THE LIMIT CYCLES OF THE HIGGINS-SELKOV SYSTEMS

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Abstract. In this paper we investigate the problem of limit cycles for general Higgins-Selkov systems with degree $n+1$. In particular, we first prove the uniqueness of limit cycles for a general Liénard system which allows for discontinuity. Then, by changing the Higgins-Selkov systems into Liénard systems, theorems and some techniques for Liénard systems can be applied. After we prove the nonexistence of limit cycles if the bifurcation parameter is outside an open interval. Finally we complete the analysis of limit cycles for the Higgins-Selkov systems showing its uniqueness.

1. Introduction and main results

In the qualitative theory of planar polynomial differential systems it is well known how difficult is to study the famous Hilbert's 16th problem, see [8, 10, 16]. Up to now there are seldom works having solved the problem of exact number of limit cycles for polynomial differential systems.

The most important physiological function of carbohydrates is to provide energy for organisms' life activities. Glucose catabolism is the main way for organisms to obtain energy. There are three main pathways for the oxidative decomposition of glucose in organisms. Among them, the anaerobic oxidation of glucose is called glycolysis. We consider the following polynomial differential system of arbitrary degree

$$\begin{aligned} \dot{x} &= 1 - xy^n, \\ \dot{y} &= ay(-1 + xy^{n-1}) \end{aligned} \tag{1}$$

which was proposed first by Higgins [7] and modified further by Selkov [13] for studying the biological nonlinear glycolytic oscillations, and was called the Higgins-Selkov system. Here n is a positive integer and a is a real parameter. Artés, Llibre and Valls in [1] characterized the global dynamics described in the Poincaré disc for system (1) as $n = 2$ and $a \in \mathbb{R}\backslash(1, 3)$. Moreover there are two conjectures stated in [1] on the the number of limit cycles of systems (1) when $a \in (1, 3)$. After Chen and Tang in [5] proved these conjectures which complete the global phase portraits of system (1) when $n = 2$.

Recently Brechmann and Rendall in [3] researched the uniqueness of limit cycles for system (1) and additionally proved that no limit cycles exist when $a \in (0, 1/(1-n))$. Llibre and Mousavi [11] classified the phase portraits of system (1) for $n = 3, 4, 5, 6$ in the Poincaré disc for all the values of the parameter a and determined in function of the parameter a the regions of the phase space with biological meaning.

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The aim of this paper is to give a clearer study and answer for the existence and the exact number of limit cycles of system (1). We have the following main results.

Theorem 1. *For every positive integer $n \geq 3$ and some bounded constant $a^* < 1$, system* (1) *has no periodic orbits when $a \in (-\infty, 1/(n-1)] \cup [a^*, +\infty)$ and has a unique limit cycle when $a \in (1/(n-1), a^*)$, which is stable and hyperbolic.*

An outline of this paper is as follows. A theorem on the uniqueness of limit cycles for general Liénard systems are presented in section 2, which we need in our study of the limit cycles of the Higgins-Selkov system. In section 3 we obtain the existence and the exact number of limit cycles of the Higgins-Selkov system and then prove our main theorem.

2. Preliminaries

In order to study the number of limit cycles for system (1), we need the following preliminary results. We first recall the uniqueness theorem of Zhifen Zhang in [14] or in [15] on the number of limit cycles of the following generalized Liénard systems

$$\begin{aligned} \dot{x} &= -\phi(y) - \hat{F}(x), \\ \dot{y} &= \hat{g}(x). \end{aligned} \tag{2}$$

Let

$$\hat{G}(x) := \int_0^x \hat{g}(s)ds.$$

Theorem 2. *Consider the generalized Liénard system* (2) *for $x \in (-\infty, +\infty)$, when $\phi(y)$, $\hat{F}(x)$ and $\hat{g}(x)$ satisfy the following conditions:*

- **(i):** *$\hat{g}(x)$ is Lipschitz in any finite interval, $x\hat{g}(x) > 0$ for all $x \neq 0$, and $\hat{G}(-\infty) = \hat{G}(+\infty) = +\infty$.*
- **(ii):** *$\hat{f}(x) = \hat{F}'(x)$ is $\mathcal{C}^0$, $\hat{F}(0) = 0$, $\hat{f}(x)/\hat{g}(x)$ is nondecreasing in $(-\infty, 0) \cup (0, +\infty)$ and $\hat{f}(x)/\hat{g}(x)$ is not a constant when $|x|$ is small.*
- **(iii):** *$\phi(y)$ is Lipschitz in any finite interval, $y\phi(y) > 0$ for all $y \neq 0$, $\phi(y)$ is nondecreasing, $\phi(-\infty) = -\infty$, $\phi(+\infty) = +\infty$, $\phi(y)$ has right-derivative $\phi'_+(0)$ and left-derivative $\phi'_-(0)$ at $y = 0$, $\phi'_-(0)\phi'_+(0) \neq 0$ when $\hat{f}(0) = 0$.*

Then system (2) *has at most one limit cycle. Moreover the limit cycle is stable when it exists.*

In fact we can find many differential systems of the form (2) but many of them do not satisfy the conditions of Theorem 2. Thus we propose the following three questions:

- **(a):** When $\hat{G}(-\infty) = \hat{G}(+\infty) \neq +\infty$ and the other conditions of Theorem 2 hold, does the conclusion of Theorem 2 still hold?
- **(b):** When either $\hat{f}(x)$ or $\hat{g}(x)$ has a discontinuity point x_0 of the second kind (i.e., $\lim_{x\to x_0+} \hat{g}(x)$ or $\lim_{x\to x_0-} \hat{g}(x)$ does not exist) and the other conditions of Theorem 2 hold, does the conclusion of Theorem 2 still hold?
- **(c):** When $\hat{g}(x)$ has a discontinuity point at $x = 0$ of the first kind (i.e., $\lim_{x\to 0+} \hat{g}(x) \neq \lim_{x\to 0-} \hat{g}(x)$) and the other conditions of Theorem 2 hold, does the conclusion of Theorem 2 still hold?

For example we have that $G(-\infty) = G(+\infty) \neq +\infty$ when $\hat{g}(x) = x/(1+x^2)^2$. Either $\hat{f}(x)$ or $\hat{g}(x)$ has a discontinuity point at $x = -1$ of the second kind when $\hat{f}(x) = 1/a - (n-1)/(x+1)^n$ or $\hat{g}(x) = x/(x+1)^n$.

Here we will show why the condition $G(-\infty) = G(+\infty) = +\infty$ is necessary in the proof of Theorem 2 of [15]. Zhang in [15] only need to research the following special Liénard system

$$\begin{aligned} \dot{u} &= -\phi(y) - \hat{F}(x(u)), \\ \dot{y} &= u, \end{aligned} \tag{3}$$

because system (2) can be changed into system (3) through the transformation $u = \sqrt{2G(x)}\mathrm{Sgn}(x)$ and $dt \to \left(\sqrt{2G(x)}\mathrm{sgn}(x)/\hat{g}(x)\right)dt$. However, the transformation is not an 1-1 transformation in $(-\infty, +\infty)$ when $G(-\infty) = G(+\infty) \neq +\infty$.

For these reasons we give the following theorem without the aforementioned conditions.

