DYNAMICS OF A FAMILY OF RATIONAL OPERATORS OF ARBITRARY DEGREE

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Abstract. In this paper we analyse the dynamics of a family of rational operators coming from a fourth-order family of root-finding algorithms. We first show that it may be convenient to redefine the parameters to prevent redundancies and unboundedness of problematic parameters. After reparametrization, we observe that these rational maps belong to a more general family $O_{a,n,k}$ of degree $n+k$ operators, which includes several other families of maps obtained from other numerical methods. We study the dynamics of $O_{a,n,k}$ and discuss for which parameters n and k these operators would be suitable from the numerical point of view.

1. Introduction

Iterative methods are the most usual tool to approximate solutions of non linear equations. These methods require at least one initial estimate close enough of the solution sought. It is known that the methods converge if the initial estimation is chosen suitably. Hence, the search of such initial conditions has became an important part in the study of iterative methods. To achieve this goal we analyse these methods as discrete dynamical systems.

The application of iterative methods to find solutions of equations of the form $f(z) = 0$, where $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, gives rise to discrete dynamical systems given by the iteration of rational functions. The best known numerical algorithm is Newton's method, whose dynamics has been widely studied (see for instance [6, 17]). Indeed, there are several results about the dynamical plane as well as the parameter plane of Newton's method applied to some concrete families of polynomials. The most studied case is Newton's method of cubic polynomials $q(z) = z(z-1)(z-\alpha)$, $\alpha \in \hat{\mathbb{C}}$ (see for instance [16, 19] and references therein).

Recently, this dynamical study has been enlarged to other numerical methods (see, for example, [9], [10], [11], [12], [13] and references therein). The dynamical properties related to an iterative method give important information about its stability. In recent studies, many authors (see [1], [7], [8], [13], [14], for example) have found interesting results from a dynamical point of view. One of the main interests in these papers has been the study of the parameter spaces associated to the families of iterative methods applied on low degree polynomials, which allows to distinguish the different dynamical behaviour.

In this paper, we consider an optimal fourth-order family of methods presented by R. Behl in [4], whose dynamics is partially studied by K. Argyros and A. Magreñán in [1]. The family of methods is given by

$$\begin{aligned} y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= x_n - \frac{((b^2-22b-27)f'(x_n) + 3(b^2+10b+5)f'(y_n))f(x_n)}{2(bf'(x_n)+3f'(y_n))(3(b+1)f'(y_n)-(b+5)f'(x_n))}, \end{aligned}$$

where b is a complex parameter. When applying these methods on quadratic polynomials of the form $z^2 + c$ (compare with Section 2) we obtain an operator which is conjugate to

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$$O_b(z) = z^4 \frac{-11 - 6b + b^2 + (-3 + 2b + b^2)z}{-3 + 2b + b^2 + (-11 - 6b + b^2)z}.$$

The goal of the paper is to analyse the main dynamical properties of this operator. Firstly, we provide a short introduction to complex dynamics. We refer to [5, 3, 15] for a more detailed introduction to the topic.

We consider the dynamical system given by the iteration of a rational map $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, where $\widehat{\mathbb{C}}$ denotes the Riemann sphere. A point $z_0 \in \widehat{\mathbb{C}}$ is called *fixed* if $R(z_0) = z_0$, and *periodic* of minimal period $p > 1$ if $R^p(z_0) = z_0$ and $R^l(z_0) \neq z_0$ for $1 \leq l < p$. Fixed points are classified depending on their multiplier $\lambda = R'(z_0)$. A fixed point z_0 is called *attractor* if $|\lambda| < 1$ (*superattractor* if $\lambda = 0$), *repulsor* if $|\lambda| > 1$, and *indifferent* or neutral if $|\lambda| = 1$. An indifferent fixed point is called *rationally indifferent* (or *parabolic*) if $\lambda = e^{2\pi i p/q}$, where $p, q \in \mathbb{N}$. Each attractive or rationally indifferent point z_0 has associated a *basin of attraction*, denoted by $\mathcal{A}(z_0)$, consisting of points $z \in \widehat{\mathbb{C}}$ whose orbit converges to z_0 under iteration of $R(z)$. The same classification can be used for periodic points of any given period p since they are fixed points of the map $R^p(z)$.

