1}{((a+1)^2+b^2)^{3/2}} + \frac{m_3}{a+1}\left(\frac{a-1}{((a-1)^2+b^2)^{3/2}} + \frac{1}{4}\right).$$

Substituting λ in the remaining equations we get that

$$\begin{aligned}
e_2 &= \frac{m_1}{4} + \frac{(a-1)m_3}{4(a+1)} - \frac{2(m_1-(a-1)m_2)}{((a+1)^2+b^2)^{3/2}} - \frac{2(a-1)((a+1)m_2+m_3)}{(a+1)\left((a-1)^2+b^2\right)^{3/2}} = 0,\\
e_4 &= \frac{m_3(a-x_5)}{4(a+1)} + \frac{2m_2(a-x_5)-m_1(x_5+1)}{((a+1)^2+b^2)^{3/2}} - \frac{(a-1)m_3(x_5+1)}{(a+1)\left((a-1)^2+b^2\right)^{3/2}} + \frac{m_1(x_5+1)}{((x_5+1)^2+y_5^2)^{3/2}} + \frac{m_3(x_5-1)}{((x_5-1)^2+y_5^2)^{3/2}}\\
&\quad + \frac{m_2(x_5-a)}{((a-x_5)^2+(b-y_5)^2)^{3/2}} + \frac{m_2(x_5-a)}{((a-x_5)^2+(b+y_5)^2)^{3/2}} = 0,\\
e_5 &= -\frac{m_2}{4b^2} + \frac{bm_3}{4(a+1)} + \frac{2bm_2}{((a+1)^2+b^2)^{3/2}} - \frac{2bm_3}{(a+1)\left((a-1)^2+b^2\right)^{3/2}} = 0,\\
e_8 &= -\frac{m_3y_5}{4(a+1)} - \frac{y_5(m_1+2m_2)}{((a+1)^2+b^2)^{3/2}} - \frac{(a-1)m_3y_5}{(a+1)\left((a-1)^2+b^2\right)^{3/2}} + \frac{m_1y_5}{((x_5+1)^2+y_5^2)^{3/2}} + \frac{m_3y_5}{((x_5-1)^2+y_5^2)^{3/2}}\\
&\quad + \frac{m_2(y_5-b)}{((a-x_5)^2+(b-y_5)^2)^{3/2}} + \frac{m_2(b+y_5)}{((a-x_5)^2+(b+y_5)^2)^{3/2}} = 0.
\end{aligned} \tag{7}$$

Notice that when $m_1 = m_2 = m_3 = m_4 = 1$, $a = 0$ and $b = 1$ the restricted kite $(4+1)$-body problem given by equations (7) becomes the restricted square $(4+1)$ body problem given in Subsection III with the positions of the masses scaled by a factor $1/\sqrt{2}$ and rotated clockwise by an angle $\pi/4$. Let $\tilde{S}^k$ for $k = 1,\ldots,13$ be the central configuration of the square *restricted* $(4+1)$*-body problem* with the primaries located at $(x_1,y_1) = (-1,0)$, $(x_2,y_2) = (0,-1)$, $(x_3,y_3) = (1,0)$,

