# *QUALITATIVE ANALYSIS OF THE ANISOTROPIC KEPLER PROBLEM*

*Memòria presentada per*
*Josefa Casasayas i Mas*
*per aspirar al grau de*
*Doctor en Ciències,*
*Facultat de Matemàtiques*
*(per la Universitat de Barcelona).*



*Departament de Teoria de Funcions*
*Facultat de Matemàtiques*
*Universitat de Barcelona*

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## 0 INTRODUCTION

The anisotropic Kepler problem was introduced by Gutzwiller as a classical mechanical system which approximates the following quantum mechanical system: the study of bound states of an electron near a donor impurity of a semiconductor. For more details on the physical connections we refer to [G1,2,3,4,5] .

As it is known the anisotropic Kepler problem exhibits many qualitative phenomena of interest in the theory of differential equations such as non- integrability and chaotic behaviour, see [G5,6] and [D2,3 ] . This paper is essentially devoted to the qualitative analysis of this problem, and also surveys the recent techniques and results from it.

The anisotropic Kepler problem is a one parameter family (of parameter $\mu$ ) of Hamiltonian systems with two degrees of freedom. The configuration space for the system is $Q=\mathbb{R}^2\setminus\{0\}$ with coordinates $q=(q_1,q_2)$, and the phase space is the tangent bundle to Q which we denote by $TQ=(\mathbb{R}^2\setminus\{0\})\times\mathbb{R}^2$. We use coordinates $p=(p_1,p_2)$ in each fiber. Then the Hamiltonian is,

$$H(q,p) = (p^t M^{-1} p)/2 + V(q),$$

where H is defined on TQ, the mass matrix $M^{-1}=\begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix}$ , and the potential energy $V(q)=-1/\|q\|$ . The associated Hamilton equations are,

$$\begin{aligned} \dot{q} &= M^{-1}p \ , \\ \dot{p} &= -q/\|q\|^3 \ . \end{aligned} \tag{1}$$

Of course, the Hamiltonian H is an integral of (1). So, orbits of (1) lie on the energy levels of H. In (II.1) we note that it is sufficient to study the cases H=-1, H=0, and H=1.

When $\mu=1$, (1) becomes the Kepler problem, which is an integrable system. It is known that when $\mu>1$ system (1) does not have any real analytic integral independent on the energy (see [D2] and [Mo]).

Note that for $\mu>1$ the $q_2$-axis is a "heavy" axis, this means that the orbits oscillate more and more rapidly about the $q_2$-axis as $\mu$ increases.

For every energy level system (1) has a singularity at q=0. It has been studied by Devaney in [D2,5] by using the blow up techniques of McGehee [Mc] . For non-negative energy levels we have another singularity at $\|q\| = \infty$; again, blow up techniques can be applied, see Lacomba-Simó in [LS] .

The blow up method replaces the singularity by an invariant boundary manifold and the system extends over it. So, the knowledge of the flow on this boundary allows to study the behaviour of the orbits near the singularity. Thus, the invariant boundaries glued to q=0 and $\|q\|=\infty$ are called the collision manifold and the infinity manifold, respectively.

In system (1) the blow up of the singularities is essential in order to make the qualitative analysis of the flow. Thus, in Chapter I we describe the global behaviour of the orbit structure of the Kepler problem by taking into account the blow up of the singularities.

The first part of Chapter II is also devoted to the singularities of the anisotropic Kepler problem. In the remaining part we analize the homothetic orbits. Since these orbits are heteroclinic and transversal, they play a major part in the qualitative analysis. Transversality was proved by Devaney for negative energy levels [D4] ; we extend it to non-negative energy levels in (II.6). In (II.7) we give the global behaviour of the flow on the collision manifold for all $\mu>1$. This improves the results of Devaney in [D2] .

As it was observed in [LS] the global orbit structure in the zero energy level can be obtained from the global flow on the collision manifold. This is shown in (III.1). The asymptotic behaviour of the orbits in the positive energy levels is given in (III.2).

In the non-negative energy levels we do not have recurrent orbits. So, the interesting case is H<0. In order to describe recurrent motions it is useful to introduce symbolic dynamics.

Gutzwiller and Devaney use symbolic dynamics to classify the possible types of orbits in the anisotropic Kepler problem, see [D5,pp.292-297] . As they said, their symbols do not take into account the symmetries of the problem. In this paper symbolic dynamics includes the symmetries, see Theorems IV.17 and IV.17' given in (IV.7).

Proofs of these theorems need the qualitative analysis of the intersection of the stable and unstable invariant manifolds of the equilibrium points of the problem with the surface of section $d/dt(\|q\|)=0$. Such an analysis is the key point of this study and it is made in the first five sections of Chapter IV.

In fact, theorems of Gutzwiller, Devaney, IV.17 and IV.17' prove the existence of a subshift with an infinite alphabet as a "subsystem" of an adequate Poincaré map for $\mu>9/8$.

In Chapter V we describe the transition from the integrable case $\mu=1$ to the chaotic one $\mu>9/8$. That is, (V.2) shows that the chaotic behaviour observed for $\mu>9/8$ is completely lost when $1\leq\mu\leq 9/8$.

In Chapter VI we study the symmetric periodic orbits with respect to the six symmetries of the problem. For the simplest ones we describe their geometry, see Theorem VI.5. Also, for each periodic sequence of the subshift given in Theorem IV.17 and IV.17', we show the existence of a symmetric periodic orbit which realizes it, see Theorem VI.6.

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