The dynamics of $R(z)$ provides a completely invariant partition of the Riemann sphere. The *Fatou set*, $\mathcal{F}(R)$, of a rational map $R(z)$ consists of the points $z \in \widehat{\mathbb{C}}$ such that the family of iterates, $\{R(z), R^2(z), \ldots, R^n(z), \ldots\}$, is normal in some open neighborhood U of z. Its complement, the *Julia set* $\mathcal{J}(R)$, consists of the points where the dynamics of $R(z)$ is chaotic. The Fatou set is open and the Julia set is closed. The connected components of the Fatou set are called *Fatou components* and are mapped among themselves under iteration. Let us notice that, while attractive fixed points are in the Fatou set, rationally indifferent fixed points are in the Julia set (see [15]). The Classification Theorem of Fatou components (see, for example, [3, 15]) establishes that all periodic Fatou components of rational maps are either basins of attraction of attractive or parabolic points, or simply connected rotation domains (Siegel disks) or doubly connected rotation domains (Herman rings).

A point $c \in \widehat{\mathbb{C}}$ is called a *critical point* of $R(z)$ if $R'(c) = 0$. Critical points are relevant in holomorphic dynamics since all periodic Fatou components are related to critical points. In particular, the basins of attraction of attracting and rationally indifferent points contain, at least, one critical point (see, for example, [3, 15]).

When we apply a numerical method to find the solutions of a given equation we obtain an iterative method. If the equation is a polynomial, the operator associated with the iterative method is a rational map. The solutions of the equation are attractive fixed points of this map. However, it may happen that an initial condition converges under iteration to an attracting cycle different from the solution of the equation. In that case, we consider that the numerical method fails. We call such attracting cycles *strange attractors*. Hence, when we study if a numerical method works adequately we need to analyse the existence of strange attractors. This can be done by studying the asymptotic behaviour of the iterates of the critical points. If such an strange attractor exists, the orbit of, at least, a critical point will accumulate on it. Therefore, in order to draw parameter planes we can iterate the different critical points and analyse their asymptotic behaviour. In Figure 1 we show the parameter plane of the operator O_b. In this figure we plot in black parameters for which a critical orbit does not converge to any of the solutions of the original equation and, hence, there may be an strange attractor. See Section 2.3 for a more detailed explanation on how this figure is produced.

As we mentioned before, the main goal of the paper is to study the dynamics of the operator O_b. To do so, we want to understand the parameter plane shown in Figure 1. In Section 2 we carry out this study. We obtain all the *strange fixed points*, that is, all fixed points which do not correspond to the solutions of the equation. We also find the analytic expressions of the regions in the parameter plane of the operator O_b where these strange fixed points are attractive and we locate these regions in the parameter space.

Once this initial study is done, we focus on two unwanted properties of the parameter plane of the family O_b. First, due to the fact that the coefficients of the rational map are quadratic (there are terms in b^2) two different parameters b_1 and b_2 may lead to the same operator $O_{b_1} = O_{b_2}$. This