Theorem 3. *Consider system* (2) *in the interval* (α, β)*, where* α *and* β *eventually can be* $-\infty$ *and* $+\infty$*, respectively. Assume that* $\phi(y)$*,* $\hat{F}(x)$ *and* $\hat{g}(x)$ *satisfy the following conditions:*

- **(i):** $\hat{g}(x) := g_0(x) + c\ \mathrm{sgn}(x)$, $x\hat{g}(x) > 0$ *for all* $x \neq 0$*, where* $c \geq 0$ *and* $g_0(x)$ *is Lipschitz in any finite interval and* $g_0(0) = 0$*.*
- **(ii):** $\hat{f}(x) = \hat{F}'(x)$ *is* $\mathcal{C}^0(\alpha, \beta)$*,* $\hat{F}(0) = 0$*,* $\hat{f}(0) \neq 0$*,* $\hat{f}(x)/\hat{g}(x)$ *is nondecreasing in* $(\alpha, 0) \cup (0, \beta)$ *and* $\hat{f}(x)/\hat{g}(x)$ *is not a constant when* $|x|$ *is small.*
- **(iii):** $\phi(y)$ *is Lipschitz in any finite interval,* $y\phi(y) > 0$ *for all* $y \neq 0$*,* $\phi(y)$ *is increasing,* $\phi(-\infty) = -\infty$*,* $\phi(+\infty) = +\infty$*,* $\phi(y)$ *has right-derivative* $\phi'_+(0)$ *and left-derivative* $\phi'_-(0)$ *at* $y = 0$*,* $\phi'_-(0)\phi'_+(0) \neq 0$ *when* $\hat{f}(0) = 0$*.*

Then system (2) *has at most one limit cycle in* (α, β)*. Moreover the limit cycle is stable when it exists.*

Proof. Since the vector field of system (2) is Lipschitz for $c = 0$, its solutions exist and are unique except at $x = 0$. Since the vector field of system (2) is discontinuous at the line $\Sigma := \{(x, y) : x = 0\}$ for $c > 0$, we need to study the dynamics on Σ and we will adapt the Filippov method, see [2, 9]. Let

$$\delta := \langle (1, 0), (-\phi(y), -c)\rangle \langle (1, 0), (-\phi(y), c)\rangle = \phi^2(y),$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product. As defined in [2, 9], the *crossing set* is

$$\Sigma_c = \{(x, y) \in \Sigma : \delta > 0\} = \{(x, y) \in \Sigma : y \neq 0\}.$$

The *sliding set* Σ_s is the complement to Σ_c, which is given by

$$\Sigma_s = \{(x, y) \in \Sigma : \delta \leq 0\} = \{(x, y) \in \Sigma : y = 0\}.$$

Therefore except at the origin, all orbits crossing any point are unique. In other words all periodic orbits are crossing.

Assume that γ is a periodic orbit of system (2). Then we have that γ is hyperbolic if $\oint_\gamma \mathrm{div}(-\phi(y) - \hat{F}(x), \hat{g}(x))dt \neq 0$, see for instance Theorem 1.23 of [6]. Moreover

γ is stable (resp. unstable) if

$$\oint_\gamma \operatorname{div}(-\phi(y) - \hat{F}(x), \hat{g}(x))dt < 0 \text{ (resp. } > 0).$$

In order to prove the uniqueness of limit cycles of system (2), assume that system (2) has at least two limit cycles, where γ_1, γ_2 are the innermost limit cycles and γ_1 lies in the bounded region surrounded by γ_2.

Actually we must have $\hat{f}(0) < 0$. Otherwise, if $\hat{f}(0) > 0$ we can obtain $\hat{f}(x) > 0$ since $\hat{f}(x)/\hat{g}(x)$ is non-decreasing and $x\hat{g}(x) > 0$. For $x > 0$ near the origin, we can get $\hat{f}(x) > 0$ by the continuity of $\hat{f}(x)$ at $x = 0$ and then $\hat{f}(x)/\hat{g}(x) > 0$, implying $\hat{f}(x)/\hat{g}(x) > 0$ for all $x > 0$ by the monotonicity of this function. Thus we have $\hat{f}(x) > 0$ for all $x > 0$. Similarly for all $x < 0$ we can also get $\hat{f}(x) > 0$. Then, by Green formula we have

$$0 = \oint_{\gamma_i} (-\phi(y) - \hat{F}(x))dy + \hat{g}(x)dx = -\oint_{\mathcal{D}_i} \hat{f}(x)dxdy,$$

which contradicts the fact that $\hat{f}(x) > 0$, where $\mathcal{D}_i$ is the bounded region surrounded by γ_i for $i = 1, 2$. Here, note that the Dulac criterion cannot be applied because the vector field of system (2) is not $\mathcal{C}^1$. Thus we get $\hat{f}(0) < 0$ if a periodic orbit exists.

Moreover we claim that the equation $\hat{f}(x) = 0$ has at most one positive root and one negative root, where a connect set of roots is viewed as one root. Otherwise, assume that $\hat{f}(x)$ has two positive zeros x_1 and x_2 such that $0 < x_1 < x_2$. Then there exists a real $x_0 \in (x_1, x_2)$ satisfying $\hat{f}(x_0)/\hat{g}(x_0) > 0 = \hat{f}(x_2)/\hat{g}(x_2)$, which contradicts the non-decreasing of $\hat{f}(x)/\hat{g}(x)$. Thus the claim is proved.

Applying Green formula again we have that system (2) has no periodic orbits when $\hat{f}(x) \le 0$ for all $x \in (\alpha, \beta)$. Therefore if system (2) exhibits periodic orbits, there is an $x_3 \in (\alpha, \beta)$ such that $\hat{f}(x_3) > 0$. In the following we divide the proof of the uniqueness of limit cycles of system (2) in three cases.

Case (I). First we consider the case $x_3 > 0$ if $\hat{f}(x_3) > 0$. Then there is a unique value $x_4 \in (0, \beta)$ such that $\hat{F}(x_4) = 0$. Moreover if there exist two different points $x_{41}, x_{42} \in (0, \beta)$ such that $\hat{F}(x_{41}) = 0$ and $\hat{F}(x_{42}) = 0$, we can get an $\tilde{x}_4 \in (x_{41}, x_{42})$ satisfying $\hat{f}(\tilde{x}_4) = 0$, which contradicts the non-decreasing of $\hat{f}(x)/\hat{g}(x)$ for $x > 0$.

We claim that any periodic orbit must surround the point $(x_4, 0)$. So no periodic orbits exist if x_4 does not exist. Let

$$E(x, y) = \frac{y^2}{2} + \int_0^x \hat{g}(s)ds, \tag{4}$$

which implies that

$$\frac{dE(x, y)}{dt} = -\hat{g}(x)\hat{F}(x).$$

It is to note that $\hat{g}(x)\hat{F}(x) < 0$ for all $x \in (\alpha, x_4)$. Assume that system (2) exhibits a periodic orbit γ, which lies in the strip $x \in (\alpha, x_4)$. Then we can find that

$$0 = \oint_\gamma dE = \oint_\gamma -\hat{g}(x)\hat{F}(x)dt > 0.$$

Thus the claim is proved.

Now we will prove that

$$\oint_{\gamma_1} \hat{f}(x)dt < \oint_{\gamma_2} \hat{f}(x)dt. \tag{5}$$

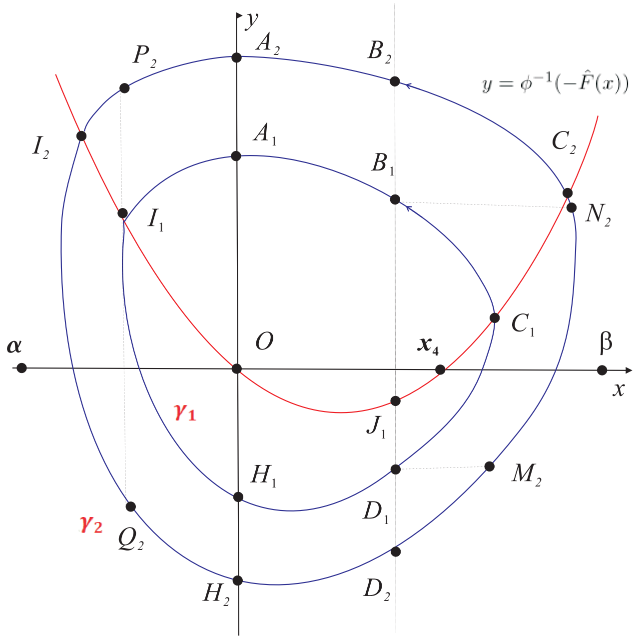

Figure 1. Limit cycles of system (2) in the Case (I).

