Asymptotic stability for block triangular maps *

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Abstract

We prove a result concerning the asymptotic stability and the basin of attraction of fixed points for block triangular maps in $\mathbb{R}^n$. This result is applied to some families of discrete dynamical systems and several types of difference equations.

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1 Introduction and main results

In this work we consider block triangular discrete dynamical systems (DDS) of the form

$$\left\{\begin{array}{rcl} x_{n+1} & = & f(x_n, u_n), \\ u_{n+1} & = & g(u_n), \end{array}\right. \tag{1}$$

where $x \in \mathbb{R}^m$, $u \in \mathbb{R}^k$ and $f : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m$, $g : \mathbb{R}^k \to \mathbb{R}^k$, and m and k are positive integers. We assume that there exists a locally attracting fiber $\{u = u_*\}$, which is invariant and with a unique fixed point on it. The existence of this locally attracting fiber is equivalent to the existence of a locally asymptotically stable fixed point of the subsystem $u_{n+1} = g(u_n)$. We will assume that on this limit fiber the dynamics given by the map $x \to f(x, u_*)$ has a globally asymptotically stable (GAS) fixed point. The problem considered here is to give conditions on the map f under which this fact forces the same behavior for all initial conditions in the whole basin of attraction of this fiber, that is (x_*, u_*) is also GAS for the DDS (1) on this basin. Next theorem is our main result. In this work $||y||$ denotes any vector norm of $y \in \mathbb{R}^\ell$.

Theorem A. *Consider the DDS* (1) *with f and g continuous and such that:*

(a) The map f is sublinear in x, that is, there exist continuous functions $M, N : \mathbb{R}^k \to \mathbb{R}^+ \cup \{0\}$ such that

$$||f(x, u)|| \leq M(u) + N(u)||x||. \tag{2}$$

(b) The point $u = u_*$ is a stable attractor for the DDS $u_{n+1} = g(u_n)$.

(c) $N_* := N(u_*) < 1$.

(d) The function $x \to f(x, u_*)$ is contractive. That is, there exists a positive real number $L < 1$ such that

$$||f(x, u_*) - f(y, u_*)|| \leq L\, ||x - y|| \text{ for all } x, y \in \mathbb{R}^m. \tag{3}$$

In particular, the map $f(\cdot, u_*)$ has a unique fixed point $x = x_*$, i.e. $f(x_*, u_*) = x_*$.

Then (x_*, u_*) is a stable attractor and any initial condition (x_0, u_0), with u_0 in the basin of attraction of $u = u_*$, is in the basin of attraction of the fixed point (x_*, u_*). Hence on this set, (x_*, u_*) is GAS.

We stress that the above result implies that the convergence is guaranteed for all (x_0, u_0) such that u_0 is in the basin of attraction of $u = u_*$, hence it is not a local result.

It is worth to remark that it is common knowledge that the sublinearity condition given in (2) of the above hypothesis (a) is necessary to have GAS type results. In Examples A, B of

Section 2 we give two DDS illustrating this fact. The first one evidences that the result can fail if, in (2), $||x||$ is replaced by $||x||^{1+\delta}$ for any $\delta > 0$.

The hypotheses (b) and (c) are natural if one wants to prove that some set of initial conditions tends to the fixed point (u_*, x_*). The hypothesis (c) and the condition (3) in (d), which is a contractivity condition, are related with the attractivity of $x = x_*$ on the invariant fiber $u = u_*$.

In the particular case that the function f in (1) is linear in x, the hypotheses in Theorem A can be simplified (for instance, observe that in this case, condition (2) implies (3)). We state this particular case in next corollary, which extends the results of [8, Prop. 2] to higher dimensions. Notice that in its statement, given a $m \times m$ matrix M and a vector norm in $\mathbb{R}^m$, $\|M\|$ is the matrix norm induced by this vector norm, which recall that it is

$$\|M\| = \max\{\|Mx\| : x \in \mathbb{R}^m \text{ with } \|x\| = 1\} = \max\left\{\frac{\|Mx\|}{\|x\|} : x \in \mathbb{R}^m \text{ with } x \neq 0\right\}.$$

Corollary 1. *Consider the DDS*

$$\left\{\begin{array}{rcl} x_{n+1} & = & f_0(u_n) + f_1(u_n)x_n, \\ u_{n+1} & = & g(u_n), \end{array}\right. \tag{4}$$

where $f_0 : \mathbb{R}^k \to \mathbb{R}^m$, $g : \mathbb{R}^k \to \mathbb{R}^k$, $f_1(u)$ is a $m \times m$ matrix, all the involved functions continuous and such that:

(i) The point $u = u_*$ is a stable attractor for the DDS $u_{n+1} = g(u_n)$ in $\mathcal{U}$.

(ii) The matrix $f_1(u_*)$ satisfies $||f_1(u_*)|| < 1$ for some matrix norm induced by a vector norm.

Then, any initial condition (x_0, u_0) such that u_0 is in the basin of attraction of $u = u_*$, is in the basin of attraction of the fixed point (x_*, u_*).

In Section 2 we prove Theorem A and Corollary 1. In Section 3 we present some examples of application to several DDS of these mains results. Finally, in Section 4 we apply them to different types of difference equations. In fact, to give more complete results in that section we complement the results of Theorem A and Corollary 1 with some related results, given in our previous work [8], which cover some cases with other simple dynamics on the attracting fiber when $n = 2$. More concretely, the new dynamics considered are when the fiber is full of fixed points, or when it is full of 2-periodic points, see Proposition 12 for more details.

2 Proof of the main results

Before proving Theorem A and Corollary 1 we present two examples which show that the sublinearity hypothesis (2) in Theorem A is essential.

Example A. Consider the planar DDS

$$\left\{\begin{array}{lcl} x_{n+1} & = & x_n/2+3\,|u_n|^{\delta}\,|x_n|^{1+\delta}/2, \\ u_{n+1} & = & u_n/2, \end{array}\right. \tag{5}$$

for $\delta \geq 0$. It satisfies all the hypothesis of Theorem A except those related to the sublinearity condition, that is (a) and (c). For it we have:

Lemma 2. *For any $\delta > 0$, the fiber $\{u = 0\}$ for the DDS (5) is globally attracting. On this fiber, $x_* = 0$ is a GAS fixed point, but the orbit with initial condition $(x_0, u_0) = (1, 1)$ is unbounded.*

Proof. It is easy to see that this orbit for $n \geq 0$ is $(x_n, u_n) = (2^n, 2^{-n})$, so it is unbounded. ∎

Notice that for $\delta = 0$ the orbit starting at $(1, 1)$ is the same, and in this case, the DDS is sublinear and satisfies (2), but Theorem A does not apply because $N_* = 2 > 1$.

The DDS (5) is a discrete counterpart of a similar example for ordinary differential equations,

$$\left\{\begin{array}{lcl} x' & = & -x+ux^2, \\ u' & = & -u, \end{array}\right.$$

which has similar qualitative properties and for the initial conditions $(x, u) = (x_0, u_0)$ with $x_0 u_0 = 2$ has the solution $(x(t), u(t)) = (x_0 \exp(t), u_0 \exp(-t))$, see [12, p. 8].

Example B. Consider the DDS

$$\left\{\begin{array}{lcl} x_{n+1} & = & x_n/2+y_n\,z_n\,u_n^5, \\ y_{n+1} & = & y_n/2+105\,x_n\,z_n\,u_n^3, \\ z_{n+1} & = & z_n/2+217\,x_n\,y_n\,u_n \\ u_{n+1} & = & u_n/2, \end{array}\right. \tag{6}$$

which can be written in the form (1) with $\mathbf{x} = (x, y, z)$, $f(\mathbf{x}, u) = (x/2+yzu^5, y/2+105xzu^3, z/2+217xyu)$ and $g(u) = u/2$. Notice that the map f has quadratic terms in $||\mathbf{x}||$.