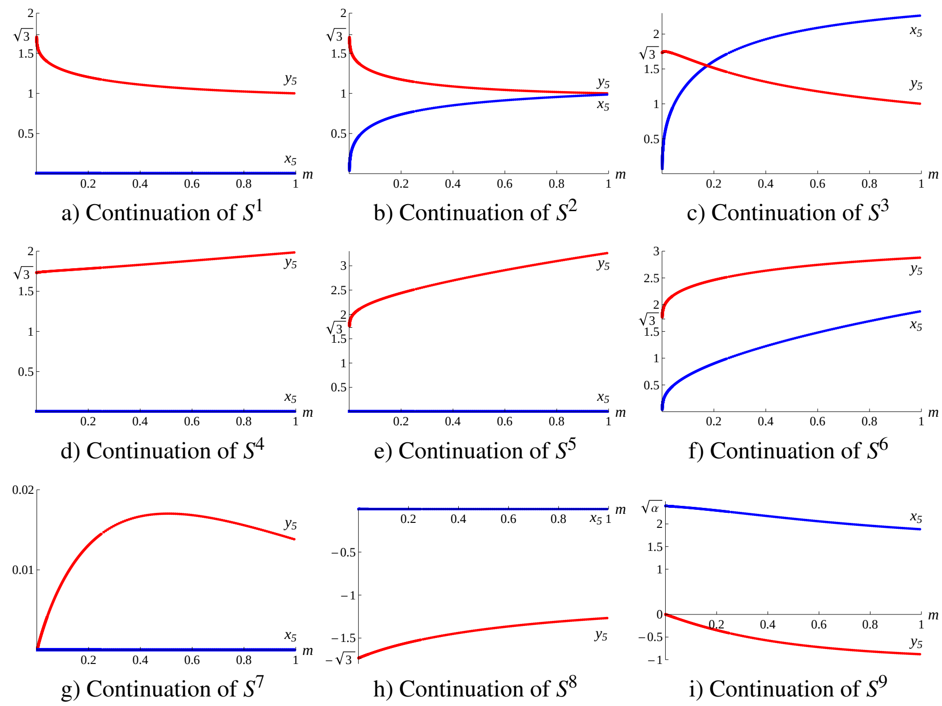


FIG. 8. Evolution of the values x_5 (in blue) and y_5 (in red) along the family of central configurations that comes from the continuation from the restricted square $(4+1)$-body problem to the restricted $(2+3)$-body problem through the family of the restricted isosceles trapezoidal $(4+1)$-body problem.

$(x_4,y_4)=(0,1)$, and the infinitesimal mass $m_5=0$ located at $\tilde{s}_k=(x_k/2+y_k/2,-x_k/2+y_k/2)$, where $s^k=(x_k,y_k)$ are the coordinates given in (3). Using the symmetry of the configurations we need only to consider the central configurations with $y_5\leq 0$; that is, $\tilde{S}^k$ for $k=1,2,3,6,7,8,9,12$. The others can be obtained by symmetry. The positions $\tilde{s}_k$ for $k=1,2,3,6,7,8,9,12$ are given by

$$
\begin{aligned}
\tilde{s}_1 &= (0,0),\\
\tilde{s}_2 &= (0.493122487600..,-0.493122487600..),\\
\tilde{s}_3 &= (1.133073906831..,-1.133073906831..),\\
\tilde{s}_6 &= (1.880455410280..,0),\\
\tilde{s}_7 &= (-0.493122487600..,-0.493122487600..)\\
\tilde{s}_8 &= (-1.133073906831..,-1.133073906831..),\\
\tilde{s}_9 &= (0,-1.880455410280..)\\
\tilde{s}_{12} &= (-1.880455410280..,0).
\end{aligned}
$$

We want to continue numerically the families of the restricted square $(4+1)$-body problem to the restricted kite $(4+1)$-body problem with either $m_1=m_3=1$ and $m_2=m_4=m$ (two pairs of equal masses), or $m_1=m$ and $m_2=m_3=m_4=1$ (three big equal masses and one smaller mass), or $m_1=1$ and $m_2=m_3=m_4=m$ (three small equal masses and one bigger mass) and where the parameter m varies from 1 to 0.

A. Kite central configurations with two pairs of equal masses

We study the central configurations of the restricted kite $(4+1)$-body problem having $m_1=m_3=1$, $m_2=m_4=m$ located at the vertices of a kite with coordinates $(x_1,y_1)=(-1,0)$, $(x_2,y_2)=(0,-b)$, $(x_3,y_3)=(1,0)$ and $(x_4,y_4)=(0,b)$ with $b>0$.

Substituting $m_1=m_3=1$, $m_2=m_4=m$ and $a=0$ into (7) we get that $e_2=0$ and

$$
\begin{aligned}
e_4 &= -\frac{x_5}{4}-\frac{2mx_5}{(b^2+1)^{3/2}}+\frac{x_5-1}{\left((x_5-1)^2+y_5^2\right)^{3/2}}+\frac{x_5+1}{\left((x_5+1)^2+y_5^2\right)^{3/2}}+\frac{mx_5}{\left((b-y_5)^2+x_5^2\right)^{3/2}}+\frac{mx_5}{\left((b+y_5)^2+x_5^2\right)^{3/2}},\\
e_5 &= \frac{2b(m-1)}{(b^2+1)^{3/2}}+\frac{b^3-m}{4b^2},\\
e_8 &= -\frac{y_5}{4}-\frac{2my_5}{(b^2+1)^{3/2}}+\frac{y_5}{\left((x_5-1)^2+y_5^2\right)^{3/2}}+\frac{y_5}{\left((x_5+1)^2+y_5^2\right)^{3/2}}+\frac{m(y_5-b)}{\left((b-y_5)^2+x_5^2\right)^{3/2}}+\frac{m(b+y_5)}{\left((b+y_5)^2+x_5^2\right)^{3/2}}.
\end{aligned}
$$

Now we continue numerically the central configurations of the restricted square $(4+1)$-body problem to the restricted kite $(4+1)$-body problem with the mass m playing the role of a parameter varying from $m=1$ to $m=0$. Note that the equations of the central configurations of the restricted kite $(4+1)$-body problem are the three equations $e_4=e_5=e_8=0$ with the three unknowns b, x_5, y_5. So we shall find numerically the solutions of that set of three equations as m varies from 1 to 0. Using the symmetry of the configurations we need only to continue the central configurations with $x_5 \geq 0$ and $y_5 \leq 0$; that is, $\tilde{S}^k$ for $k=1,2,3,6,9$. The others can be obtained by symmetry. Let J be as in (6) without the row related with the equation e_2 and the column related to the variable a, and taking e_5 instead of e_6. For each class $\tilde{S}^k$ we provide the values of the determinant of J, that we denote by $|J^k|$ for $k=1,2,3,6,9$ and we get

$$
\begin{aligned}
|J^1| &= -6.558360386504..,\\
|J^2| &= 35.988878543651..,\\
|J^3| &= -1.859830027973..,\\
|J^6| &= |J^9| = 1.975327597366..
\end{aligned}
$$

Since all these determinants are different from zero, from the Implicit Function Theorem, the central configuration $\tilde{S}_k$ for $k=1,2,3,6,9$ can be continued to a family of central configurations with values of m sufficiently close to 1.

