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ORIGINAL PAPER

Nilpotent bi-center in continuous piecewise $\mathbb{Z}_2$-equivariant cubic polynomial Hamiltonian systems

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Abstract One of the classical and difficult problems in the theory of planar differential systems is to classify their centers. Here we classify the global phase portraits in the Poincaré disk of the class continuous piecewise differential systems separated by one straight line and formed by two cubic Hamiltonian systems with nilpotent bi-center at $(\pm 1, 0)$. The main tools for proving our results are the Poincaré compactification, the index theory, and the theory of sign lists for determining the exact number of real roots or negative real roots of a real polynomial in one variable.

Keywords Nilpotent · bi-center · Hamiltonian · Phase portrait

Mathematics Subject Classification 34C25

1 Introduction and statement of the main results

The problem of distinguishing focus and center in the qualitative theory of planar differential equations is known as the *center-focus problem*. This classical problem started investigation by Poincaré [43,44] in 1881 and Dulac [20] in 1908, and nowadays the center-focus problem remains as one of main subjects in the qualitative theory of planar polynomial differential systems.

We say that a singular point p of a planar differential system is a *center* if it has a neighborhood U filled with periodic orbits with the unique exception of this singular point.

If a planar polynomial differential system has a *linear type center* or a *nilpotent center* or a *degenerate center* at the origin of coordinates, after making a time rescaling and a linear change of variables, this differential system can be written as

$$(\dot{x}, \dot{y}) = \left\{ \begin{array}{c} (-y,\ x) \\ (y,\ 0) \\ (0,\ 0) \end{array} \right\} + (f(x, y), g(x, y)), \qquad (1)$$

respectively. Here the dot denotes derivative with respect to time t, and $f(x, y)$ and $g(x, y)$ are real polynomials without constant and linear terms.

The center-focus problem for quadratic polynomial differential systems has been solved, see [6,7,20,29,30,45,52]. There are partial results in the classification of the centers for cubic polynomial differential systems, see for instance [15,38,49,53–55], but the center-focus problem for general cubic polynomial differential systems still remains open.

Recently Colak *el at.* [17,18] studied the phase portraits of some cubic Hamiltonian differential systems with a linear type center or a nilpotent center at the

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