Proposition 3. *For the DDS (6), the fiber $\{u = 0\}$ is globally attracting. On this fiber, $\mathbf{x}_* = 0$ is a GAS fixed point. However, there exist initial conditions giving rise to unbounded orbits.*

Proof. Observe that, trivially, $u_* = 0$ is a GAS fixed point of the subsystem $u_{n+1} = g(u_n)$, and also that $\mathbf{x}_* = 0$ a GAS fixed point of $\mathbf{x}_{n+1} = f(\mathbf{x}_*, 0)$. Let us find some initial conditions of unbounded orbits. The method of construction follows the ideas given in [6] to give counterexamples of the discrete Markus Yamabe conjecture. First we observe that the components of the family of maps

$$H(x, y, z, u) = \left(x/2+ayzu^5, y/2+bxzu^3, z/2+cxyu, u/2\right),$$

where a, b and c are real parameters to be determined, are quasi-homogeneous polynomials with degree of homogeneity 2,3,4 and -1 respectively and weights 2,3,4 and -1 in the variables x,y,z and u respectively. This is so, because for instance, if $H_1(x, y, z, u) = x/2 + ayzu^5$, it holds that

$$H_1(\lambda^2 x, \lambda^3 y, \lambda^4 z, \lambda^{-1} u) = \lambda^2 H_1(x, y, z, u),$$

and similarly with all the other components. In fact, this is the reason to choose the above expression for the map H. By imposing that

$$x_n = 2^{2n} x_0,\ y_n = 2^{3n} y_0,\ z_n = 2^{4n} z_0 \text{ and } u_n = 2^{-n} u_0, \tag{7}$$

are solutions of system $(x_{n+1}, y_{n+1}, z_{n+1}, u_{n+1}) = H(x_n, y_n, z_n, u_n)$ we arrive to the following system of equations

$$\begin{cases} 2ay_0z_0u_0^5 - 7x_0 = 0, \\ 2bx_0z_0u_0^3 - 15y_0 = 0, \\ 2cx_0y_0u_0 - 31z_0 = 0. \end{cases}$$

By solving it we get

$$x_0 = \frac{2aAB}{7u_0^2}, \quad y_0 = \frac{A}{u_0^3}, \quad z_0 = \frac{B}{u_0^4},$$

where A is any solution of $4ac\,\xi^2 - 217 = 0$ and B is any solution of $4ab\,\xi^2 - 105 = 0$. Setting $a = 1$, $b = 105$ and $c = 217$ we get that the DDS of the statement has unbounded solutions for any initial condition

$$(x_0, y_0, z_0, u_0) = \left(\pm\frac{1}{14u_0^2}, -\frac{1}{2u_0^3}, \mp\frac{1}{2u_0^4}, u_0\right) \text{ or } (x_0, y_0, z_0, u_0) = \left(\pm\frac{1}{14u_0^2}, \frac{1}{2u_0^3}, \pm\frac{1}{2u_0^4}, u_0\right)$$

with $u_0 \neq 0$. In fact, these unbounded orbits are the ones given in (7) for any $u_0 \neq 0$ and (x_0, y_0, z_0) given above. ∎

Let us prove our main results. We will need the following technical lemmas.

Lemma 4. *Let $f : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^\ell$ be a continuous function. Consider $K \subseteq \mathbb{R}^m$ a compact set, and fix $u_* \in \mathbb{R}^k$. Given any norm in $\mathbb{R}^\ell$, define*

$$d_{K,u_*}(u) = \max_{x\in K} ||f(x, u) - f(x, u_*)||. \tag{8}$$

Then $\lim_{u\to u_*} d_{K,u_*}(u) = 0$.

Proof. Take any norm in $\mathbb{R}^k$ and denote it also as $||\cdot||$. Given $\varepsilon > 0$, we will find $\delta > 0$ such that if $||u - u_*|| \le \delta$ then $d_{K,u_*}(u) \le \varepsilon$. Indeed, if we consider the continuous function f restricted to some the compact set $K \times C$ where $C = \{u : ||u - u_*|| \le c\}$ for some suitable $c > 0$, then on this set f is uniformly continuous. Hence, for any $\varepsilon > 0$ there exists $\delta > 0$ such that if

$||(x,u)-(x',u')|| \leq \delta$ then $||f(x,u)-f(x',u')|| \leq \varepsilon$ for all $(x,u),(x',u') \in K \times C$. Here, in $\mathbb{R}^m \times \mathbb{R}^k$, we consider for instance $||(x,u)|| = ||x|| + ||u||$. Taking $x' = x$ and $u' = u_*$ we get the result.

∎

Lemma 5. *Set $e_0 \geq 0$, $0 < L < 1$ and let $\{\alpha_n\}_n$ be any non-negative sequence. Assume that the sequence $\{e_n\}_n$ satisfies that for all $n \geq 0$,*

$$e_{n+1} \leq \alpha_n + L\, e_n.$$

Therefore, if $\lim_{n\to\infty} \alpha_n = 0$ then $\lim_{n\to\infty} e_n = 0$.

Proof. Fixed any $\varepsilon > 0$ it suffices to prove that there exists $K = K(\varepsilon)$ such for all $n \geq K$ it holds that $e_{n+1} < \varepsilon$. Let A be such that $\alpha_n < A$ for all n. Take $N \in \mathbb{N}$ such that

$$L^N \leq \frac{(1-L)\varepsilon}{3A} \quad \text{and} \quad \alpha_n \leq \frac{(1-L)\varepsilon}{3} \quad \text{for} \quad n \geq N.$$

Let $M \in \mathbb{N}$ be such that $L^{M+1}e_0 \leq \varepsilon/3$. Then for $n \geq K := \max(2N-1, M)$ it holds that

$$\begin{aligned}
e_{n+1} &\leq \alpha_n + Le_n \leq \alpha_n + L\alpha_{n-1} + L^2 e_{n-1} \leq \alpha_n + L\alpha_{n-1} + L^2\alpha_{n-2} + L^3 e_{n-2} \leq \cdots \leq \\
&\leq \alpha_n + L\alpha_{n-1} + L^2\alpha_{n-2} + \cdots + L^n\alpha_0 + L^{n+1}e_0 \leq \\
&\leq \Big(\alpha_n + L\alpha_{n-1} + \cdots + L^{N-1}\alpha_{n-N+1}\Big) + \Big(L^N\alpha_{n-N} + \cdots + L^n\alpha_0\Big) + \frac{\varepsilon}{3} \leq \\
&\leq \Big(1 + L + \cdots + L^{N-1}\Big)\frac{(1-L)\varepsilon}{3} + \Big(L^N + L^{N+1} + \cdots + L^n\Big)A + \frac{\varepsilon}{3} \leq \frac{\varepsilon}{3} + \frac{L^N A}{1-L} + \frac{\varepsilon}{3} \leq \varepsilon.
\end{aligned}$$

∎

Proof of Theorem A. By hypothesis (e), the point (x_*, u_*) is a fixed point of the DDS given by (1). Take (x_0, u_0) with u_0 in the basin of attraction of u_* for the DDS generated by g. The proof that $\{(x_n, u_n)\}_n$ tends to (x_*, u_*) will be done in two steps. Firstly we will show that the sequence $\{x_n\}_n$ is bounded and secondly we will prove its convergence.