Now we make the continuation from the *restricted square* $(4+1)$*-body problem* to the *restricted* $(2+3)$*-body problem* with masses $m_1=m_3=1$ at $(-1,0)$ and $(1,0)$ and three infinitesimal masses $m_2=m_4=m_5=0$ through the family of the restricted kite $(4+1)$-body problem with two pairs of equal masses. In order to continue the central configurations we have used the same methodology as for the isosceles trapezoid.

The central configurations $\tilde{S}^k$ for $k \in \{3,6,9\}$ can be continued to a family of central configurations of the restricted kite $(4+1)$-body problem for $m \in [1,0)$ (the determinant is never zero along the families) that tends to the configurations of the restricted $(2+3)$-body problem with the three infinitesimal masses located at: q^4 for $\tilde{S}^3$ and $\tilde{S}^9$; and q^3 for $\tilde{S}^6$.

Let $m^1 = 0.673935906579..$ and $m^2 = 0.186739432174...$ In the continuation of $\tilde{S}^1$ and $\tilde{S}^2$ appear two bifurcation values m^1 and m^2. The central configuration $\tilde{S}^1$ can be continued to a family of central configurations with $x_5=y_5=0$ for $m \in [1,m^1)$. At $m=m^1$, $b=1.239315257944..$ and $x_5=y_5=0$ there is a subcritical pitchfork bifurcation so that when m decreases three families of central configurations bifurcate from this family, all remain on $x_5=0$. More precisely, from the three bifurcated families one remains at $x_5=y_5=0$ for $m \in (m^1,0)$, and the other two, which are symmetric with respect to the y_5-axis, can be continued from m^1 to m^2. We continue the bifurcated family with $y_5 \leq 0$ to m^2. At $m=m^2$, $b=1.61028690479..$, $x_5=0$ and $y_5=0.967261480331..$ there is a supercritical pitchfork bifurcation so that when m decreases three families of central configurations coalesce in one. More precisely, the family coming from the bifurcation of $\tilde{S}^1$ and the two families, symmetric with respect to the x_5-axis, coming from $\tilde{S}^2$ and $\tilde{S}^7$ coalesce into a unique family that can be continued for $m \in (m^2,0)$. This family remain on

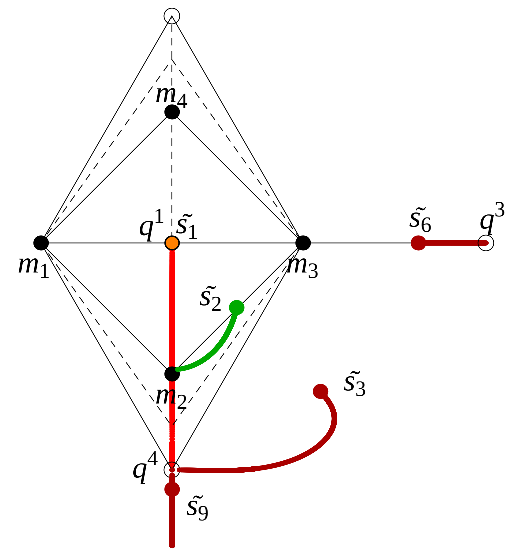


FIG. 9. Continuation from the restricted square $(4+1)$-body problem to the restricted $(2+3)$-body problem through the family of the restricted kite $(4+1)$-body problem with two pairs of equal masses.

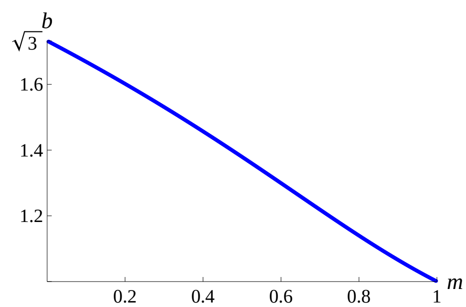


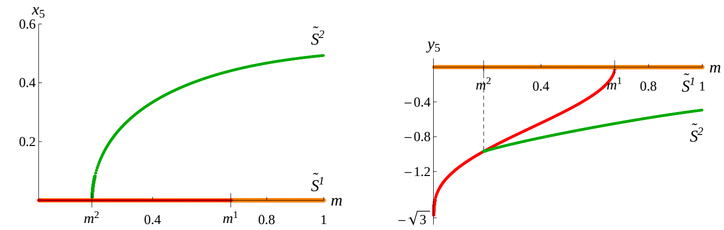
FIG. 10. Evolution of b along the family of central configurations that comes from the continuation from the restricted square $(4+1)$-body problem to the restricted $(2+3)$-body problem through the family of the restricted kite $(4+1)$-body problem with two pairs of equal masses.