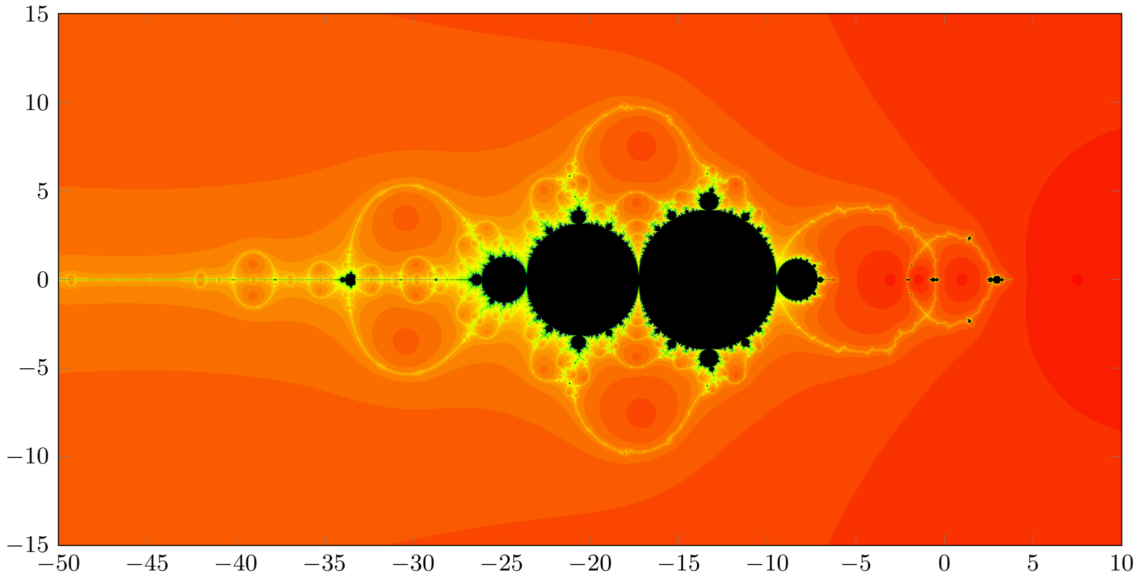


FIGURE 1. Parameter plane of the operator O_b.

is usually an unwanted feature when studying a parameter plane. Because of this, in Section 2.3 we show how to reparametrize this family, obtaining a new operator

$$O_a(z) = z^4 \frac{z-a}{1-az}.$$

The other unwanted feature of the dynamical plane of the operator O_b is the unboundedness of 'bad' parameters. In Figure 1 we can see an 'antenna' of parameters that spreads through the negative real axis for which a critical orbit does not converge to the solutions of the equation. This is an unwanted feature since may leave out of the numerical picture parameters for which relevant dynamics, such as convergence to strange attractors, take place. In Section 3 we prove that the antenna observed in the parameter plane of O_b is actually unbounded. However, this unbounded antenna becomes a bounded set of parameters for O_a (see Proposition 12 and Figure 4). Hence, the reparametrised operator O_a possesses none of the unwanted features of the operator O_b and, therefore, is a much better model to study the dynamics of the family of methods presented by R. Behl in [4]. In Section 3 we also study the dynamics of the operator O_a and analyse the relation between the parameter planes of O_b and O_a.

To finish the paper, in Section 4, we study the dynamics of a generalised version of the operator O_a. We study this generalised operator since O_a is somehow similar to other operators that may be obtained from applying numerical methods to quadratic polynomials. We considered the generalised family of operators

$$O_{a,n,k}(z) = z^n \left(\frac{z-a}{1-az}\right)^k.$$

The map $O_{a,n,k}$ coincides with O_a for $n=4$ and $k=1$. Moreover, for $n=3$ and $k=1$ this operator is obtained from the Chebyshev-Halley family of numerical methods (see [10]). This operator is also obtained from a family of root finding algorithms for $n=6$ and $k=2$ in [2]. For certain combinations of n and k, the dynamics of $O_{a,n,k}$ is very similar to the one of O_a. However, if $n-k \leq 1$ the operator $O_{a,n,k}$ possesses some complicated dynamics which would not be desirable if obtained from a numerical method. We finish the section proving that the operators $O_{a,n,k}$ do not have Herman rings. This is an important characteristic since Herman rings would provide open sets of initial conditions for which the numerical method fails.