Consider the two limit cycles $\gamma_1 = A_1B_1\widehat{C_1D_1H_1}I_1A_1$ and $\gamma_2 = A_2B_2\widehat{C_2D_2H_2}I_2A_2$ of Figure 1. Notice that the limit cycle γ_i intersects the graphic of the function $y = \phi^{-1}(-\hat{F}(x))$ at the points C_i and I_i for $i = 1, 2$, respectively. Since

$$\oint_{\gamma_1} \hat{g}(x)dt = \oint_{\gamma_1} dy = 0 = \oint_{\gamma_2} dy = \oint_{\gamma_2} \hat{g}(x)dt,$$

we only need to prove

$$\oint_{\gamma_1} f_1(x)dt < \oint_{\gamma_2} f_1(x)dt, \tag{6}$$

which is equivalent to (5), where

$$f_1(x) := \hat{f}(x) + b\hat{g}(x) \tag{7}$$

for any constant $b \in \mathbb{R}$. It is clear that $f_1(x)/\hat{g}(x)$ is still non-decreasing if $f(x)/\hat{g}(x)$ is non-decreasing. Fixing

$$b = -\hat{f}(x_{I_1})/\hat{g}(x_{I_1}) < 0,$$

we have $f_1(x_{I_1}) = 0$. Moreover we have $f_1(x)/\hat{g}(x) \geq 0$ for $x_{I_1} < x < 0$ and $f_1(x)/\hat{g}(x) \leq 0$ for $x < x_{I_1}$, because $f_1(x_{I_1})/\hat{g}(x_{I_1}) = 0$ and $f_1(x)/\hat{g}(x)$ is non-decreasing. Thus $f_1(x) \leq 0$ if $x_{I_1} < x < 0$, and $f_1(x) \geq 0$ if $x < x_{I_1}$, because $\hat{g}(x) < 0$ for $x < 0$.

Denote by $P = (x_P, y_P)$ for an arbitrary point P. We can find a point $J_1(x_{J_1}, y_{J_1})$ in the curve $y = \phi^{-1}(-\hat{F}(x))$ such that $f_1(x_{J_1}) = 0$ and $x_{J_1} \in (0, x_{C_1})$. Otherwise,

$f_1(x) < 0$ for all $x \in (0, x_{C_1})$, and if the point J_1 does not exist then $f_1(x) \le 0$ for all $x \in (x_{I_1}, x_{C_1})$. Thus we obtain

$$\oint_{\gamma_1} f_1(x)dt < 0. \tag{8}$$

However the origin is a source and the periodic orbit γ_1 is internally stable because $f_1(x) < 0$ for small x, implying $\oint_{\gamma_1} f_1(x)dt \ge 0$. It induces a contradiction with the inequality (8). Thus the point J_1 exists. Moreover we have $f_1(x) \ge 0$ for $x > x_{J_1}$, and $f_1(x) \le 0$ for all $0 < x < x_{J_1}$, because $\hat{g}(x) > 0$ for $x > 0$ and $f_1(x)/\hat{g}(x)$ is non-decreasing.

Assume that the line $x = x_{J_1}$ intersects with the graphic of the function $y = \phi^{-1}(-\hat{F}(x))$ at the points B_i and D_i for $i = 1, 2$, respectively. Notice that

$$x_{B_1} = x_{B_2} = x_{D_1} = x_{D_2} = x_{J_1}.$$

Let $y = y_1(x)$ and $y = y_2(x)$ be the orbit segments $\widehat{A_1B_1}$ and $\widehat{A_2B_2}$, respectively. Since $y_1 < y_2$ and the function $\phi(x)$ is increasing, we have $\phi(y_1) < \phi(y_2)$. Then we have

(9)
$$\begin{aligned}\int_{\widehat{B_1A_1}} f_1(x)dt - \int_{\widehat{B_2A_2}} f_1(x)dt &= \int_0^{x_{B_1}} \frac{f_1(x)}{\phi(y_1) + \hat{F}(x)}dx - \int_0^{x_{B_2}} \frac{f_1(x)}{\phi(y_2) + \hat{F}(x)}dx \\ &= \int_0^{x_{B_1}} \frac{f_1(x)(\phi(y_2) - \phi(y_1))}{(\phi(y_1) + \hat{F}(x))(\phi(y_2) + \hat{F}(x))} < 0.\end{aligned}$$

It is similar to prove that

$$\begin{aligned}&\int_{\widehat{H_1D_1}} f_1(x)dt - \int_{\widehat{H_2D_2}} f_1(x)dt < 0, \\ &\int_{\widehat{A_1I_1}} f_1(x)dt - \int_{\widehat{A_2P_2}} f_1(x)dt < 0, \\ &\int_{\widehat{I_1H_1}} f_1(x)dt - \int_{\widehat{Q_2H_2}} f_1(x)dt < 0,\end{aligned} \tag{10}$$

where $P_2, Q_2 \in \gamma_2$ and $x_{P_2} = x_{Q_2} = x_{I_1}$.

Let $x = x_1(y)$ and $x = x_2(y)$ be the orbit segments $\widehat{D_1C_1B_1}$ and $\widehat{D_2C_2B_2}$, respectively. Then we have

$$\begin{aligned}\int_{\widehat{D_1C_1B_1}} f_1(x)dt - \int_{\widehat{D_2C_2B_2}} f_1(x)dt &< \int_{\widehat{D_1C_1B_1}} \hat{f}(x)dt - \int_{\widehat{M_2N_2}} \hat{f}(x)dt \\ &= \int_{y_{M_2}}^{y_{N_2}} \left(\frac{\hat{f}(x_1)}{\hat{g}(x_1)} - \frac{\hat{f}(x_2)}{\hat{g}(x_2)} \right) dy < 0,\end{aligned} \tag{11}$$

where $M_2, N_2 \in \gamma_2$, $y_{M_2} = y_{D_1}$ and $y_{N_2} = y_{B_1}$. Since $f_1(x) \ge 0$ for all $x < x_{I_1}$, we have

$$\int_{\widehat{P_2I_2Q_2}} f_1(x)dt > 0. \tag{12}$$

Therefore (6) holds from (9)-(12). Notice that the origin is a source and the periodic orbit γ_1 is internally stable. Thus $\oint_{\gamma_1} \hat{f}(x)dt \ge 0$. It follows from (5) that $\oint_{\gamma_2} \hat{f}(x)dt > 0$. Consequently γ_2 is stable and hyperbolic. By the Poincaré-Bendixson Theorem (see for instance Corollary 1.30 of [6]), it is impossible the existence of two consecutive stable limit cycles. Therefore system (2) has at most two limit cycles. Moreover, γ_1 is semi-stable and γ_2 is stable if they exist.

In order to induce a contradiction for the case that γ_1 is semi-stable, we construct an auxiliary vector field $(-\phi(y) - \tilde{F}(x), \hat{g}(x))$, where $\tilde{F}(x) = \hat{F}(x) + \epsilon R(x)$ and

$$R(x) := \begin{cases} 0, & \text{if } x \leq x_4, \\ \hat{F}(x), & \text{if } x > x_4, \end{cases}$$

for small $|\epsilon|$. We can check that the vector field $(-\phi(y) - \tilde{F}(x), \hat{g}(x))$ is rotated with respect to the parameter ϵ; see [16, Chapter 4.3] or [12]. Consider the following system

$$\begin{aligned} \dot{x} &= -\phi(y) - \tilde{F}(x), \\ \dot{y} &= \hat{g}(x). \end{aligned} \tag{13}$$

System (13) is exactly system (2) if $\epsilon = 0$. Moreover, we can check that system (13) still satisfies all assumptions of Theorem 3. In other words system (13) has at most two limit cycles. Further, we can find that γ_1 will split into at least two limit cycle for $\epsilon < 0$ by [16, Theorem 3.4 of Chapter 4]. Then system (13) can have three limit cycles, a contradiction with the previous result. Therefore we have proven that system (2) has at most one periodic orbit in the case $x_3 > 0$ if $\hat{f}(x_3) > 0$.