$x_5=0$ and tends to the central configuration of the restricted $(2+3)$-body problem with the three infinitesimal masses located at q^4.

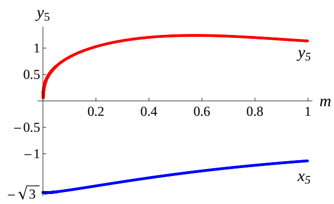
The position of the five masses along the continued families is plotted in Figure 9. In Figure 10 we plot b as function of m along the continued families and in Figure 11 we plot x_5 and y_5.

B. Restricted kite central configurations of the $(4+1)$-body problem with three big equal masses

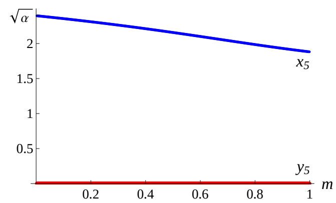
Now we consider the central configurations of the restricted kite $(4+1)$-body problem having $m_1=m$ and $m_2=m_3=$



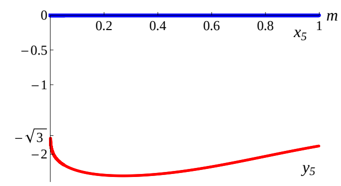
a) Continuation of $\tilde{S}^1$ and $\tilde{S}^2$



b) Continuation of $\tilde{S}^3$



c) Continuation of $\tilde{S}^6$



d) Continuation of $\tilde{S}^9$

FIG. 11. Evolution of the values x_5 and y_5 along the family of central configurations that comes from the continuation from the restricted square $(4+1)$-body problem to the restricted $(2+3)$-body problem through the family of the restricted kite $(4+1)$-body problem with two pairs of equal masses.

$m_4 = 1$ located at the vertices of a kite with coordinates $(x_1,y_1) = (-1,0)$, $(x_2,y_2) = (a,-b)$, $(x_3,y_3) = (1,0)$ and $(x_4,y_4) = (a,b)$ with $b > 0$. The equations of these central configurations are four equations $e_2 = e_4 = e_5 = e_8 = 0$ with e_i given by (7) and taking $m_1 = m$ and $m_2 = m_3 = m_4 = 1$.

We want to continue numerically the central configurations $\tilde{S}^k$ of the restricted square $(4+1)$-body problem through the family of the restricted kite $(4+1)$-body problem with three equal masses as the parameter m varies from 1 to 0. We shall find numerically the solutions of the set of four equations $e_2 = e_4 = e_6 = e_8 = 0$ with the four unknowns a,b,x_5,y_5 as m varies from 1 to 0. Let J be as in (6) by taking e_5 instead of e_6. For each class $\tilde{S}^k$ with $k = 1,2,3,6,7,8,9,12$ we provide the values of the determinant of J, that we denote by $|J^k|$ for $k = 1,2,3,6,7,8,9,12$ (again, the other cases can be obtained by symmetry) and we get

$$
\begin{aligned}
&|J^1| = -21.829024604302..,\\
&|J^2| = |J^7| = 119.786359533887..,\\
&|J^3| = |J^8| = -6.190308712523..\\
&|J^6| = |J^9| = |J^{12}| = 6.574733955334..
\end{aligned}
$$

Now we do the continuation from the restricted square $(4+1)$-body problem with $m_1 = m_2 = m_3 = m_4 = 1$ located at $(x_1,y_1) = (-1,0)$, $(x_2,y_2) = (0,-1)$, $(x_3,y_3) = (1,0)$ and $(x_4,y_4) = (0,1)$ to the restricted equilateral triangular $(3+2)$-body problem with one infinitesimal mass $m_1 = 0$ located at $(-1,0)$ and three masses $m_2 = m_3 = m_4 = 1$ at the vertices of an equilateral triangle with $(x_3,y_3) = (1,0)$ through the family of the restricted kite $(4+1)$-body problem with three equal masses. Since all the determinants $|J^k|$ are different from zero, all the configurations $\tilde{S}^k$ can be continue to a family of central configurations of the restricted kite $(4+1)$-body problem with m close to one. We continue these families of central configurations by using the same methodology as in the previous cases.