2. The optimal fourth-order family $O_b(z)$

In this section we carry out a dynamical study of the optimal fourth-order family of methods presented by R. Behl [4] given by:

$$\begin{aligned} y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= x_n - \frac{((b^2-22b-27)f'(x_n)+3(b^2+10b+5)f'(y_n))f(x_n)}{2(bf'(x_n)+3f'(y_n))(3(b+1)f'(y_n)-(b+5)f'(x_n))} \end{aligned}$$

where b is a complex parameter.

We study the dynamics of this family applied on a degree two polynomial $p(z) = z^2 + c$, $c \in \mathbb{C}$. The operator we obtain by applying the previous family on $p(z)$ is

$$W_b(z) = z\frac{(9+14b+b^2)c^2 - 2(-13+2b+3b^2)cz^2 + (-7-2b+b^2)z^4}{4((1+b)c^2 - (-1+4b+b^2)cz^2 + (-6-b+b^2)z^4)}.$$

This operator is conjugate to

$$O_b(z) = z^4\frac{-11-6b+b^2+(-3+2b+b^2)z}{-3+2b+b^2+(-11-6b+b^2)z} \tag{1}$$

by means of the conjugation map $h(z) = \frac{z+i\sqrt{c}}{z-i\sqrt{c}}$. This conjugation map sends one of the roots of the polynomial $p(z)$ to zero and the other one to infinity. Moreover, $h(\infty) = 1$.

Let us observe that the expression of the operator (1) is simplified when $-3+2b+b^2 = 0$, and when $-3+2b+b^2 = \pm(-11-6b+b^2)$. Then, the operator (1) has degree five, except for the following cases:

- For $b = -3$ or $b = 1$, we have that $O_1(z) = z^3$ and $O_{-3}(z) = z^3$; in these cases, the operator has degree three.
- For $b = -1$ or $b = 1 \pm 2\sqrt{2}$ we have that $O_{-1}(z) = z^4$ and $O_{1\pm2\sqrt{2}}(z) = -z^4$; in these cases, the operator has degree four.

Moreover, there exist values of the parameters for which the operator increases its order of convergence:

Proposition 1. *The operator (1) has order of convergence 5 for $b = 3 \pm 2\sqrt{5}$.*

Proof. If $b = 3 \pm 2\sqrt{5}$, then $-11-6b+b^2 = 0$ and the operator becomes $O_b(z) = z^5$. □

2.1. Fixed points.

The first step in the dynamical study of operator $O_b(z)$ consists of calculating its fixed and critical points. As we will see, the number and the stability of the fixed and critical points depend on the parameter b. It is known that any rational map of degree d has $d+1$ fixed points (counting multiplicity) and $2d-2$ critical points (counting multiplicity) (see [3], for example). Therefore, our operator has 6 fixed and 8 critical points, except for the values of the parameters studied above that lead to an operator of lower degree.

Fixed points satisfy $O_b(z) = z$. The fixed points obtained are $z = 0$, $z = \infty$, $z = 1$ (if $b \neq 1 \pm 2\sqrt{2}$), $z = -1$ (if $b \neq -1$) and

$$z_\pm = \frac{11+6b-b^2 \pm \sqrt{(5+10b+b^2)(17+2b-3b^2)}}{2(b-1)(b+3)}$$

if $b \neq -3$ and $b \neq 1$.

The points $z = 0$ and $z = \infty$ are associated to the roots of the quadratic polynomial $p(z) = z^2+c$ and are superattractive fixed points for all parameter values. The other fixed points $z = \pm 1$ and $z = z_\pm$ are called *strange fixed points*, since they do not correspond to the roots of the original polynomial. We can observe that $z_+ z_- = 1$. The next proposition describes the parameters for which z_+ and z_- collide and, hence, the number of strange fixed points decreases.