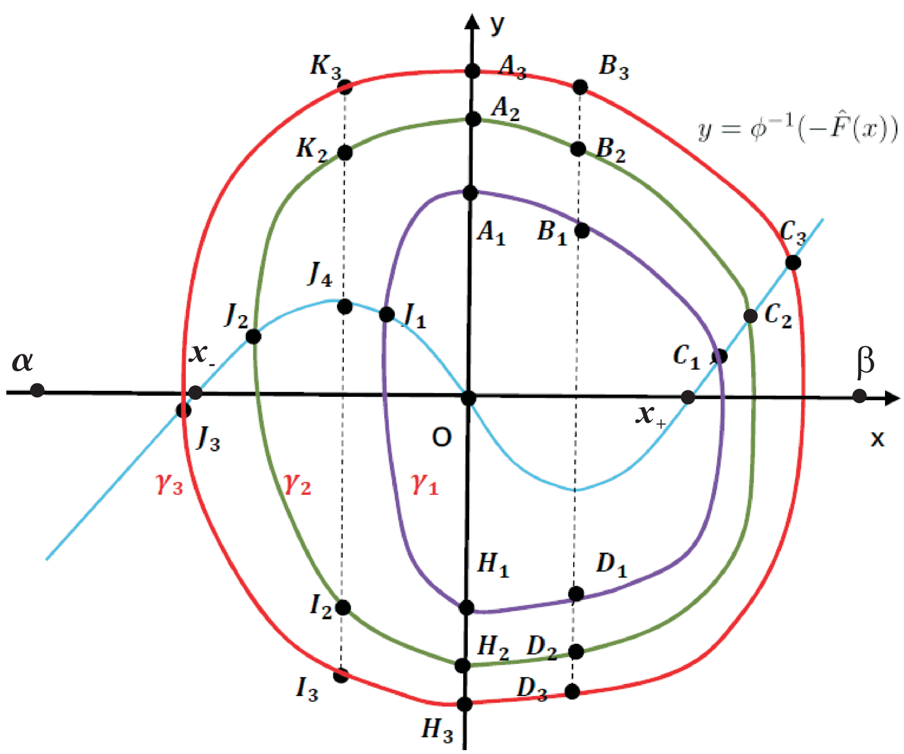

Figure 2. Limit cycles of system (2) in the Case (III).

Case (II). Second, we consider the case that there must be $x_3 < 0$ if $\hat{f}(x_3) > 0$. Since the proof is similar to the Case (I), we omit it.

Case (III). We consider the case that x_3 can be negative or positive if $\hat{f}(x_3) > 0$. We claim that the equation $\hat{F}(x) = 0$ has either one non-zero root or two non-zero roots. Notice that the equation $\hat{F}(x) = 0$ can not have two positive roots or two negative roots. Otherwise if there exist two different points $x_{41}, x_{42} \in (0, \beta)$ or $\in (\alpha, 0)$ such that $\hat{F}(x_{41}) = 0$ and $\hat{F}(x_{42}) = 0$, we can get a point $\tilde{x}_4 \in (x_{41}, x_{42})$ satisfying $\hat{f}(\tilde{x}_4) = 0$, which contradicts the non-decreasing of the function $\hat{f}(x)/\hat{g}(x)$. On the other hand, if the equation $\hat{F}(x) = 0$ has not non-zero roots, we get $dE/dt \geq 0$ for $x \in (\alpha, \beta)$ from (4), implying that no periodic orbits exist. If the equation $\hat{F}(x) = 0$ has a unique non-zero root, we consider that the non-zero root is $x_+ \in (0, \beta)$ for simpcility. If system (2) has a periodic orbit, it must surround $(x_+, 0)$. Otherwise this is a contradiction with the fact that $dE/dt \geq 0$. When equation $\hat{F}(x) = 0$ has one positive root $x_+ \in (0, \beta)$ and a negative root $x_- \in (\alpha, 0)$, if system (2) has a periodic orbit, it must surround at least one of the points $(x_+, 0)$ and $(x_-, 0)$. Otherwise again we have a contradiction with the fact that $dE/dt \geq 0$. Without loss of generality we can assume that any limit cycle surrounds $(x_+, 0)$.

Assume that system (2) has three limit cycles $\gamma_1, \gamma_2, \gamma_3$ as the ones shown in Figure 2, where γ_1 is the innermost one, γ_3 surrounds γ_1 and γ_2, the points $A_i, B_i, C_i, D_i, H_i, I_i J_i, K_i \in \gamma_i$, the periodic orbit γ_i intersects the graphic of the function $y = \phi^{-1}(-\hat{F}(x))$ at the points C_i and J_i for $i = 1, 2, 3$, respectively. Notice that

$$\begin{array}{lll} x_{B_i} = x_{D_i}, & x_{A_i} = x_{H_i} = 0, & x_{K_2} = x_{K_3} = x_{I_2} = x_{I_3} = x_{J_4}, \\ f(x_{B_i}) = 0, & f(x_{J_4}) = 0, & \alpha < x_{J_3} < x_{J_2} < x_{J_4} < x_{J_1} < 0, \end{array}$$

for $i = 1, 2, 3$.

In a similar way to the proof of Case (I) we shall obtain that system (2) has at most one periodic orbit in the strip $x \in (x_{J_4}, \beta)$. Moreover the periodic orbit is stable if it exists. Now we shall prove that

$$\oint_{\gamma_1} \hat{f}(x)dt < \oint_{\gamma_2} \hat{f}(x)dt < \oint_{\gamma_3} \hat{f}(x)dt. \tag{14}$$

Notice that the function $y = \phi(x)$ has the same properties as in Case (I) when $x > 0$, as it is shown in Figures 1 and 2. Thus we can obtain that

$$\begin{array}{l} \int_{\widehat{H_1C_1A_1}} \hat{f}(x)dt < \int_{\widehat{H_2C_2A_2}} \hat{f}(x)dt, \\ \int_{\widehat{H_2C_2A_2}} \hat{f}(x)dt < \int_{\widehat{H_3C_3A_3}} \hat{f}(x)dt. \end{array} \tag{15}$$

To prove the first inequality of (14) it suffices to prove the inequality (6). Using the auxiliary function $f_1(x)$ in (7) for the Case (I) again, we can prove that

$$\int_{\widehat{A_1J_1H_1}} f_1(x)dt < \int_{\widehat{A_2J_2H_2}} f_1(x)dt. \tag{16}$$

From (15) and (16) we get

$$\oint_{\gamma_1} \hat{f}(x)dt < \oint_{\gamma_2} \hat{f}(x)dt. \tag{17}$$

Moreover we can calculate that

$$\begin{aligned}
&\int_{\widehat{A_2K_2}} \hat{f}(x)dt - \int_{\widehat{A_3K_3}} \hat{f}(x)dt < 0,\\
&\int_{\widehat{K_2J_2I_2}} \hat{f}(x)dt - \int_{\widehat{K_3J_3I_3}} \hat{f}(x)dt < 0,\\
&\int_{\widehat{I_2H_2}} \hat{f}(x)dt - \int_{\widehat{I_3H_3}} \hat{f}(x)dt < 0
\end{aligned} \tag{18}$$

doing a similar calculation as in Case (I) for $x > 0$. Thus, from (15) and (18), we get the second inequality of (14), i.e.

$$\oint_{\gamma_2} \hat{f}(x)dt < \oint_{\gamma_3} \hat{f}(x)dt. \tag{19}$$

It follows from (17) and (19) that (14) holds. Since the origin is a source, we have

$$\oint_{\gamma_1} \hat{f}(x)dt \geq 0, \quad \text{implying} \quad \oint_{\gamma_2} \hat{f}(x)dt > 0 \quad \text{and} \quad \oint_{\gamma_3} \hat{f}(x)dt > 0,$$

from the inequality (14). However it is impossible to have two consecutive stable limit cycles. Therefore system (2) cannot have three periodic limit cycles and there are at most two limit cycles.

We divide the rest of the proof in two subcases. First, we consider the subcase that γ_1 only surrounds one of the points $(x_+, 0)$ and $(x_-, 0)$. In Case (I) we have proved that for this kind of periodic orbits, as γ_1, at most one can exist and it is stable. Thus its consecutive periodic orbit γ_2 is internally unstable and then $\oint_{\gamma_2} \hat{f}(x)dt \leq 0$. Moreover the inequality (17) holds and γ_2 is stable, indicating a contradiction. Therefore the periodic orbit γ_2 does not exist and system (2) has exact one periodic orbit γ_1 if such periodic orbit exists.