The central configurations $\tilde{S}^k$, for $k = 2,3,6,8,9,12$ can be continued to a family of central configurations of the restricted kite $(4+1)$-body problem for $m \in [1,0)$ (the determinant along the family is never zero) that tends to the configuration of the restricted equilateral triangular $(3+2)$-body problem with the three equal masses at the vertices of the equilateral triangle T and the infinitesimal mass m_1 located at p^1. Moreover the infinitesimal mass m_5 can be located at: p^1 for $\tilde{S}^8$ and $\tilde{S}^{12}$; p^4 for $\tilde{S}^6$; p^1 rotated by an angle $2\pi/3$ (i.e. p^{1*}) for $\tilde{S}^3$; p^2 rotated by an angle $2\pi/3$ (i.e. p^{2*}) for $\tilde{S}^2$; and p^4 rotated by an angle $4\pi/3$ (i.e. p^{4**}) for $\tilde{S}^9$.

Let $m^3 = 0.0438759786648...$ The central configurations $\tilde{S}^1$ and $\tilde{S}^7$ can be continued for $m \in [1,m^3)$. At $m = m^3$ with $a = 0.138286458360..$, $b = 0.674663595536$, $y_5 = 0$ and $x_5 = 0.749916350749$ there is a supercritical pitchfork bifurcation so that when m decreases three families of central configurations (the family comming from the continuation of $\tilde{S}^1$ and the two symmetric families with respect to the y_5-axis comming from the continuation of $\tilde{S}^7$ and $\tilde{S}^{10}$) coalesce in one. This unique family of central configurations remains at $y_5 = 0$ for $m \in [m^3,0)$, and tends to the configuration of the restricted equilateral triangular $(3+2)$-body problem with three equal masses at the vertices of the equilateral triangle T and the two infinitesimal masses m_1 and m_5 located at p^1.

The position of the five masses along the continued families is plotted in Figure 12. In Figure 13 we plot b as function of m along the continued families and in Figure 14 we plot x_5 and y_5.

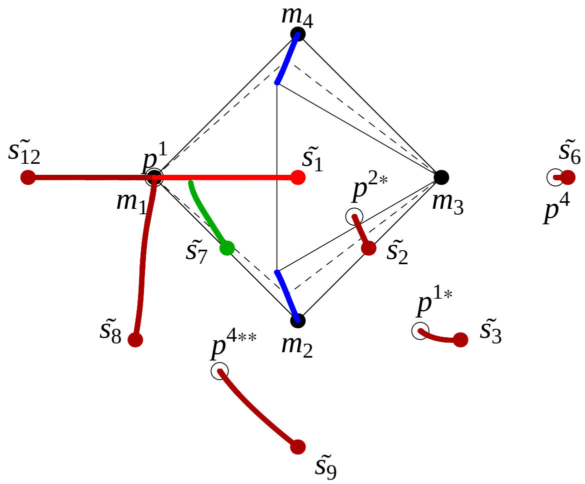


FIG. 12. Continuation from the restricted square $(4+1)$-body problem to the restricted equilateral triangular $(3+2)$-body problem through the family of the restricted kite $(4+1)$-body problem with three big equal masses.

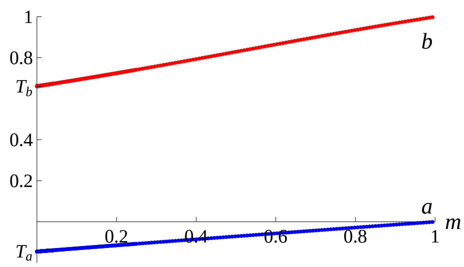


FIG. 13. Evolution of a and b along the family of central configurations that comes from the continuation from the restricted square $(4+1)$-body problem to the restricted equilateral triangular $(3+2)$-body problem through the family of the restricted kite $(4+1)$-body problem with three big equal masses.

C. Restricted kite central configurations of the $(4+1)$-body problem with three small equal masses

Now we consider the central configurations of the restricted kite $(4+1)$-body problem having $m_1 = 1$ and $m_2 = m_3 = m_4 = m$ located at the vertices of a kite with coordinates $(x_1, y_1) = (-1, 0)$, $(x_2, y_2) = (a, -b)$, $(x_3, y_3) = (1, 0)$ and $(x_4, y_4) = (a, b)$ with $b > 0$. The equations of these central configurations are $e_2 = e_4 = e_5 = e_8 = 0$ with e_i given by (7) and taking $m_1 = 1$ and $m_2 = m_3 = m_4 = m$.

We want to continue numerically the central configurations $\tilde{S}^k$ of the restricted square $(4+1)$-body problem to the restricted $(1+3)+1$-body problem with $m_1 = 1$ located at $(-1, 0)$ and three infinitesimal masses $m_2 = m_3 = m_4 = 0$ at a central configuration of the $(1+3)$-body problem with $(x_3, y_3) = (1, 0)$ through the family of the restricted kite $(4+1)$-body problem with three equal masses as the parameter m varies from 1 to 0. As in Subsection V B, all the configurations $\tilde{S}^k$ can be continue to a family of central configurations of the restricted kite $(4+1)$-body problem with m close to one. We continue these families of central configurations proceeding as in the previous cases.