Proposition 2. *The number of fixed points of operator (1) decreases for $b = \frac{1}{3}(1 \pm 2\sqrt{13})$ or $b = -5 \pm 2\sqrt{5}$.*

Proof. For $b = \frac{1}{3}(1 \pm 2\sqrt{13})$ we have that $z_+ = z_- = -1$ and for $b = -5 \pm 2\sqrt{5}$ we have that $z_+ = z_- = -1$; then, the number of fixed points is reduced and their multiplicity is increased. □

In addition, if $b = -1$ we have that $O_{-1}(z) = z^4$ and that $z = -1$ is a pre-periodic point of the fixed point $z = 1$. If $b = 1 \pm 2\sqrt{2}$, then $O_{1\pm 2\sqrt{2}}(z) = -z^4$ and $z = 1$ is a pre-periodic point of $z = -1$.

Strange fixed points can be attractors; so, a first step in the dynamical study of iterative methods is the search of the regions in the parameter plane corresponding to values of the parameters for which the strange fixed points are attractors. The stability of such fixed points is studied from the derivative of the operator, given by:

$$O'_b(z) = \tag{2}$$

$$= 4z^3 \frac{(b-1)(b+3)(-11-6b+b^2) + 2(51+42b+4b^2-2b^3+b^4)z + (b-1)(b+3)(-11-6b+b^2)z^2}{((b-1)(b+3) + (-11-6b+b^2)z)^2}.$$

The stability of the strange fixed points is studied in the following propositions.

Proposition 3. *Let us write $b = \alpha + i\beta$. For $b \neq -1$, the strange fixed point $z = -1$ satisfies the following statements.*

a) The fixed point $z = -1$ is attractive if

$$-9 - 2\sqrt{17} < \alpha < -5 - 2\sqrt{2} \qquad \text{or} \quad -5 + 2\sqrt{2} < \alpha < -9 + 2\sqrt{17}$$

and

$$-\sqrt{-81 - 14\alpha - \alpha^2 + 4\sqrt{2(7+\alpha)(29+5\alpha)}} \quad < \beta < \quad \sqrt{-81 - 14\alpha - \alpha^2 + 4\sqrt{2(7+\alpha)(29+5\alpha)}}.$$

Moreover, it is a superattractor if $b = -7 \pm 2\sqrt{10}$.

b) The point $z = -1$ is indifferent if

$$-9 - 2\sqrt{17} < \alpha < -5 - 2\sqrt{2} \qquad \text{or} \quad -9 + 2\sqrt{17} < \alpha < -5 + 2\sqrt{2}$$

and

$$\beta \quad = \quad \pm\sqrt{-81 - 14\alpha - \alpha^2 + 4\sqrt{2(7+\alpha)(29+5\alpha)}}$$

c) The fixed point $z = -1$ is repulsive for any other value of the complex parameter b.

Proof. The fixed point $z = -1$ is indifferent on the curve defined by

$$|O'_b(-1)| = 1$$

that is,

$$\left| \frac{9 + 14b + b^2}{4 + 4b} \right| = 1.$$

Writing $b = \alpha + i\beta$ and simplifying the previous expression we obtain

$$65 + 220\alpha + 198\alpha^2 + 28\alpha^3 + \alpha^4 + (162 + 28\alpha + 2\alpha^2)\beta^2 + \beta^4 = 0.$$

For $b = -7 \pm 2\sqrt{10}$ we have that $O'_b(-1) = 0$ and $z = -1$ is a superattractor for this value of b. As this value is inside the curve previously defined, the fixed point $z = -1$ is attractive inside the curve defined by the previous expression and it is repulsive outside the curve and the above statements are proved. □

