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ORIGINAL PAPER

A dynamic Parrondo's paradox for continuous seasonal systems

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Abstract We show that planar continuous alternating systems, which can be used to model systems with seasonality, can exhibit a type of Parrondo's dynamic paradox, in which the stability of an equilibrium, common to all seasons is reversed for the global seasonal system. As a byproduct of our approach we also prove that there are locally invertible orientation preserving planar maps that cannot be the time-1 flow map of any smooth planar vector field.

Keywords Continuous dynamical systems with seasonality · Non-hyperbolic critical points · Local asymptotic stability · Parrondo's dynamic paradox

Mathematics Subject Classification Primary 37C75 · 34D20; Secondary 37C25

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1 Introduction and main results

For dynamical systems given by differential equations, alternating systems take the form

$$\begin{cases} \dot{\mathbf{x}}(t) = X_1(\mathbf{x}(t)) \text{ for } t \text{ such that } t \pmod{T} \in [0, T_1), \\ \dot{\mathbf{x}}(t) = X_2(\mathbf{x}(t)) \text{ for } t \text{ such that } t \pmod{T} \in [T_1, T_1+T_2), \\ \vdots \\ \dot{\mathbf{x}}(t) = X_n(\mathbf{x}(t)) \text{ for } t \text{ such that } t \pmod{T} \in \\ \qquad [T_1 + \cdots + T_{n-1}, T_1 + \cdots + T_n), \end{cases} \tag{1}$$

where $T = \sum_{j=1}^{n} T_j$, with $T_j > 0$ for $j = 1, 2, \ldots, n$, and X_j being class $\mathcal{C}^1$ vector fields. They can be used to model continuous seasonal systems with n seasons of durations $T_1, T_2, \ldots, T_n$. It is not necessary to recall the importance of these kind of systems in mathematical biology, for instance in population models for which the seasonality has an effect in the reproduction and mortality rates due to environmental circumstances or to human intervention like harvesting, see [12,19,20] and references therein (and [5,9,14,15] for discrete examples); or also in epidemiological models with periodic contact rate, see [3] and the references therein.

The existence of chaotic behaviors in this kind of systems has been reported, see for instance the proof of the existence of topological horseshoes in the Poincaré maps associated with the flow of a 2-seasonal Lotka–Volterra system of type (1) in [18]. The study of chaotic dynamics in continuous seasonal systems is a challenge that will continue to require a lot of attention. In this work we expose a collateral aspect that appears,