The central configurations $\tilde{S}^k$ with $k = 3, 6, 7, 8, 9, 12$ can be continued to a family of central configurations of the restricted kite $(4+1)$-body problem with three small equal masses for $m \in [1, 0)$ (the determinant along the families is never zero) that tends to the configuration of the restricted $((1+3)+1)$-body problem with $m_1 = 1$ located at $(-1, 0)$, $m_2 = m_3 = m_4 = 0$ located at τ^2, $(1, 0)$ and τ^4 respectively. Moreover, the infinitesimal mass m_5 is located at: t^2 for $\tilde{S}^3$; at $(1, 0)$ for $\tilde{S}^6$; at τ^2 for $\tilde{S}^7$ and $\tilde{S}^9$; at t^1 for $\tilde{S}^8$; and at t^5 for $\tilde{S}^{12}$.

In the continuation of the families $\tilde{S}^1$ and $\tilde{S}^2$ we get two bifurcation values one at $m = m^4 = 0.461860745797..$ with $a = 0.127510274911..$, $b = 1.228892081922..$, $x_5 = 0.147603375472..$ and $y_5 = 0$, and another one at $m = m^5 = 0.255403199450..$ with $a = 0.217767653987..$, $b = 1.342479413641..$, $x_5 = 0.436776292663..$ and $y_5 = -0.550311565370...$

The central configuration $\tilde{S}^2$ can be continued to a family of central configurations of the restricted kite $(4+1)$-body problem with three small equal masses for $m \in [1, m^5)$. At $m = m^5$ this family passes through a simple fold bifurcation and it can still be continued going back until $m = m^4$. The central configuration $\tilde{S}^1$ also can be continued for $m \in [1, m^4)$. At $m = m^4$ there is a subcritical pitchfork bifurcation that when decreasing m three solutions bifurcate, the one coming from the continuation of $\tilde{S}^1$ (with always $y_5 = 0$) and the two new symmetric families with respect to the x_5-axis which are the families that come from the continuation of $\tilde{S}^2$ and its symmetric configuration $\tilde{S}^4$. The family coming from the continuation of $\tilde{S}^1$, which remains at $y_5 = 0$, is also defined for $m \in [m^4, 0)$ and tends to the configuration of the restricted circle $((1+3)+1)$-body problem with m_2 located at τ^2, m_4 located at τ^4 and m_5 located at $(1, 0)$ (m_5 coalesces with m_3).

The position of the five masses along the continued families is plotted in Figure 15. In Figure 16 we plot b as function of m along the continued families and in Figure 17 we plot x_5 and y_5.

AUTHOR DECLARATIONS

The authors have no conflicts to disclose.

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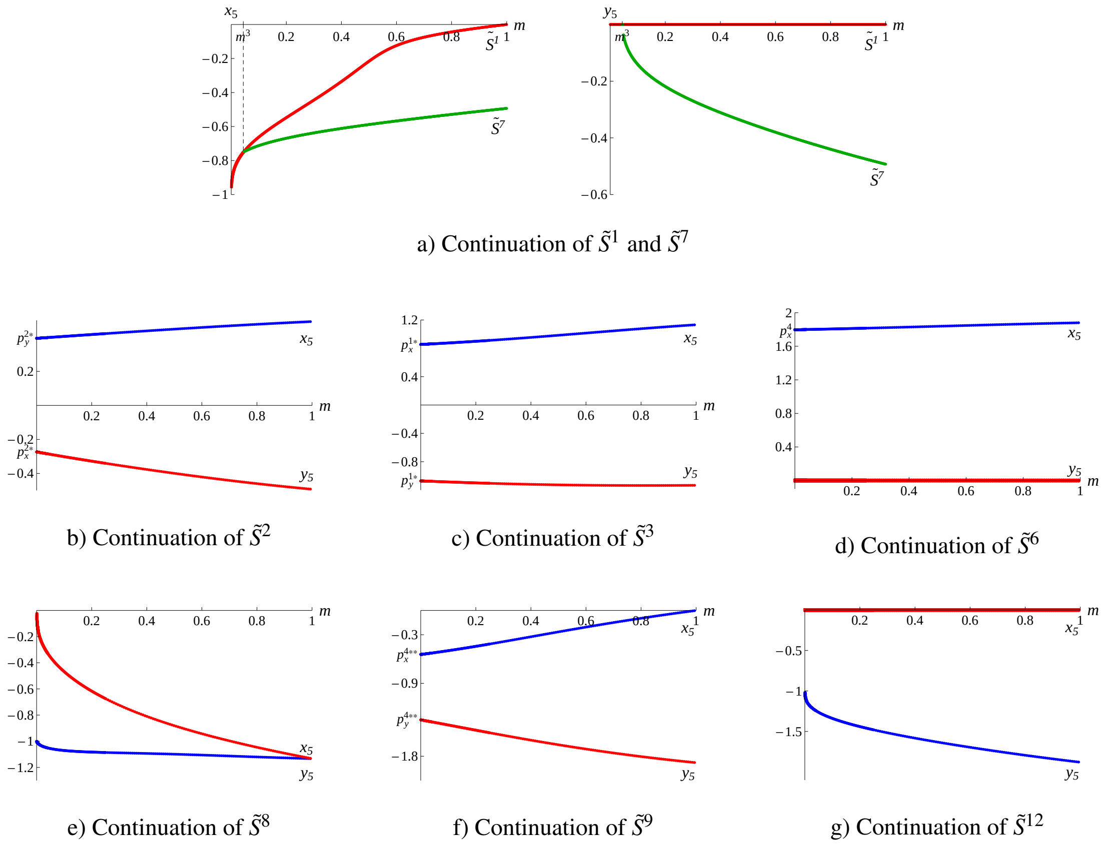