Now we consider the subcase that system (2) has no such kind of periodic orbits like γ_1. We assume that system (2) has two periodic orbits γ_2 and γ_3, which surround both points $(x_+, 0)$ and $(x_-, 0)$. Since the origin is a source, we have

$$\oint_{\gamma_2} \hat{f}(x)dt \geq 0, \quad \text{implying} \quad \oint_{\gamma_3} \hat{f}(x)dt > 0,$$

by the inequality (19). Therefore γ_2 is semi-stable and γ_3 is stable. Using the auxiliary vector field (13) again, we can get that system (13) still satisfies all conditions of this theorem and has at most two limit cycles. However by the rotated properties of system (13), the semi-stable γ_2 will split into at least two limit cycles for $\epsilon \neq 0$ by [16, Theorem 3.4 of Chapter 4]. Then system (13) can have three limit cycles, a contradiction. Thus we have proven that system (2) has at most one periodic orbit in the case (III) and the proof is completed. □

Remark 4. *The conditions $\phi(-\infty) = -\infty$, $\phi(+\infty) = +\infty$ are needed only if (α, β) is unbounded. If (α, β) is a bounded interval, these conditions can be deleted in Theorem 3.*

Notice that the vector field is Lipschitz if $c = 0$ in Theorem 3. Thus the results of Theorem 3 also hold when system (2) is Lipschitz or further smooth.

The modified Liénard system (21) *of the Higgins-Selkov* (1) *is Lipschitz except at the line $x = 1$, which is a discontinuity point of the second kind for the functions in the system. So we need to apply Theorem* 3 *for showing the uniqueness of the limit cycles.*

3. Proof of Theorem 1

Notice that system (1) cannot have periodic orbits when $a \leq 0$, because the unique equilibrium $(1, 1)$ is a saddle as $a < 0$ or $\dot{y} \equiv 0$ as $a = 0$. Thus in the following we only consider the case $a > 0$. Moreover, the periodic orbits of system (1) must lie in the region

$$\{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\},$$

since the x-axis is invariant and $\dot{x}|_{x=0} = 1$.

In order to simplify the computations and the analysis we do the following change of coordinates

$$(x, y, t) \to \left(\frac{y_1 - x_1}{a}, x_1, \frac{t_1}{a}\right),$$

changing system (1) into

$$\begin{aligned} \dot{x}_1 &= -x_1 + \frac{x_1^n y_1}{a} - \frac{x_1^{n+1}}{a}, \\ \dot{y}_1 &= 1 - x_1. \end{aligned} \tag{20}$$

Obviously the periodic orbit of system (20) only exists in the region $x_1 > 0$, because $\dot{y}_1 = 1 > 0$ and $\dot{x}_1 = 0$ on the line $x_1 = 0$. Moreover system (20) can be changed into the following Liénard system

$$\begin{aligned} \dot{x} &= y - F(x), \\ \dot{y} &= -g(x), \end{aligned} \tag{21}$$

where

$$F(x) := \frac{1}{(x+1)^{n-1}} + \frac{x}{a} - 1 \quad \text{and} \quad g(x) := \frac{x}{a(x+1)^n},$$

doing the transformation $(x_1, y_1, t_1) \to (x + 1, ay + (a + 1), t/(x + 1)^n)$. Here we only need to consider $x > -1$ for the problem of limit cycles of system (21), because system (21) is equivalent to system (1) as $a > 0$ and $x > -1$.

From [3] system (21) has no periodic orbits when $a \in (0, 1/(n-1))$. In the following we prove that system (21) may have periodic orbits only if $a > 1/(n-1)$. Here we cannot use the methods of [5], because it is difficult to decide when the equations

$$\frac{g(z_1)}{f(z_1)} = \frac{g(z_2)}{f(z_2)} \text{ and } F(z_1) = F(z_2),$$

have solutions or not for an arbitrary integer n. We need use a new method and technique.

Proposition 5. *For $a > 0$, the amplitude of the stable or unstable limit cycle of system* (21) *surrounding the origin varies monotonically with respect to the parameter a.*

Proof. Notice that we can change system (21) into the following equivalent form

$$\begin{aligned}\dot{x} &= y - \check{F}(x),\\ \dot{y} &= -ag(x)\end{aligned} \tag{22}$$

by the transformation of coordinates $(x, y, t) \to \Big(x, y/\sqrt{a}, \sqrt{a}t\Big)$, where

$$F(x) := \sqrt{a}\Big(\frac{1}{(x+1)^{n-1}} - 1\Big) + \frac{x}{\sqrt{a}}.$$

From the calculation in [11], we have the value of the following determinant

$$\begin{vmatrix} y - \left(\sqrt{a_1}\Big(\frac{1}{(x+1)^{n-1}} - 1\Big) + \frac{x}{\sqrt{a_1}}\right) & -\frac{x}{(x+1)^n} \\ y - \left(\sqrt{a_2}\Big(\frac{1}{(x+1)^{n-1}} - 1\Big) + \frac{x}{\sqrt{a_2}}\right) & -\frac{x}{(x+1)^n} \end{vmatrix}$$

$$= (\sqrt{a_1} - \sqrt{a_2})\left(\frac{x\Big(1-(x+1)^{n-1}\Big)}{(x+1)^{2n-1}} - \frac{1}{\sqrt{a_1 a_2}}\frac{x^2}{(x+1)^n}\right) \le 0$$

for $a_2 < a_1$, $a_2, a_1 \in (0, +\infty)$ and $x > -1$.

Thus the vector field of system (22) is a generalized rotated vector field (see [16, Chapter 4.3] or [12]) with respect to the parameter a if $x > -1$. Moreover from [16, Theorem 3.5, Chapter 4], the amplitude of the stable or unstable limit cycle of system (22) surrounding the origin varies monotonically with respect to the positive parameter a. □

Proposition 6. *System* (21) *has no periodic orbits when* $a \in (-\infty, 1/(n-1)]$.

Proof. By [1, 5], system (21) has no periodic orbits for $a \le 1$ when $n = 2$. In the rest of this proof, we only consider $n \ge 3$. We only need to consider system (21) and its limit cycles in the region $x > -1$. Assume that

$$F(x_1) = F(x_2),\ G(x_1) = G(x_2) \tag{23}$$

for $n \ge 3$ and $-1 < x_1 < 0 < x_2$, where

$$G(x) := \int_0^x g(s)ds = \frac{1}{a(n-1)(n-2)} - \frac{nx - x + 1}{a(x+1)^{n-1}(n-1)(n-2)}.$$

It follows from (23) that

$$\frac{1}{(x_1+1)^{n-1}} + \frac{x_1}{a} - 1 = \frac{1}{(x_2+1)^{n-1}} + \frac{x_2}{a} - 1, \tag{24}$$

(25)
$$\frac{1}{(2-n)}\frac{1}{(x_1+1)^{n-2}} + \frac{1}{(n-1)}\frac{1}{(x_1+1)^{n-1}} = \frac{1}{(2-n)}\frac{1}{(x_2+1)^{n-2}} + \frac{1}{(n-1)}\frac{1}{(x_2+1)^{n-1}}.$$

From (24) we have

$$\frac{1}{(x_1+1)^{n-1}} - \frac{1}{(x_2+1)^{n-1}} = -\frac{x_1 - x_2}{a} \tag{26}$$

$$\Leftrightarrow \quad \frac{1}{(x_1+1)^{n-2}} = -\frac{(x_1+1)(x_1-x_2)}{a} + \frac{x_1+1}{(x_2+1)^{n-1}}. \tag{27}$$

Furthermore it follows from (25) and (26) that

$$\begin{aligned}\frac{1}{(2-n)}\left(\frac{1}{(x_1+1)^{n-2}}-\frac{1}{(x_2+1)^{n-2}}\right) &= \frac{1}{(n-1)}\left(\frac{1}{(x_2+1)^{n-1}}-\frac{1}{(x_1+1)^{n-1}}\right)\\ &= \frac{x_1-x_2}{a(n-1)}. \end{aligned} \tag{28}$$