Clearly, by hypothesis (b), given any $\varepsilon > 0$ it holds that $||u_n - u_*|| \leq \varepsilon$ when $n \geq n_0$ for some $n_0 = n_0(\varepsilon)$. Therefore, changing the initial condition, if necessary, we can assume that $||u_n - u_*|| \leq \varepsilon$ for all $n \geq 0$. We take ε such that

$$\bar{N} := \max_{\{u\,:\,||u-u_*||\leq\varepsilon\}} N(u) < 1,$$

where $N(u)$ is given in item (a). This is possible by the hypothesis in item (c) and by the continuity of N. Similarly, define

$$\bar{M} := \max_{\{u\,:\,||u-u_*||\leq\varepsilon\}} M(u).$$

Hence, applying recurrently (2) we get the boundeness of $\{x_n\}_n$, because

$$||x_{n+1}|| \leq \bar{M} + \bar{N}\,||x_n|| \leq \bar{M} + \bar{N}\bar{M} + \bar{N}^2||x_{n-1}|| \leq \bar{M} + \bar{N}\bar{M} + \bar{N}^2\bar{M} + \bar{N}^3||x_{n-2}|| \leq \cdots \leq$$
$$\leq \bar{M}\big(1 + \bar{N} + \bar{N}^2 + \cdots + \bar{N}^n\big) + \bar{N}^{n+1}||x_0|| \leq \frac{\bar{M}}{1-\bar{N}} + ||x_0||.$$

To prove the convergence, we define $e_n := ||x_n - x_*||$ and we will show that the limit of $\{e_n\}_n$ is 0. By using the fact that $\{x_n\}_n$ is bounded, we can assume that it is confined in some compact set K, and consider the function $d_{K,u_*}(u)$ defined in (8). By using also inequality (3) we get that

$$\begin{aligned} e_{n+1} &= ||x_{n+1} - x_*|| = ||f(x_n, u_n) - f(x_*, u_*)|| \\ &\leq ||f(x_n, u_n) - f(x_n, u_*)|| + ||f(x_n, u_*) - f(x_*, u_*)|| \leq \\ &\leq d_{K,u_*}(u_n) + L\,||x_n - x_*|| = d_{K,u_*}(u_n) + L\,e_n. \end{aligned}$$

If we denote by $\alpha_n = d_{K,u_*}(u_n)$, by Lemma 4 we have that $\lim_{n\to\infty} \alpha_n = 0$. Hence, since $L < 1$, we can apply Lemma 5 obtaining that $\lim_{n\to\infty} e_n = 0$, as we wanted to prove. ∎

Proof of Corollary 1. By hypothesis the DDS (4) satisfies hypothesis (b) of Theorem A. Using the vector norm whose induced matrix norm satisfies (*ii*), we get:

$$||f(x,u)|| \leq ||f_0(u) + f_1(u)\,x|| \leq ||f_0(u)|| + ||f_1(u)\,x|| \leq ||f_0(u)|| + ||f_1(u)||\,||x||.$$

So taking $M(u) = ||f_0(u)||$ and $N(u) = ||f_1(u)||$ we have that the DDS (4) also satisfies the hypothesis (a) and (c) of the main result. Also, straightforwardly one gets

$$||f(x,u_*) - f(y,u_*)|| = ||f_1(u_*)(x-y)|| \leq ||f_1(u_*)||\,||(x-y)||,$$

hence hypothesis (d) of Theorem A is fulfilled. Then Corollary 1 is obtained as a direct consequence of the main result. ∎

3 Some examples of application of Theorem A

Example C. Consider the DDS defined in $Q^+ = \{(x,u) \in \mathbb{R}^2, \text{ with } x > 0, u > 0\}$:

$$\left\{ \begin{array}{rcl} x_{n+1} & = & \dfrac{f_0(u_n)}{1+x_n} + \dfrac{f_1(u_n)\,x_n}{1+x_n}, \\ u_{n+1} & = & \dfrac{(\alpha+1)\,u_n}{1+u_n}, \end{array} \right. \tag{9}$$

where $\alpha > 0$, $f_0 > 0, f_1 > 0$ and $f_0(\alpha) + f_1(\alpha) < 1$. As a consequence of Theorem A we have:

Proposition 6. *Any orbit of DDS (9) with initial condition $(x_0, u_0) \in Q^+$ converges to the fixed point (x_*, u_*) given by*

$$x_* = \left(f_1(\alpha) - 1 + \sqrt{(1 - f_1(\alpha))^2 + 4f_0(\alpha)}\right)/2, \quad \textit{and} \quad u_* = \alpha,$$

which is a stable attractor in Q^+.

Proof. The result is a consequence of Theorem A by taking $||\cdot|| = ||\cdot||_2$. To see this, we check that all the four hypotheses of the theorem are satisfied. Indeed, since we are considering $x > 0$, observe that the sublinearity condition 2 stated in hypothesis (a) is straightforwardly satisfied with $M(u) = f_0(u)$ and $N(u) = f_1(u)$.

(b) The Riccati difference equation $u_{n+1} = (\alpha+1)u_n/(1+u_n)$ has two fixed points at $u = 0$ and $u_* = \alpha$. Since $\alpha > 0$, this last fixed point is GAS in $\mathbb{R}^+ = \{x : x > 0\}$.

(c) Since $f_0(\alpha) + f_1(\alpha) < 1$, then $N_* = f_1(u_*) = f_1(\alpha) < 1$.

(d) A computation shows that,

$$\begin{aligned} ||f(x,u_*) - f(y,u_*)|| &= \left|\frac{f_0(u_*)(y-x) + f_1(u_*)(x-y)}{(1+x)(1+y)}\right| \le |f_0(u_*)(y-x) + f_1(u_*)(x-y)| \\ &\le (f_0(u_*) + f_1(u_*))\,|x-y|. \end{aligned}$$

Setting $L := f_0(u_*) + f_1(u_*)$, by hypothesis we obtain $L < 1$, and therefore hypothesis (d) is satisfied. ∎

For instance, next result is a corollary of the above proposition.

Corollary 7. *The DDS, defined on Q^+,*

$$\begin{cases} x_{n+1} &= \dfrac{x_n + u_n}{2(1+x_n)}, \\ u_{n+1} &= \dfrac{(\alpha+1)\,u_n}{1+u_n}, \end{cases}$$

with $\alpha > 0$, has the GAS fixed point (x_*, α) with $x_* = (-1 + \sqrt{1+8\alpha})/4$.

Before introducing next example we need to prove a preliminary result. As usual, for $x = (x_1, \ldots, x_m)$ and $1 \le p \in \mathbb{R}$,

$$||x||_p = \sqrt[p]{|x_1|^p + \cdots + |x_m|^p}, \quad \text{and} \quad ||x||_\infty = \max_{i=1,\ldots,m} |x_i|.$$

Lemma 8. *For $x, y \in \mathbb{R}^m$ set $x * y = (x_1y_1, x_2y_2, \ldots, x_my_m)$. Then, for any vector norm $||\cdot||$ on $\mathbb{R}^m$,*

$$||x * y|| \le R\,||x||_\infty ||y||, \tag{10}$$

where R is a positive constant that depends on the norm. In particular, when the norm is any p norm, $1 \le p \in \mathbb{R} \cup \{\infty\}$ then $R = 1$, but for other norms R can be bigger that 1.

Proof. For any p norm, $1 \leq p \in \mathbb{R} \cup \{\infty\}$ it is easy to prove that (10) holds with $R = 1$. For instance, when $p \geq 1$ is finite,

$$||x * y||_p \leq \sqrt[p]{\sum_{i=1}^{m} |x_i|^p |y_i|^p} \leq \max_{i=1,\ldots,m} |x_i| \sqrt[p]{\sum_{i=1}^{m} |y_i|^p} = ||x||_\infty ||y||_p.$$

Since any two norms in $\mathbb{R}^m$ are equivalent, taking for instance the 2-norm, there exist two positive constants K_1 and K_2 such that for any $y \in \mathbb{R}^m$, $K_1||y||_2 \leq ||y|| \leq K_2||y||_2$. Hence,

$$||x * y|| \leq K_2||x * y||_2 \leq K_2||x||_\infty ||y||_2 \leq \frac{K_2}{K_1}||x||_\infty ||y||,$$

and taking $R = K_2/K_1$ the inequality (10) follows.