FIG. 14. Evolution of the values x_5 and y_5 along the family of central configurations that comes from the continuation from the restricted square $(4+1)$-body problem to the restricted equilateral triangular $(3+2)$-body problem through the family of the restricted kite $(4+1)$-body problem with three big equal masses.

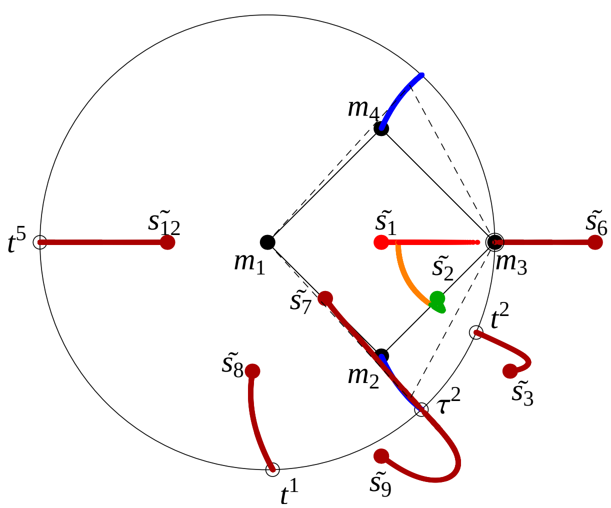

FIG. 15. Continuation from the restricted square $(4+1)$-body problem to the restricted $((1+3)+1)$-body problem through the family of the restricted kite $(4+1)$-body problem with three small equal masses.

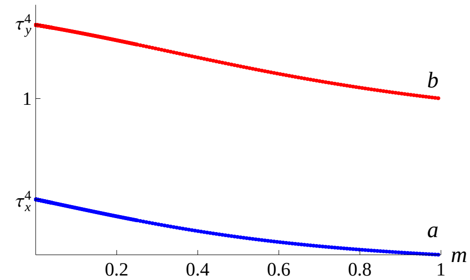

FIG. 16. Evolution of a and b along the family of central configurations that comes from the continuation from the restricted square $(4+1)$-body problem to the restricted $((1+3)+1)$-body problem through the family of the restricted kite $(4+1)$-body problem with three small equal masses.

DATA AVAILABILITY STATEMENT

The data that supports the findings of this study are available within the article.

[1] R.F. ARENSTORF, *Central configurations of four bodies with one inferior mass*, Celest. Mech. Dyn. Astr. **28** (1982), 9–15.