Moreover we calculate from (27) and (28) that

$$\begin{aligned}
&\frac{1}{(2-n)}\left(-\frac{(x_1+1)(x_1-x_2)}{a}+\frac{x_1+1}{(x_2+1)^{n-1}}-\frac{1}{(x_2+1)^{n-2}}\right)=\frac{x_1-x_2}{a(n-1)},\\
&\frac{1}{(2-n)}\left(-\frac{(x_1+1)(x_1-x_2)}{a}+\frac{1}{(x_2+1)^{n-1}}(x_1-x_2)\right)=\frac{x_1-x_2}{a(n-1)},\\
&\frac{1}{(2-n)}\left(-\frac{(x_1+1)}{a}+\frac{1}{(x_2+1)^{n-1}}\right)=\frac{1}{a(n-1)},\\
&x_1=\frac{a}{(x_2+1)^{n-1}}-\frac{1}{n-1}.
\end{aligned} \tag{29}$$

Substituting (29) into (24) we have

$$\frac{1}{(\frac{a}{(x_2+1)^{n-1}}+\frac{n-2}{n-1})^{n-1}}-\frac{1}{a(n-1)}-\frac{x_2}{a}=0. \tag{30}$$

Let

$$H(x,a):=\frac{1}{(\frac{a}{(x+1)^{n-1}}+\frac{n-2}{n-1})^{n-1}}-\frac{x}{a}-\frac{1}{a(n-1)}. \tag{31}$$

Then we have that

$$\frac{dH}{dx}(x,a)=\frac{a(n-1)^2}{H_1^n(x,a)}-\frac{1}{a}, \tag{32}$$

where

$$H_1(x,a)=\Big(\frac{a}{(x+1)^{n-1}}+\frac{n-2}{n-1}\Big)(x+1).$$

Now we consider the case $a=1/(n-1)$. From (31) and (32) we get

$$H(x,1/(n-1))=\frac{1}{(\frac{1}{(n-1)(x+1)^{n-1}}+\frac{n-2}{n-1})^{n-1}}-(n-1)x-1$$

and

$$\frac{dH}{dx}(x,1/(n-1))=\frac{n-1}{H_1^n(x,1/(n-1))}-(n-1),$$

where

$$H_1(x,1/(n-1))=\Big(\frac{1}{(n-1)(x+1)^{n-1}}+\frac{n-2}{n-1}\Big)(x+1).$$

Then we claim that

$$\frac{dH}{dx}(x,1/(n-1))<0.$$

Actually, we have

$$\begin{aligned}\frac{dH_1}{dx}(x,1/(n-1)) &= \Big(\frac{1}{(n-1)(x+1)^{n-2}}+\frac{n-2}{n-1}(x+1)\Big)' \\ &= \frac{n-2}{n-1}\Big(\frac{-1}{(x+1)^{n-1}}+1\Big)>0,\end{aligned}$$

for $n \geq 3$ and $x \geq 0$, implying $\min\{H_1(x,1/(n-1))\}_{x\geq 0} = H_1(0,1/(n-1)) = 1$. Thus we get $H'(x,1/(n-1)) \leq 0$ and then

$$\max\{H(x,1/(n-1))\}_{x\geq 0} = H(0,1/(n-1)) = 0.$$

In other words equation (30) has no solutions for $x_2 > 0$, and then equations (23) have no solutions $\{x_1, x_2\}$ such that $-1 < x_1 < 0 < x_2$ if $n \geq 3$ and $a = 1/(n-1)$. Thus from continuity we have $F(x_1) > F(x_2)$, or $F(x_1) < F(x_2)$ if $G(x_1) = G(x_2)$. Moreover we have that $F(0) = 0$ and $xg(x) > 0$. Therefore by Proposition 2.1 of [4], system (21) has no periodic orbits for $a = 1/(n-1)$.

Now consider the case $a < 1/(n-1)$. When $a \leq 0$, either the unique equilibrium $(1,1)$ of system (1) is a saddle or the system has an invariant line through equilibrium $(1,1)$, which implies non-existence of periodic orbits. The vector field of equivalent system (22) of system (21) is a generalized rotated vector field with respect to a for $x > -1$ and $a > 0$ by the proof of Proposition 5. Moreover the amplitude of the stable or unstable limit cycle surrounding the origin of (21) varies monotonically with respect to a. Assume that system (21) exhibits limit cycles for $a = a_0 \in (0, 1/(n-1))$, where γ is the innermost limit cycle. Since the origin of (21) is stable, then γ is internally unstable. Note that the amplitude of an unstable limit cycle decreases as a increases by [16, Theorem 3.5, Chapter 4]. When a increases from $a = a_0$ to $a = 1/(n-1)$, the origin keeps stability. Therefore, γ does not disappear for $a = 1/(n-1)$. This is a contradiction to our above analysis as $a = 1/(n-1)$, and the proof is completed. ☐

When

$$a = a_n := \frac{2^n-1}{2^n-2},$$

we give the following lemma for the region where periodic orbits exist. Obviously, $a_n > 1$ for $n \geq 3$.

Lemma 7. *When $a = a_n$ for $n \geq 3$, periodic orbits of system* (21) *only exist in the strip $x \in (-1, 1.6)$.*

Proof. Note that any periodic orbit of system (1) must lie in the first quadrant and the y-axis of system (1) is changed into the line $y = x-1$ of system (21). Therefore, the periodic orbits of system (21) cannot intersect the line $y = x - 1$.

Assume that Γ is a periodic orbit of system (21) and Γ intersects with the curve $y = F(x)$ at the point $(x^*, F(x^*))$ in the right half-plane. Then $x \leq x^*$ as $(x,y) \in \Gamma$. If $x^* \leq 1$, we have that Γ lies in the strip $x \in (-1, 1]$ and this lemma is proven. In the following, we consider the case $x^* > 1$.

Let $y = \tilde{y}(x) < F(x)$ denote the orbit segment of Γ as $0 \leq x \leq x^*$. For $x \geq 1$ and $a = a_n$, we have that $\tilde{y}(x) > x - 1$ and

$$\frac{d\tilde{y}(x)}{dx} = \frac{g(x)}{F(x)-\tilde{y}(x)} > \frac{g(x)}{F(x)-x+1} \geq x, \tag{33}$$

which implies

$$\tilde{y}(x) < \frac{1}{2}(x^2 - (x^*)^2) + \tilde{y}(x^*) \tag{34}$$

for $1 \leq x < x^*$. Actually, the inequality $g(x)/(F(x)-x+1) \geq x$ in (33) is equivalent to the inequality

$$a \geq \frac{\varphi_1(x)}{\varphi_2(x)}, \tag{35}$$

where $\varphi_1(x) = x - \frac{1}{(x+1)^n}$ and $\varphi_2(x) = x - \frac{1}{(x+1)^{n-1}}$. Notice that for $x \geq 1$ we get the maximum value of the positive function $\varphi_1(x) - \varphi_2(x)$ at $x = 1$, implying that the function $\varphi_1(x)/\varphi_2(x)$ has its maximum value a_n also at $x = 1$ and the inequality (33) is obtained.

We can find that the curve

$$\Upsilon : y = \frac{1}{2}x^2 - \frac{1}{2}$$

is tangent to the line $y = x - 1$ at the point $(1, 0)$. Moreover, the curves Υ and $y = F(x)$ have a unique intersection point at $(\tilde{x}^*, \tilde{y}^*)$ for $x \geq 1$. Actually, from $F(x) = y = \frac{1}{2}x^2 - \frac{1}{2}$ and $a = a_n$ we get that

$$P(x) := \frac{1}{(x+1)^{n-1}} + \frac{(2^n - 2)x}{2^n - 1} - \frac{x^2}{2} - \frac{1}{2} = 0.$$

Applying

$$P'(x) = (1-n)\frac{1}{(x+1)^n} + \frac{2^n - 2}{2^n - 1} - x < 0$$

for $x \geq 1$, $P(1) = (2^{n-1} - 1)/(2^{2n-1} - 2^{n-1}) > 0$ and $P(1.6) = 2.6^{1-n} - 0.18 - 1.6/(2^n - 1) < 0$, we get a unique value $\tilde{x}^*$ such that $P(\tilde{x}^*) = 0$ and $1 < \tilde{x}^* < 1.6$.