Let us define a norm on $\mathbb{R}^2$ for which $R > 1$ in (10). For any $x = (x_1, x_2)$, introduce the polynomial $P_x(t) = x_1 + x_2 t$. Then

$$||x|| := ||P_x|| = \sqrt{\int_0^1 P_x^2(t)\,\mathrm{d}t} = \sqrt{x_1^2 + x_1 x_2 + x_2^2/3}\,.$$

where, in the above line, $||P_x||$ is the $L^2([0, 1])$ norm of P_x (recall that $||f||^2 = \int_0^1 f^2(t)\,\mathrm{d}t$). Because of this definition it is easy to see that $||\cdot||$ is a norm on $\mathbb{R}^2$. For instance, $||x + y|| = ||P_{x+y}|| = ||P_x + P_y|| \leq ||P_x|| + ||P_y|| = ||x|| + ||y||$.

By taking $x = (-1, 1)$ and $y = (-1, 2)$, then $x * y = (1, 2)$, $||x||_\infty = 1$, $||y|| = \sqrt{1/3}$ and $||x * y|| = \sqrt{13/3}$. Hence $R \geq ||x * y||/(||x||_\infty ||y||) = \sqrt{13} > 1$, as we wanted to show. ∎

Example D. Consider $x = (x_1, \ldots, x_m)$, $u = (u_1, \ldots, u_k)$ and the DDS given by

$$\left\{ \begin{array}{rcl} x_{n+1} & = & f(x_n, u_n) = f_0(u_n) + f_1(x_n, u_n), \\ u_{n+1} & = & g(u_n). \end{array} \right. \tag{11}$$

where $f_0 : \mathbb{R}^k \longrightarrow \mathbb{R}^m$, $f_1(x, u) = (at_1(u)F_1(x), at_2(u)F_2(x), \ldots, at_m(u)F_m(x))$ defined from $\mathcal{U} \times \mathbb{R}^k$, with $\mathcal{U} \subseteq \mathbb{R}^m$ into $\mathbb{R}^m$, $g : \mathbb{R}^k \longrightarrow \mathbb{R}^k$, all the functions differentiable, and $a > 0$. Moreover, we will consider $u_{n+1} = g(u_n)$ given by a map g from $\mathbb{R}^k$ into itself associated to a k-th order recurrence and having a GAS fixed point. Some concrete examples are given in [9]. We prove:

Proposition 9. *Consider the DDS* (11) *and assume that it satisfies the following hypotheses:*

(a) There exist positive constants A, B and K such that the map $F(x) = (F_1(x), \ldots, F_m(x))$ satisfies $||F(x)|| \leq A + B||x||$ and $||F(x) - F(y)|| \leq K\,||x - y||$ for all $x, y \in \mathcal{U} \subseteq \mathbb{R}^m$.

(b) The map g can be written as $g(u) = (u_2, \ldots, u_k, h(u_1, \ldots, u_k))$, where h is such for all $u \in \mathbb{R}^k$, $\sum_{i=1}^{k} \left|\frac{\partial h(u)}{\partial u_i}\right| < 1$, and $u_* = (z, \ldots, z) \in \mathbb{R}^k$, where $v = z$ is the unique solution of the equation $h(v, \ldots, v) = v$.

Then there exists $a_0 > 0$ such that for all $a < a_0$ it has a fixed point (x_*, u_*) which is GAS in $\mathcal{U} \times \mathbb{R}^k$.

Proof. Let us see that the system (11) satisfies the hypothesis of Theorem A. Indeed, set $T(u) = \max_{i=1,\ldots,m}(|t_i(u)|)$. Then, by using Lemma 8, we get

$$||f(x,u)|| \leq ||f_0(u)|| + aT(u)\,||F(x)|| \leq ||f_0(u)|| + aAT(u) + aBT(u)||x||.$$

Hence, setting $M(u) = ||f_0(u)|| + aAT(u)$ and $N(u) = aBT(u)$ we get that the sublinearity hypothesis (a) of Theorem A is satisfied.

The hypotheses on the map g ensure, from [9, Thm. 1], that the fixed point u_* of the subsystem $u_{n+1} = g(u_n)$ is a GAS in $\mathbb{R}^k$. Hence the hypothesis (b) of Theorem A holds.

Observe that $N_* = aBT(u_*)$. Hence taking $a < a_1 = (BT(u_*))^{-1}$ the hypothesis (b) of Theorem A is satisfied. To verify hypothesis (d), observe that by using Lemma 8, straightforwardly one gets

$$||f(x,u_*) - f(y,u_*)|| \leq aRT(u_*)||F(x) - F(y)|| \leq aRKT(u_*)||x - y||,$$

for some constant R given in its statement. Hence, for all $a < a_2 = (RKT(u_*))^{-1}$ hypothesis (d) is satisfied. Thus, setting $a_0 = \min(a_1, a_2)$, for all $a < a_0$ there exists a fixed point (x_*, u_*) which is a GAS in $\mathcal{U} \times \mathbb{R}^k$. ∎

Next we give concrete examples of functions F and h for which system (11) is under the hypotheses of Proposition 9.

We start giving a family of functions F. Fix σ and τ two permutations of the set $\{1, 2, \ldots, m\}$. Then take

$$F(x) = \left(\mathrm{e}^{-x_{\sigma_1}} + x_{\tau_1}, \mathrm{e}^{-x_{\sigma_2}} + x_{\tau_2}, \ldots, \mathrm{e}^{-x_{\sigma_m}} + x_{\tau_m}\right). \tag{12}$$

This map is sublinear in $\mathcal{U} = (0,\infty)^m$ because $||F(x)|| \leq ||(1,1,\ldots,1)|| + ||x||$. Hence we can set the constants A and B in Proposition 9 to be $A = ||(1,1,\ldots,1)||$ and $B = 1$.

Next, if we find $K > 0$ such that $||DF(x)|| < K$ for all $x \in \mathcal{U}$, where the norm of the matrix is the one induced by the vector norm, then for all $x, y \in \mathcal{U}$ we have $||F(x) - F(y)|| \leq K\,||x - y||$, see for instance [14, Thm. 9.19].

In our case observe that for each row of $DF(x)$, either there is a unique non-zero entry $-\mathrm{e}^{-x_{\sigma_i}} + 1$, if $\sigma_i = \tau_i$; or there are only two non-zero entries $\mathrm{e}^{-x_{\sigma_i}}$ and 1, respectively if $\sigma_i \neq \tau_i$. In any case, the sum of the squares of the elements of each row is less than 4 for all $x \in \mathcal{U}$. Hence, by using that $||A||^2 \leq \sum_{i,j} a_{ij}^2$ for any matrix norm ([14, Eq. (6), p. 211]), we obtain that: $||DF(\mathbf{x})|| \leq \sqrt{4m} = K$. Therefore for this family of maps F the hypothesis (a) of Proposition 9 holds.

Finally, a simple example of function h, that provides a g under the hypothesis (b) of Proposition 9 with $u_* = (0, \dots, 0)$ is

$$h(u_1, \dots, u_k) = \frac{u_1 + \cdots + u_k}{(k+1)\sqrt{1 + (u_1 + \cdots + u_k)^2}}. \tag{13}$$

This is so, because $v = 0$ is the unique solution of equation $h(v, \dots, v) = v$ and

$$\sum_{i=1}^{k} \left| \frac{\partial h(u)}{\partial u_i} \right| = \sum_{i=1}^{k} \frac{1}{(k+1)\,(1 + (u_1 + \cdots + u_k)^2)^{\frac{3}{2}}} = \frac{k}{(k+1)(1 + (u_1 + \cdots + u_k)^2)^{\frac{3}{2}}} \le \frac{k}{k+1} < 1.$$

In summary, consider the DDS defined in (11) with the functions F and h given in (12) and (13). Then, by Proposition 9, there is a computable value a_0 such that for all $a < a_0$, there exists a fixed point $(x_*, 0)$ which is a GAS in $\mathcal{U} \times \mathbb{R}^k$.