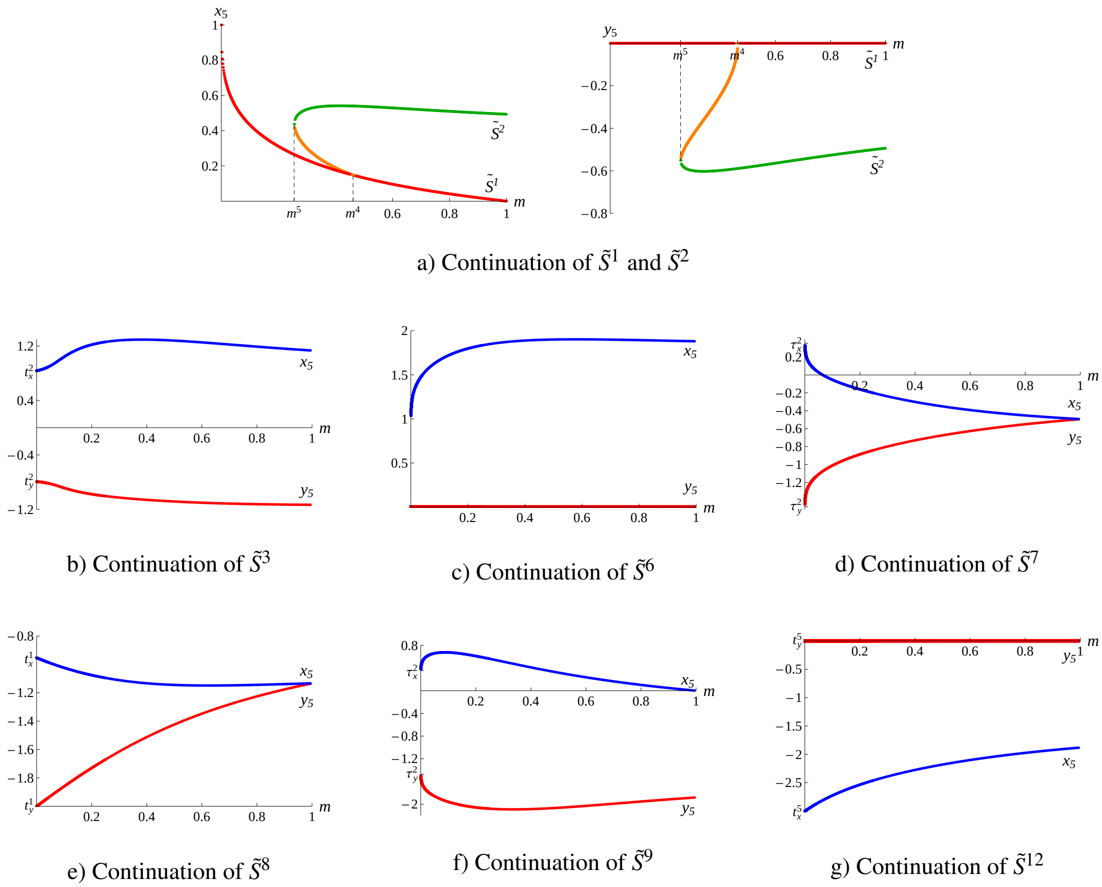

FIG. 17. Evolution of the values x_5 and y_5 along the family of central configurations that comes from the continuation from the restricted square $(4+1)$-body problem to the restricted $((1+3)+1)$-body problem through the family of the restricted kite $(4+1)$-body problem with three small equal masses.

[2]D. Bang and B. Elmabsout, *Representations of complex functions, means on the regular n-gon and applications to gravitational potential*, J.Phys.Gen. **36** (2003), 11435–11450.

[3]J. Casasayas, J. Llibre and A. Nunes, *Central configurations of the planar* $1+N$*-body problem*, Celest. Mech. Dyn. Astr. **60** (1994), 273–288.

[4]A. Chenciner, *Collisions totales, mouvements complètement paraboliques et réduction des homothéthies dans le problème des n corps*, Regular Chaotic Dynam. **3** (1998), 93–106.

[5]A.C. Fernandes, B.A. Garcia, J. Llibre and L.F. Mello, *New central configurations of the* $(n+1)$*-body problem*, J. of Geometry and Physics **124** (2018), 199–207.

[6]G. Gómez, J. Llibre, R. Martínez and C. Simó, *Dynamics and Mission Design Near Libration Points. Vol. I Fundamentals: The case of collinear libration points*, World Scientific Monograph Series in Mathematics, Vol. **2**, World Scientific, Singapur, 2001.

[7]G. Gómez, J. Llibre, R. Martínez and C. Simó, *Dynamics and Mission Design Near Libration Points. Vol. II Fundamentals: The case of triangular libration points*, World Scientific Monograph Series in Mathematics, Vol. **3**, World Scientific, Singapur, 2001.

[8]J. Llibre, *Posiciones de equilibrio relative del problema de* 4 *cuerpos*, Pub. Sec. Mat. U.A.B, 73–86, 1977.

[9]K.R. Meyer, G.R. Hall and D. Offin, *Introduction to Hamiltonian dynamical systems and the N-body problem*, second edition, Applied Mathematical Sciences, Vol. **90**, Springer, New York, 2009.

[10]D. Saari and N.D. Hulkower, *On the manifolds of total collapse orbits and of completely parabolic orbits for the n-body problem*, J. Diff. Equations **41** (1981), 27–43.

[11]A.A. Santos and C. Vidal, *Symmetry of the restricted* $(4+1)$*-body problem with equal masses*, Regular Chaotic Dynam. **12** (2007), 27–38.

[12]C. Simó, *Dynamical systems, numerical experiments and supercomputing*, Mem. Real Acad. Cienc. Artes Barcelona 61 **1** (2003), 336.

[13]S. Smale, *Topology and mechanics. I*, Invent. Math. **10** (1970), 305–331.

[14]S. Smale, *Topology and mechanics. II. The planar n-body problem*, Invent. Math. **11** (1970), 45–64.

[15]V.G. Szebehely, *Theory of orbits, the restricted problem of three bodies*, Academic Press, 1967.

[16]A. Wintner, *The Analytic Foundations of Celestial Mechanics*, Princeton University Press, Princeton, N.J., 1944.