In the following, we prove that $x^* < \tilde{x}^*$. Otherwise, if $x^* \geq \tilde{x}^*$, we have $-(x^*)^2/2 + \tilde{y}(x^*) \leq -1/2$, inducing

$$\tilde{y}(x) < \frac{1}{2}(x^2 - (x^*)^2) + \tilde{y}(x^*) \leq \frac{1}{2}x^2 - \frac{1}{2}$$

for $x = 1$ by (34). Hence, we can obtain that the curve $y = \tilde{y}(x)$ has intersection points with the line $y = x - 1$, indicating a contradiction. Therefore, $x^* < \tilde{x}^*$. In other words, the periodic orbits of system (21) must lie in the region $x \in (-1, 1.6)$. The lemma is proven. □

Proposition 8. *System* (21) *has no periodic orbits when* $a \geq a_n$.

Proof. By [1, 5], there is a $a^* \in (1, 3)$ such that system (21) has no periodic orbits for $a > a^*$ when $n = 2$. When $n = 3$ and $a = a_n$, we can calculate numerically that the function $H(x, a)$ in (31) has not zeros as x lies between $\tilde{x}^*$ and the positive zero of $F(x)$, where the curves Υ and $y = F(x)$ intersect at $(\tilde{x}^*, \tilde{y}^*)$ for $x \geq 1$, as shown in the proof of Lemma 7. By Proposition 2.1 of [4] system (21) has no periodic orbits when $n = 3$ and $a = a_n$. In the rest of this proof, we only need consider the case $n \geq 4$.

From (32), we have that

$$\frac{dH}{dx}(x,a)=\frac{a(n-1)^2}{H_1^n(x,a)}-\frac{1}{a}$$

and

$$\frac{dH_1(x,a)}{dx}=-\frac{a(n-2)}{(x+1)^{n-1}}+\frac{n-2}{n-1}\begin{cases}<0, & 0<x<x_0,\\ =0, & x=x_0,\\ >0, & x>x_0,\end{cases}$$

where

$$x_0=\sqrt[n-1]{a(n-1)}-1>0$$

as $a\geq a_n>1$. Further, we obtain

$$\min_{x>-1}\{H_1(x,a)\}=H_1(x_0,a),$$

implying

$$\max_{x>-1}\{\frac{dH}{dx}(x,a)\}=\frac{dH}{dx}(x_0,a)=\frac{n-1}{\sqrt[n-1]{a(n-1)}}-\frac{1}{a}>0.$$

Moreover, we have

$$\lim_{x\to+\infty}\frac{dH}{dx}(x,a)<0$$

and

$$\frac{dH}{dx}(0,a)=\frac{a(n-1)^2}{\left(a+\dfrac{n-2}{n-1}\right)^n}-\frac{1}{a}=\frac{H_2(a)}{a\left(a+\dfrac{n-2}{n-1}\right)^n}<0,$$

where

$$H_2(a)=a^2(n-1)^2-\left(a+\frac{n-2}{n-1}\right)^n,$$

because

$$\begin{aligned}\frac{dH_2(a)}{da}&=2a(n-1)^2-n\left(a+\frac{n-2}{n-1}\right)^{n-1}<\frac{dH_2(a)}{da}|_{a=1}<0,\\ \frac{d^2H_2(a)}{da^2}&=2(n-1)^2-n(n-1)\left(a+\frac{n-2}{n-1}\right)^{n-2}<\frac{d^2H_2(a)}{da^2}|_{a=1}<0,\\ \frac{d^3H_2(a)}{da^3}&=-n(n-1)(n-2)\left(a+\frac{n-2}{n-1}\right)^{n-3}<0.\end{aligned}$$

Notice that

$$H(0,a)=\frac{1}{(a+\dfrac{n-2}{n-1})^{n-1}}-\frac{1}{a(n-1)}<0$$

from (31) and a similar discussion as $dH(0,a)/dx<0$.

If the inequality $H(x,a)<0$ always holds as $a\geq 1$, we can get that system (21) has no periodic orbits by Proposition 2.1 of [4] and a similar discussion as the proof of Proposition 6. So, in the following, we consider the case that there exists a value $x_*>x_0$ such that

$$\max_{x>-1}\{H(x,a)\}=H(x_*,a)>0.$$

Without loss of generality we assume that

$$H(x_0,a)=1-\frac{x_0}{a}-\frac{1}{a(n-1)}>0.$$

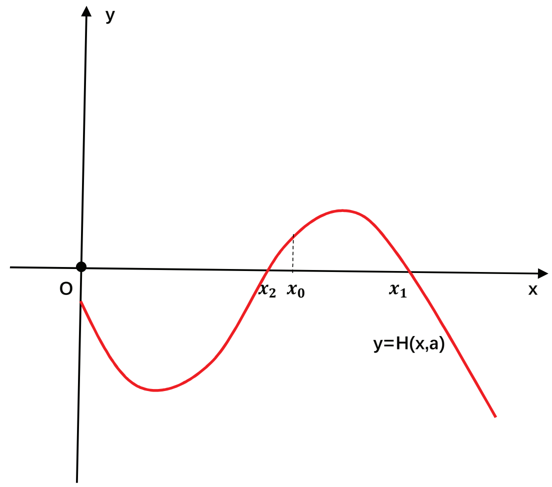

Figure 3. $y = H(x,a)$

When $H(x_0,a) \le 0$, we can research by a similar way. Thus, there are values x_1 and x_2 such that $x_1 > x_2 > 0$ and $H(x,a) > 0$ (resp. < 0) for $x \in (x_2,x_1)$ (resp. $x \in (0,x_2) \cup (x_1,+\infty)$) when $n \ge 4$, as shown in Figure 3.

The function $F'(x)$ has a unique zero at $x = \sqrt[n]{a(n-1)} - 1 > 0$ and there exists a unique positive x_3 such that $F(x_3) = 0$ and $F(x) > 0$ (resp. < 0) for $x \in (-1,0) \cup (x_3,+\infty)$ (resp. $x \in (0,x_3)$). Letting $z_0 = \sqrt[n-1]{a(n-1)}$ we can find that $z_0 > 1$ and

$$\begin{aligned} F(x_0) &= \frac{1}{a(n-1)} + \frac{\sqrt[n-1]{a(n-1)} - 1}{a} - 1 \\ &= \frac{1}{z_0^{n-1}} - 1 + \frac{z_0 - 1}{a} \\ &= (1 - z_0)\left(\frac{1 + z_0 + \cdots + z_0^{n-2}}{z_0^{n-1}} - \frac{1}{a}\right) \\ &= (1 - z_0)\left(\frac{1 + z_0 + \cdots + z_0^{n-2}}{z_0^{n-1}} - \frac{n-1}{z_0^{n-1}}\right) < 0, \end{aligned}$$

indicating $x_0 < x_3$.

Consider the case $a = a_n > 1$. Calculating the equation $F(x_3) = 0$ from (21), we can let

$$x_3 = \frac{(n - k(n))a_n}{n-1}, \tag{36}$$

where $k \in (1, 1.5)$ and $k = k(n)$ is decreasing in n. Specially, $x_3 = 1$ as $n \to +\infty$ and $x_3 \approx 0.92$ as $n = 4$. Further, by (36) we have that

$$\begin{aligned} H(x_3,a_n) &= \frac{1}{(1 + \frac{(k-1)a_n - 1}{n-1})^{n-1}} + \frac{k - n - 1/a_n}{n-1} \\ &= \frac{1 - H_3(n)}{(1 + \frac{(k-1)a_n - 1}{n-1})^{n-1}} > 0, \end{aligned}$$

where

$$\begin{aligned} H_3(n) &= (1-\frac{k-1-1/a_n}{n-1})(1+\frac{(k-1)a_n-1}{n-1})^{n-1} \\ &= \Big(1+(1-\frac{1}{a_n})\frac{((k-1)a_n-1)}{n-1}-\frac{((k-1)a_n-1)^2}{a_n(n-1)^2}\Big)(1+\frac{(k-1)a_n-1}{n-1})^{n-2} \\ &< 1, \end{aligned}$$

because $a_n > 1$ and $(k-1)a_n - 1 < 0$. Thus, we can get $x_0 < x_3 < x_1$. From Lemma 7, any limit cycle of system (21) must lie in the region $\{(x, y) \in \mathbb{R}^2 | x < 1.6\}$. Hence, we consider the solution of (23) satisfying $x_3 < x < 2$ for $x > 0$. In order to prove $H(x, a_n) > 0$ for $x_3 < x < 2$, we only need to show $H(2, a_n) > 0$ by Figure 3. We can compute that