Example E. In this example we study a system with a kind of diffusive coupling.

Proposition 10. *Consider the DDS*

$$\left\{ \begin{array}{rcl} x_{n+1} & = & u_n \, (a\, x_n + b\, h(x_n - y_n)) \,, \\ y_{n+1} & = & v_n \, (a\, y_n - b\, h(x_n - y_n)) \,, \\ u_{n+1} & = & g_1(u_n, v_n), \\ v_{n+1} & = & g_2(u_n, v_n), \end{array} \right. \tag{14}$$

where $h : \mathbb{R} \to \mathbb{R}$ is differentiable and being h and h' bounded functions such that $h(0) = 0$. Assume that there exists a stable attracting point (u_*, u_*) in a certain non-empty set $\mathcal{U}$ for the subsystem $(u_{n+1}, v_{n+1}) = g(u_n, v_n)$ with $g = (g_1, g_2)$. Then, there exist a and b small enough, such that $(0, 0, u_*, u_*)$ is a stable attractor of (14) in $\mathbb{R}^2 \times \mathcal{U}$.

The explicit conditions that must satisfy the parameters a and b to guarantee the statement of the above result are given in Remark 11 below.

Proof. To prove the above statement, we set $\mathbf{x} = (x, y)$ and $\mathbf{u} = (u, v)$ and

$$f(\mathbf{x}, \mathbf{u}) = f(x, y, u, v) = (u\,(a\,x + b\,h(x - y)), v\,(a\,y - b\,h(x - y)))\,,$$

and we check that the hypotheses of Theorem A, by taking the $||\cdot||_2$ norm, are satisfied. For the sake of simplicity, we skip the subscript 2 in the rest of the proof.

Let K_1 and K_2 be positive constants such that $|h(z)| \le K_1$ and $|h'(z)| \le K_2$ for all $z \in \mathbb{R}$. By using that $x^2u^2 + y^2v^2 \le (x^2 + y^2)(u^2 + v^2)$ we have

$$||f(\mathbf{x}, \mathbf{u})|| \le |a|\; ||(ux, vy)|| \;+\; |b|\, K_1\, ||(u, -v)|| \le |a|\, ||\mathbf{x}||\, ||\mathbf{u}|| \;+\; |b|\, K_1 ||\mathbf{u}||.$$

Setting $M(\mathbf{u}) = |b|\, K_1 ||\mathbf{u}||$ and $N(\mathbf{u}) = |a|\, ||\mathbf{u}||$ we get that condition (a) of Theorem A is verified. Observe that hypothesis (b) is verified by the initial assumptions on the system.

Set $\mathbf{u}_* = (u_*, u_*)$. If a is such that $N(\mathbf{u}_*) = |a|\,||\mathbf{u}_*|| = \sqrt{2}\,|a|\,|u_*| < 1$ condition (c) is satisfied.

To check condition (d) of Theorem A we use again that if $||Df(\mathbf{x}, \mathbf{u}_*)|| < L$ for all $\mathbf{x} \in \mathbb{R}^2$ then $||f(\mathbf{x}, \mathbf{u}_*) - f(\mathbf{y}, \mathbf{u}_*)|| \leq L\,||\mathbf{x} - \mathbf{y}||$, see [14, Thm. 9.19]. In our case

$$Df(\mathbf{x}, \mathbf{u}_*) = u_* \begin{pmatrix} a + bh'(x-y) & -bh'(x-y) \\ -bh'(x-y) & a + bh'(x-y) \end{pmatrix}.$$

Hence, by using once more that $||A||^2 \leq \sum_{i,j} a_{ij}^2$ for any matrix norm ([14, Eq. (6), p. 211]), we have

$$||Df(\mathbf{x}, \mathbf{u}_*)|| \leq \sqrt{2}\,|u_*|\,\sqrt{(a + bh'(x-y))^2 + b^2\,(h'(x-y))^2} \leq \sqrt{2}|u_*|\sqrt{a^2 + 2b^2\,K_2^2 + 2|a|\,|b|\,K_2}.$$

Setting $L := \sqrt{2}\,|u_*|\,\sqrt{a^2 + 2b^2 K_2^2 + 2|a|\,|b|\,K_2}$, we get that if a and b are small enough, $L < 1$ and, therefore, $f(x, u_*)$ is contractive and condition (d) of Theorem A is verified. Since $f(\mathbf{x}, \mathbf{u}_*)$ has the unique fixed point $(\mathbf{x}, \mathbf{u}) = (0, 0, u_*, u_*)$, the result follows. ∎

Remark 11. *The existence of a GAS fixed point for system* (14) *in* $\mathbb{R} \times \mathcal{U}$ *is guaranteed if* a *and* b *are small enough. From the above arguments, we get the following explicit necessary conditions:*

- $\sqrt{2}\,|a|\,|u_*| < 1$, *and*
- $\sqrt{2}\,|u_*|\,\sqrt{a^2 + 2b^2\,K_2^2 + 2|a|\,|b|\,K_2} < 1$, *where* $\max_{z\in\mathbb{R}} |h'(z)| = K_2$.

4 Applications to difference equations

In this section we apply our results on planar triangular maps given in Corollary 1 and an extension given in [8] to study the asymptotic behavior of several difference equations.

First we state a general proposition that collects both results. On one side the hypotheses are stronger, because the maps are assumed to be of class $\mathcal{C}^1$ and the fixed point that gives rise to the attracting fiber is assumed to be hyperbolic, but on the other hand it covers different situations for the dynamics on this attracting fiber. In short, before, the attracting fiber had a single GAS fixed point, but this result also covers other cases with simple dynamics: either when the fiber is full of fixed points or when the map restricted to it is a linear involution of the form $x \to k - x$, full of 2-periodic points. See [8] for the details on the reason why more regularity for the functions, or the hyperbolicity condition, are needed.

Proposition 12. *Consider the planar DDS*

$$\begin{cases} x_{n+1} = & f_0(u_n) + f_1(u_n)\,x_n, \\ u_{n+1} = & g(u_n), \end{cases} \tag{15}$$

with $f_0, f_1, g \in \mathcal{C}^1(\mathcal{U})$, where $\mathcal{U}$ is a neighborhood of $u = u_*$ and suppose that $u = u_*$ is a hyperbolic attractor of g. Then, for all initial conditions (x_0, u_0) with $u_0 \in \mathcal{U}$ in the basin of attraction of $u = u_*$, it holds:

(a) If $|f_1(u_*)| < 1$, then $\lim\limits_{n\to\infty}(x_n, u_n) = (f_0(u_*)/(1 - f_1(u_*)), u_*)$.

(b) If $f_0(u_*) = 0$ and $f_1(u_*) = 1$, then $\lim\limits_{n\to\infty}(x_n, u_n) = (\ell(x_0, u_0), u_*)$ for some $\ell(x_0, u_0) \in \mathbb{R}$.

(c) If $f_1(u_*) = -1$, then $\lim\limits_{n\to\infty}(x_{2n}, u_{2n}) = (\ell(x_0, u_0), u_*)$ and $\lim\limits_{n\to\infty}(x_{2n+1}, u_{2n+1}) = (f_0(u_*) - \ell(x_0, u_0), u_*)$ for some $\ell(x_0, u_0) \in \mathbb{R}$.

Furthermore, in cases (b) and (c), for any point $(\bar{x}, u_*) \in \{u = u_*\}$ there exist initial conditions (x_0, u_0) in the basin of attraction of the attracting fiber such that $(\bar{x}, u_*)$ is one of the acummulation points of the orbit $\{x_n, u_n\}_{n\geq 0}$.

In the sequel, in each section we consider several types of difference equations for which Proposition 12 can be applied.

4.1 Multiplicative difference equations

We consider the next family of second order multiplicative difference equations

$$x_{n+2} = x_n h(x_n x_{n+1}), \tag{16}$$

where $h : \mathcal{U} \to \mathbb{R}$ is a $\mathcal{C}^1$ function defined in an open set of $\mathcal{U} \subseteq \mathbb{R}$, which we call of multiplicative type. Multiplying both sides of (16) by x_{n+1}, and setting $u_n = x_n x_{n+1}$ we get that it can be written as

$$\begin{cases} x_{n+1} &= u_n/x_n, \\ u_{n+1} &= u_n\, h(u_n). \end{cases} \tag{17}$$

It has the associated map $F(x, u) = (u/x, u\, h(u))$. If we consider the map $F^2(x, u) = F \circ F(x, u)$ we obtain $F^2(x, u) = (h(u)\, x, u\, h(u)\, h(u\, h(u)))$. Hence, by calling $z_n = x_{2n}$ and $v_n = u_{2n}$ (respectively $z_n = x_{2n+1}$ and $v_n = u_{2n+1}$) we get the DDS

$$\begin{cases} z_{n+1} &= h(v_n)\, z_n, \\ v_{n+1} &= v_n\, h(v_n)\, h(v_n\, h(v_n)), \end{cases} \tag{18}$$

which is of the form (15). Hence we can apply Proposition 12 to it and study the behavior of (x_{2n}, u_{2n}) (respectively (x_{2n+1}, u_{2n+1})). Notice that when we consider the initial conditions (x_0, u_0) (resp. (x_1, u_1)) and we apply system (18) iteratively we get (x_{2n}, u_{2n}) (resp. (x_{2n+1}, u_{2n+1})).

Example F. Among all the recurrences of type (16), we are going to consider the one given by

$$x_{n+2} = \frac{x_n}{a + b x_n x_{n+1}}, \quad ab \neq 0,\ a, b \in \mathbb{R}. \tag{19}$$

The behavior of the solutions for the case $ab = 0, a^2 + b^2 \neq 0$, is very simple. The global behavior of the difference equation (19) is already completely understood after the work [5]. Their approach is based on the computation of the explicit solutions of (19). Previously, particular cases of this equation have been studied in several works, see [1, 2, 3, 4, 10, 15]. The reader is also referred to [5] for a summary of these previous references concerning this equation.

Difference equation (19) has the form (16), where $h(u) = 1/(a + bu)$. Furthermore, it corresponds with the DDS of type (17):

$$\begin{cases} x_{n+1}x_n &= u_n, \\ u_{n+1} &= \dfrac{u_n}{a + bu_n}. \end{cases}$$

This DDS has two invariant fibers given by $\{u = 0\}$ and $\{u = (1 - a)/b\}$ whose stability is determined by the parameter a, since setting $g(u) = uh(u) = u/(a + bu)$ we have $g'(0) = 1/a$ and $g'((1 - a)/b) = a$. The global dynamics of (19) in its *good set* (that is, the set of all initial conditions for which the dynamical system is well defined, [13]) $\mathcal{G} = \mathbb{R} \setminus \{\cup_{n\geq 0} g^{-n}(-a/b)\}$, is determined by the graph of g (notice that in another framework we could consider this recurrence defined in $\bar{\mathbb{R}} = [-\infty, \infty]$, and then the good set would be $\bar{\mathbb{R}}$). First it is necessary to know the dynamics of the recurrence $u_{n+1} = g(u_n)$ which, since g is a Möbius transformation, is well known. For instance, using [7, Cor. 7] we get:

Lemma 13. *Consider the real one-dimensional difference equation given by*

$$u_{n+1} = \frac{u_n}{a + bu_n}, \quad ab \neq 0. \tag{20}$$

Then the points $u = 0$ and $u = (1 - a)/b$ are fixed points. Furthermore setting $\mathcal{G} = \mathbb{R} \setminus \{\cup_{n\geq 0} g^{-n}(-a/b)\}$ we have:

(a) If $|a| > 1$, then for any initial condition $(1 - a)/b \neq u_0 \in \mathcal{G}$, $\lim_{n\to\infty} u_n = 0$.

(b) If $|a| < 1$, then for any initial condition $0 \neq u_0 \in \mathcal{G}$, $\lim_{n\to\infty} u_n = (1 - a)/b$.

(c) If $a = -1$, then for any initial condition in $\mathcal{G}$, the sequence $\{u_n\}$ is 2-periodic.

(d) If $a = 1$, then for any initial condition in $\mathcal{G}$, $\lim_{n\to\infty} u_n = 0$.

Hence as a consequence of the above lemma and Proposition 12, as well as some other ad hoc arguments, we obtain an alternative proof of the result in [5]:

Theorem 14. *Consider the difference equation (19). Then the following statemens hold:*

(a) If $x_0 = x_1 = 0$, then the sequence $x_n = 0$ for all $n \in \mathbb{N}$.

(b) *If $|a|>1$, then for any initial condition x_0, x_1 such that $(1-a)/b \neq x_0x_1 \in \mathcal{G}$, $\lim_{n\to\infty} x_n = 0$. If $x_0x_1 = (1-a)/b$ then $\{x_n\}$ is 2-periodic.*

(c) *If $|a|<1$, then for any initial condition x_0, x_1 such that $x_0x_1 \in \mathcal{G}$ we have: if $x_0x_1 \neq 0$ and $x_0x_1 \neq (1-a)/b$ then $\{x_n\}$ tends to a 2-periodic orbit $\{\ell_0(x_0,x_1), \ell_1(x_1,x_2)\}$ such that $\ell_0(x_0,x_1)\ell_1(x_1,x_2) = (1-a)/b$; if $x_0x_1 = (1-a)/b$ then $\{x_n\}$ is 2-periodic; and if $x_0x_1 = 0$ then $\lim_{n\to\infty} |x_n| = \infty$.*

(d) *If $a=-1$, then for any initial condition x_0, x_1 such that $x_0x_1 \in \mathcal{G}$ we have: if $x_0x_1 \neq 0$ and $x_0x_1 \neq 2/b$ then the solution $\{x_n\}$ is unbounded; if $x_0x_1 = 2/b$ then $\{x_n\}$ is 2-periodic; and if $x_0x_1 = 0$ then $\{x_n\}$ is 4-periodic.*

(e) *If $a=1$, then for any initial condition x_0, x_1 such that $0 \neq x_0x_1 \in \mathcal{G}$, $\lim_{n\to\infty} x_n = 0$. If $x_0x_1 = 0$ then $\{x_n\}$ is 2-periodic.*

Proof. Statement (a) is trivial. In order to prove (b) we consider $|a|>1$. Then, by item (a) of Lemma 13, $u=0$ is an attractor of the recurrence (20) in $\mathcal{G} \setminus \{u = (1-a)/b\}$. We now note that for equation (19), the DDS (18) writes as

$$\left\{\begin{array}{ll} z_{n+1} & = \dfrac{1}{a+bv_n} z_n, \\ v_{n+1} & = \dfrac{v_n}{a^2+b(1+a)v_n}. \end{array}\right. \tag{21}$$

Applying Proposition 12 to system (21) we deduce that $z_n \to 0$. It implies that $x_{2n} \to 0$ and $x_{2n+1} \to 0$ too. Hence, for each initial condition such that $(1-a)/b \neq x_0x_1 \in \mathcal{G}$ $\lim_{n\to\infty} x_n = 0$. On the other hand, substituting $x_0x_1 = (1-a)/b$ in Equation (19) we get $x_2 = x_0$, obtaining a 2-periodic orbit.