$$\frac{\partial(aH(2,a))}{\partial a} = \frac{1-\dfrac{1}{1/(n-1)+3^{n-1}(n-2)/(a(n-1)^2)}}{\Big(a3^{1-n}+(n-2)/(n-1)\Big)^{n-1}} > 0$$

as $n > 3$ and $a \in [1, a_n + \varepsilon]$ for small enough $\varepsilon > 0$. It implies that the function $aH(2, a)$ is increasing in a for $a \in [1, a_n + \varepsilon]$. Thus from $H(2, 1) > 0$ we can get $H(2, a_n) > 0$. Moreover, we have

$$H(2,1) = \frac{1}{(3^{1-n}+\dfrac{n-2}{n-1})^{n-1}} - 2 - \frac{1}{n-1} = \frac{1-\hat{H}(n)}{(3^{1-n}+\dfrac{n-2}{n-1})^{n-1}},$$

where

$$\hat{H}(n) = (2+\frac{1}{n-1})(3^{1-n}+\frac{n-2}{n-1})^{n-1}.$$

In the following, we prove that $\hat{H}(n) < 1$ and then $H(2, 1) > 0$. First, we get

$$\begin{aligned} H_4(n) &= \ln \sqrt[n-1]{\hat{H}(n)} \\ &= \frac{1}{n-1}\ln(2+\frac{1}{n-1}) + \ln(3^{1-n}+1-\frac{1}{n-1}) \\ &= u\ln(2+u) + \ln(3^{-\frac{1}{u}}+1-u), \end{aligned}$$

where $u = \frac{1}{n-1} \in (0, \frac{1}{3}]$. Noticing that

$$\hat{H}(4) \leq 0.814 < 1,\ \hat{H}(5) \leq 0.76 < 1,\ \hat{H}(6) \leq 0.74 < 1,$$
$$\hat{H}(7) \leq 0.738 < 1,\ \hat{H}(8) \leq 0.736 < 1$$

and $\hat{H}(n) \to \frac{2}{e}$ as $n \to +\infty$. We can only consider the case $n \geq 9$ and $u \in (0, \frac{1}{8}]$ for proving the inequality $\hat{H}(n) < 1$. Moreover,

$$\begin{aligned} H_4'(n) &= \ln(2+u) + \frac{u}{2+u} + \frac{\frac{3^{-\frac{1}{u}}\ln 3}{u^2}-1}{3^{-\frac{1}{u}}+1-u} \\ &= \xi_1(u) + \xi_2(u), \end{aligned}$$

where

$$\xi_1(u) = \ln(2+u) + \frac{u}{2+u} - 1,$$
$$\xi_2(u) = \frac{3^{-\frac{1}{u}} - u + \frac{3^{-\frac{1}{u}}\ln 3}{u^2}}{3^{-\frac{1}{u}} + 1 - u} =: \frac{\xi_3(u)}{3^{-\frac{1}{u}} + 1 - u}.$$

In addition, for $u \in (0, \frac{1}{8}]$ we get

$$\begin{aligned}\xi_1'(u) &= \frac{1}{2+u} + \frac{2}{(2+u)^2} > 0,\\ \xi_3'(u) &= -1 + \frac{3^{-\frac{1}{u}}\ln 3}{u^2} + \frac{3^{-\frac{1}{u}}\ln^2 3}{u^4} - 2\frac{3^{-\frac{1}{u}}\ln 3}{u^3}\\ &= (-1 + \frac{3^{-\frac{1}{u}}\ln^2 3}{u^4}) + \frac{3^{-\frac{1}{u}}\ln 3}{u^2}(1 - \frac{2}{u}) < 0.\end{aligned}$$

It follows that

$$\xi_1(u) \le \xi_1(\frac{1}{8}) = \ln(17/8) - 16/17 < 0,$$
$$\xi_3(u) \le \lim_{u\to 0+} \xi_3(u) = 0,$$

indicating $H_4'(n) < 0$ and then $\hat{H}(n) < 1$. Therefore, we get $H(2,1) > 0$ and then $H(2,a_n) > 0$, implying that $H(x,a_n) > 0$ for $x \in [x_3, 2]$. In other words, the equations (23) have no roots for $a = a_n$. By Proposition 2.1 of [4] system (21) has no periodic orbits when $n \ge 3$ and $a = a_n$. Since the amplitude of the stable or unstable limit cycle of system (21) varies monotonically in a by Proposition 5, system (21) has no periodic orbits when $n \ge 3$ and $a \ge a_n$. $\square$

When $a \in (1/(n-1), a_n)$ we will study the existence and uniqueness of limit cycle in the following proposition.

Proposition 9. *There exists a constant $a^* \in (1/(n-1), a_n)$ such that system (21) has a unique limit cycle when $a \in (1/(n-1), a^*)$ and no periodic orbits when $a \in (a^*, +\infty)$. Moreover the limit cycle is stable and hyperbolic, and the amplitude of the limit cycle increases as a increases.*

Proof. By [16, Theorem 3.5, Chapter 4] and Proposition 5 the amplitude of the stable limit cycle surrounding the origin of (21) is monotonous with respect to a as $x > -1$ and $a > 0$. From [11] the Hopf bifurcation occurs and a stable limit cycle appears when a varies from $1/(n-1)$ to $1/(n-1) + \epsilon$, where $\epsilon > 0$ is small. The amplitude of the stable limit cycle is sufficiently small for small enough $\epsilon > 0$. Thus the amplitude of the stable limit cycle increases as a increases.

On the other hand, system (21) has no periodic orbits when $a \in (-\infty, 1/(n-1)] \cup [a_n, +\infty)$ by Propositions 6 and 8, and has a unique finite equilibrium at the origin. Therefore, there exists $a^* \in (1/(n-1), a_n)$ such that the amplitude of the stable limit cycle approaches infinity when $a = a^* - \epsilon$ for sufficiently small $\epsilon > 0$ by the continuity of the vector field and the monotonous properties of amplitude of the stable limit cycle in parameter a.

In the following we will prove the uniqueness of periodic orbits when $a \in (1/(n-1), a^*)$. From system (21) we calculate

$$\frac{d\big(f(x)/g(x)\big)}{dx} = a\frac{n-1}{x^2} + \frac{((n-1)x-1)(x+1)^{n-1}}{x^2} = \frac{Q(x)}{x^2}, \tag{37}$$

where $Q(x) := a(n-1) + ((n-1)x-1)(x+1)^{n-1}$, and

$$Q'(x) := n(n-1)x(x+1)^{n-2}. \tag{38}$$

Notice that $Q(0) = a(n-1) - 1 > 0$ since $a > 1/(n-1)$. Moreover, it is easy to see that $Q(x) > 0$ when $x \in [1/(n-1), +\infty)$. When $x \in [0, 1/(n-1))$, we get $Q'(x) \geq 0$ from (38). Thus

$$\min\{Q(x)\}_{x\in[0,1/(n-1))} = Q(0) > 0,$$

inducing $Q(x) > 0$ in this case. When $x \in (-1, 0]$ we get $Q'(x) \leq 0$ from (38). Thus,

$$\min\{Q(x)\}_{x\in(-1,0]} = Q(0) > 0,$$

also inducing $Q(x) > 0$ in this case. Therefore we obtain $Q(x) > 0$ when $x \in (-1, +\infty)$ and then the function $f(x)/g(x)$ is increasing from (37) when $x \in (-1, 0) \cup (0, +\infty)$.

Moreover we can verify that $xg(x) > 0$ for all $x \neq 0$, $F(0) = 0$, $F'(0) \neq 0$ and $f(x)/g(x)$ is not a constant when $|x|$ is small. Hence all conditions in Theorem 3 hold and we can get that system (21) has at most one limit cycle in $(-1, +\infty)$. Moreover the limit cycle is stable if it exists. Notice that for our system (21), the function $\phi(y) = y$ in the general system (2). The proposition is proven. □

From Propositions 5-9 we can obtain Theorem 1.

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