(c) If $|a|<1$, then by Lemma 13 (b), for all $u_0 \neq u_* := (1-a)/b$ the sequence $u_n \to u_*$. Since $f_1(u_*) = 1$ and u_* is a hyperbolic attractor of (20), we can use Proposition 12 to assert that the sequence v_n converges to a point which depend on the initial condition. Then, if we take the initial condition $(z_0,v_0) = (x_0,u_0)$ (resp. $(z_0,v_0) = (x_1,u_1)$) we have that $\lim_{n\to\infty} u_{2n} = \lim_{n\to\infty} v_n = \overline{\ell_0}(x_0,u_0) := \ell_0(x_0,x_1)$ (resp. $\lim_{n\to\infty} u_{2n+1} = \lim_{n\to\infty} v_n = \overline{\ell_1}(x_1,u_1) := \ell_1(x_1,x_2)$). Then, since $u_n = x_nx_{n+1}$, the condition $\ell_0(x_0,x_1)\,\ell_1(x_1,x_2) = u_*$ must be satisfied. The other assertions of statement (c) are easily deduced from (19).

(d) If $a=-1$ and $x_0x_1 \neq 0$, $x_0x_1 \neq 2/b$ then

$$u_{n+2} = u_n, \quad x_{2n} = \frac{x_0}{(bx_0x_1-1)^n} \quad \text{and} \quad x_{2n+1} = x_1(bx_0x_1-1)^n.$$

Hence x_n is unbounded. If $x_0x_1 = 2/b$, then $x_{2n} = x_0$, $x_{2n+1} = x_1$ and $\{x_n\}$ is 2-periodic. If $x_0 = 0$, then $x_{2n} = 0$, $x_{2n+1} = x_1(-1)^n$ and $\{x_n\}$ is 4-periodic. If $x_1 = 0$, then $x_{2n} = (-1)^n x_0$, $x_{2n+1} = 0$ and $\{x_n\}$ is 4-periodic.

(e) Similarly, if $a = 1$ and $x_0 x_1 = 0$ then $\{x_n\}$ is 2-periodic. Consider now $x_0 x_1 \neq 0$. The DDS (18) is

$$\left\{ \begin{array}{ll} z_{n+1} & = \left(\dfrac{1}{1+bv_n} \right) z_n, \\ v_{n+1} & = \dfrac{v_n}{1+2bv_n}. \end{array} \right.$$

A straightforward computation shows that its second component has the following explicit solution $v_n = v_0/(1 + 2\, b\, v_0\, n)$. Hence its first equation writes as

$$z_{n+1} = \left(\frac{1}{1 + \frac{bv_0}{2\,b\,v_0\,n+1}} \right) z_n,$$

and we obtain the explicit solution

$$z_{n+1} = \left(\prod_{j=0}^{n} \frac{1}{1 + \frac{bv_0}{2\,b\,v_0\,j+1}} \right) z_0.$$

So if $n > n_0$ for a suitable n_0, we can take logarithms in the above equations, obtaining:

$$\ln|z_{n+1}| = -\sum_{j=0}^{n} \ln \left| 1 + \frac{b\,v_0}{2\,b\,v_0\,j+1} \right| + \ln|z_0| \sim -\sum_{j=0}^{n} \left| \frac{bv_0}{2\,b\,v_0\,j+1} \right|.$$

Observe that, for any value of b and v_0 we have $\lim\limits_{n\to\infty} \sum\limits_{j=0}^{n} \left| \frac{bv_0}{2\,b\,v_0\,j+1} \right| = +\infty$, hence $\lim\limits_{n\to\infty} z_n = 0$, and the result follows. ∎

The next third order difference equation can be studied using the same approach.

Example G The exact solutions of the third order difference equation

$$x_{n+3} = \frac{x_{n+2} x_n}{x_{n+1}(a + b x_{n+2} x_n)},$$

have been obtained in [11]. A complete analysis of the dynamics associated to this equation can be done using a similar approach as in Example F. Setting $g(u) = u/(a + bu)$, we have that the subsequences $\{x_{2k}\}$ and $\{x_{2k+1}\}$ can be studied using the DDS

$$\left\{ \begin{array}{ll} y_{n+1} y_n & = v_n, \\ v_{n+1} & = g^2(v_n) = \dfrac{v_n}{b(a+1)v_n + a^2}, \end{array} \right.$$

where $y_n = x_{2n+i}$, and $v_n = u_{2n+i}$ for $i = 0, 1$. Hence, the sequences $\{x_{2k+i}\}$ can be straightforwardly characterized using the behavior of the real Möebius recurrence $v_{n+1} = v_n/(b(a+1)v_n + a^2)$.

The next example shows that the approach in this section can be applied to other families of difference equations not in the class (16). We will not develop the analysis in this paper, since our main purpose is only to illustrate a range of applications of our results.

Example H. Consider the difference equation

$$x_{n+2} = \frac{x_n^\gamma h(x_{n+1}x_n^\gamma)}{x_{n+1}^{\gamma-1}},$$

with $x_n \in \mathbb{R}^+$, $\gamma \in \mathbb{R}$ and being h a $\mathcal{C}^1$ positive function. By multiplying both sides of the equation by x_{n+1}^γ; setting $u_n = x_{n+1}x_n^\gamma$, and taking $y_n = \ln x_n$ we get the DDS of type (15)

$$\begin{cases} y_{n+1} &= \ln(u_n) - \gamma y_n, \\ u_{n+1} &= u_n h(u_n), \end{cases}$$

that can be studied again by using the results of Proposition 12.

4.2 Additive difference equations

In this section we consider the second order difference equations

$$x_{n+2} = -bx_{n+1} + g(x_{n+1} + bx_n), \tag{22}$$

the we call that we will call additive. They can be studied via the associated map $F(x, y) = (y, -by + g(y + bx))$ which preserves the fibration $\mathcal{F} = \{y + bx = c,\, c \in \mathbb{R}\}$. So if $u = u^*$ is a fixed point of h, the map preserves the fiber $y + bx = u^*$.

Setting $u_n = x_{n+1} + bx_n$, we get the DDS

$$\begin{cases} x_{n+1} &= u_n - bx_n, \\ u_{n+1} &= g(u_n), \end{cases} \tag{23}$$

which is of type (15) with $f_0(u) = u$ and $f_1(u) \equiv -b$.

It is easy to observe that if $|b| > 1$, or $b = -1$ and $u^* \neq 0$ then there are iterates of map F on the invariant fiber $y + bx = u^*$ which are unbounded, and therefore these cases are out of our scope. In fact it is straightforward to obtain the following result.

Lemma 15. *Consider the difference equation (22) where g is a continuous function defined in an open set $\mathcal{U} \subseteq \mathbb{R}$. Let $u^* \in \mathcal{U}$ be a fixed attracting point $u_{n+1} = g(u_n)$. Then, for each initial condition x_0, x_1 such that $x_1 + bx_0 = u^*$, we have $\lim\limits_{n\to\infty} x_n = u^*/(1 + b)$ if $|b| < 1$; the orbits are 2-periodic if $b = 1$; are fixed points if $b = -1$ and $u^* = 0$; and there are unbounded orbits if $b = -1$ and $u^* \neq 0$, or $|b| > 1$.*

For the rest of initial conditions, the dynamics can be studied using Proposition 12, obtaining:

Proposition 16. *Consider the difference equation (22) being $g \in \mathcal{C}^1(\mathcal{U})$ function defined in an open set $\mathcal{U} \subseteq \mathbb{R}$. Let $u^* \in \mathcal{U}$ be a hyperbolic attracting point of $u_{n+1} = g(u_n)$. Then, for all initial conditions x_0, x_1 such that $u_0 = x_1 + bx_0$ is in the basin of attraction of $u = u^*$, we have:*

(a) If $|b| < 1$, then $\lim\limits_{n\to\infty} x_n = u^*/(1+b)$.

(b) If $b = 1$ then $\{x_n\}$ tends to a 2-periodic orbit $\{\ell(x_0, x_1), u_* - \ell(x_0, x_1)\}$.

(c) If $b = -1$, $u^* = 0$ then there exists $\ell(x_0, x_1)$ such that $\lim\limits_{n\to\infty} x_n = \ell(x_0, x_1)$.

Proof. (a) If $|b| < 1$, then system (23) is under the hypothesis of Proposition 12, hence

$$\lim_{n\to\infty} x_n = \frac{f_0(u_*)}{1 - f_1(u_*)} = \frac{u^*}{1+b}.$$

To prove statement (b) we apply Proposition 12 to DDS (23). Since $f_1(u) = -1$ we get that $\lim\limits_{n\to\infty}(x_{2n}, u_{2n}) = (\ell(x_0, u_0), u_*)$ and $\lim\limits_{n\to\infty}(x_{2n+1}, u_{2n+1}) = (u_* - \ell(x_0, u_0), u_*)$. Since $u_0 = bx_0 + x_1$, $\ell(x_0, u_0) = \ell(x_0, x_1)$.

Finally, if $b = -1$ and $u_* = 0$, then by Proposition 12 again, $\lim\limits_{n\to\infty} x_n = \ell(x_0, x_1)$. ∎

Example I. In an analogous way as in the previous example, by adding ax_{n+1} in both sides of the difference equation

$$x_{n+2} = ax_n + (1-a)x_{n+1} + f(x_{n+1} + ax_n),$$

and setting $u_n = x_{n+1} + ax_n$, we get that the it can be studied via the DDS of type (15)

$$\begin{cases} x_{n+1} &= u_n - ax_n, \\ u_{n+1} &= u_n + f(u_n). \end{cases}$$

4.3 Other higher order difference equations

A similar approach can be applied to several higher order multiplicative-type of difference equations. Consider the k-th order difference equation:

$$x_{n+k} = x_n\, h(x_n x_{n+1} \cdots x_{n+k-1}). \tag{24}$$

Some straightforward computations using the associated map

$$F(x_0, \ldots, x_{k-1}) = (x_1, \ldots, x_{k-1}, x_0 h(x_0 \cdots x_{k-1})),$$

show that the sets $x_0 x_1 \cdots x_{k-1} = 0$ and $x_0 x_1 \cdots x_{k-1} = u_*$ for $u^* \neq 0$ such that $h(u^*) = 1$ (if it exists) are invariant, and also lead to the following result:

Lemma 17. *Consider the difference equation (24), being h a $\mathcal{C}^1(\mathcal{U})$ function, for some open set $\mathcal{U} \subseteq \mathbb{R}$ containing 0. Given some initial conditions $x_0, \ldots, x_{k-1}$ we have:*

(i) If $x_i = 0$ for all $i = 0, \ldots, k-1$ then $x_n = 0$ for all $n \in \mathbb{N}$.

(ii) Suppose that $x_0 x_1 \cdots x_{k-1} = 0$ and $x_0^2 + \cdots + x_{k-1}^2 \neq 0$. If additionally $h(0) = -1$ then $\{x_n\}$ is $2k$-periodic; if $h(0) = 1$ then $\{x_n\}$ is k-periodic; if $|h(0)| < 1$ then $\lim\limits_{n\to\infty} x_n = 0$; and if $|h(0)| > 1$ then $\lim\limits_{n\to\infty} |x_n| = \infty$.

(iii) If there exists $u^* \neq 0$ in $\mathcal{U}$ such that $h(u^*) = 1$, and $x_0 x_1 \cdots x_{k-1} = u^*$ then $\{x_n\}$ is k-periodic (non-minimal).

Proposition 18. *Consider the difference equation (24) being h a $\mathcal{C}^1(\mathcal{U})$ function, for some open set $\mathcal{U} \subseteq \mathbb{R}$. If $0 \in \mathcal{U}$ and $|h(0)| < 1$, then $u = 0$ is an attractor for the DDS $u_{n+1} = u_n h(u_n)$, and for all initial condition $x_0, \ldots, x_{k-1}$ such that $0 \neq u_0 = x_0 \ldots x_{k-1}$ is in the basin of attraction of $\{u = 0\}$ we have that $\lim\limits_{n\to\infty} x_n = 0$.*

Proof. Multiplying both sides of (24) by $x_{n+k-1} \cdots x_{n+1}$ and setting $u_n = x_n x_{n+1} \cdots x_{n+k-1}$, we obtain that equation (24) can be written as

$$\left\{\begin{array}{ll} x_{n+k} & = h(u_n) x_n, \\ u_{n+1} & = m(u_n) := u_n h(u_n), \end{array}\right. \Longrightarrow \left\{\begin{array}{ll} x_{n+k} & = h(u_n) x_n, \\ u_{n+k} & = g(u_n) := m^k(u_n), \end{array}\right.$$

where m^k denotes the composition of m, k times, that is $m^j(u) = m(m^{j-1}(u))$. From the expressions, and after renaming $z_{i,n} = x_{kn+i}$ and $v_{i,n} = u_{kn+i}$ we get k systems of type (15) associated to each initial condition (x_i, u_i) for $i = 0, \ldots, k-1$.

$$\left\{\begin{array}{ll} z_{i,n+1} & = h(v_{i,n}) z_{i,n}, \\ v_{i,n+1} & = g(v_{i,n}). \end{array}\right. \tag{25}$$

The result follows because each of these systems is under the hypotheses of Proposition 12. ∎

Proposition 19. *Consider the difference equation (24) being h a $\mathcal{C}^1$ function defined in an open set $\mathcal{U} \subseteq \mathbb{R}$. Let $u_a \in \mathcal{U}$ be a hyperbolic attractor of $g(u) = u h(u)$. Then, for all initial condition $(x_0, \ldots, x_{k-1})$ such that $u_a \neq u_0 = x_0 \cdots x_{k-1}$ is in the basin of attraction of $\{u = u_a\}$ we have that:*

(a) If $h(u_a) = 1$, then the solution of (24) tends to a k-periodic orbit.

(b) If $u_a = 0$ and $h(0) = -1$, then the solution $\{x_n\}$ tends to a $2k$-periodic orbit.

In both cases, the period is not necessarily minimal.

Proof. We consider again the systems (25) which, under the current hypotheses, also satisfy the ones of Proposition 12. Hence if $h(u_a) = 1$, then for $i = 0, \dots, k-1$ there exists ℓ_i depending on x_i, u_i, that is depending on $x_i, x_{i+1}, \dots, x_{i+k-1}$, such that each sequence $\{z_{i,n}\}$ satisfies $\lim_{n\to\infty} z_{i,n} = \ell_i$. If $h(0) = -1$, then there exist $\ell_{i,0}$ and $\ell_{i,1}$ such that $\lim_{j\to\infty} z_{i,2j} = \ell_{i,0}$, $\lim_{j\to\infty} z_{i,2j+1} = \ell_{i,1}$ and $(\ell_{i,0}, \ell_{i,1})$ is a $2k$-periodic orbit. Hence the result follows. ∎

Example J. Consider the difference equation

$$x_{n+k} = x_n + f\left(\sum_{i=n}^{n+k-1} x_i\right).$$

Adding the term $\sum_{i=n+1}^{n+k-1} x_i$ in both sides; setting $u_n = x_n + x_{n+1} + \cdots + x_{n+k-1}$, and after renaming $z_{i,n} = x_{kn+i}$ and $v_{i,n} = u_{kn+i}$ we get k systems of type (15) associated to each initial condition (x_i, u_i) for $i = 0, \dots, k-1$.

$$\begin{cases} z_{i,n+1} &= f(v_{i,n}) + z_{i_n}, \\ v_{i,n+1} &= g(v_{i,n}), \end{cases}$$

where here $g(v) = m^k(v)$ with $m(v) = v + f(v)$. Again, under suitable hypotheses we can apply Proposition 12 to each of these DDS and obtain similar results on GAS.

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