Proposition 4. *Let us write $b = \alpha + i\beta$. For $b \neq 1 \pm 2\sqrt{2}$, the strange fixed point $z = 1$ satisfies the following statements.*

a) The fixed point $z=1$ is attractive if

$$\frac{1}{3}(1-2\sqrt{13}\leq\alpha\leq\frac{1}{5}(3-2\sqrt{41}) \quad or \quad \frac{1}{3}(1+2\sqrt{13}\leq\alpha\leq\frac{1}{5}(3+2\sqrt{41})$$

and

$$-\sqrt{\frac{-95+14\alpha-15\alpha^2+4\sqrt{2(35-160\alpha+173\alpha^2)}}{15}}<\beta$$

$$\beta<\sqrt{\frac{-95+14\alpha-15\alpha^2+4\sqrt{2(35-160\alpha+173\alpha^2)}}{15}}.$$

Moreover, it is a superattractor if $b=-2$ or $b=3$.

b) The fixed point $z=1$ is indifferent if

$$\frac{1}{3}(1-2\sqrt{13}\leq\alpha\leq\frac{1}{5}(3-2\sqrt{41}) \quad or \quad \frac{1}{3}(1+2\sqrt{13}\leq\alpha\leq\frac{1}{5}(3+2\sqrt{41})$$

and

$$\beta \;=\; \pm\sqrt{\frac{-95+14\alpha-15\alpha^2+4\sqrt{2(35-160\alpha+173\alpha^2)}}{15}}.$$

c) The fixed point $z=1$ is repulsive for any other value of the complex parameter b.

Proof. The fixed point $z=1$ is indifferent on the curve defined by

$$|O_b'(1)|=1$$

that is,

$$\left|\frac{4(-6-b+b^2)}{-7-2b+b^2}\right|=1.$$

Writing $b=\alpha+i\beta$ and simplifying the previous expression we obtain:

$$(-17-2\alpha+3\alpha^2)(-31-6\alpha+5\alpha^2)+(190-28\alpha+30\alpha^2)\beta^2+15\beta^4=0.$$

that correspond to two closed curves. For $b=-2$ or $b=3$ we have that $O_b'(-1)=0$ and $z=-1$ is a superattractor for this value of b. Then, the fixed point $z=-1$ is attractive inside the curves defined by the previous expression and it is repulsive outside these curves and the above statements are proved. □

Proposition 5. *Let us write $b=\alpha+i\beta$. For $b\neq-3$ and $b\neq1$, the strange fixed points $z_\pm$ are indifferent if*

$$\begin{aligned}(5+10\alpha+\alpha^2)(-17-2\alpha+3\alpha^2)(-67-204\alpha-26\alpha^2+36\alpha^3+5\alpha^4) &\quad + \\ 4(8259+5994\alpha+1225\alpha^2-4\alpha^3+709\alpha^4+186\alpha^5+15\alpha^6)\beta^2 &\quad + \\ 2(6069+1508\alpha+1606\alpha^2+372\alpha^3+45\alpha^4)\beta^4+4(299+62\alpha+15\alpha^2)\beta^6+15\beta^8 &\quad = \quad 0\end{aligned}$$

Proof. The previous expression is obtained by substituting $b=\alpha+i\beta$ in

$$|O_b'(z_\pm)|=1\Rightarrow\left|\frac{4(-19+b(8+b)(-6+b^2))}{(-3+2b+b^2)^2}\right|=1.$$

and simplifying. □

The regions defined in the previous propositions can be observed in Figures 1 and 2. In Figure 2, the red curves correspond to $|O_b'(-1)|=1$, the blue curves correspond to $|O_b'(1)|=1$ and the magenta curves correspond to $|O_b'(z_\pm)|=1$.

From Figure 2 it is easy to check that the strange fixed points $z_\pm$ are attractive for values of the parameter b inside the magenta curves as they are superattractors for the points $b=-2+\sqrt{5}\pm\sqrt{10-2\sqrt{5}}$ and $b=-2-\sqrt{5}\pm\sqrt{10+2\sqrt{5}}$, that are inside these curves. Moreover, for $b=\frac{1}{3}\left(1\pm2\sqrt{13}\right)$ we have $z_\pm=1$ and for $b=-5\pm2\sqrt{5}$ we have $z_\pm=-1$. These are the values

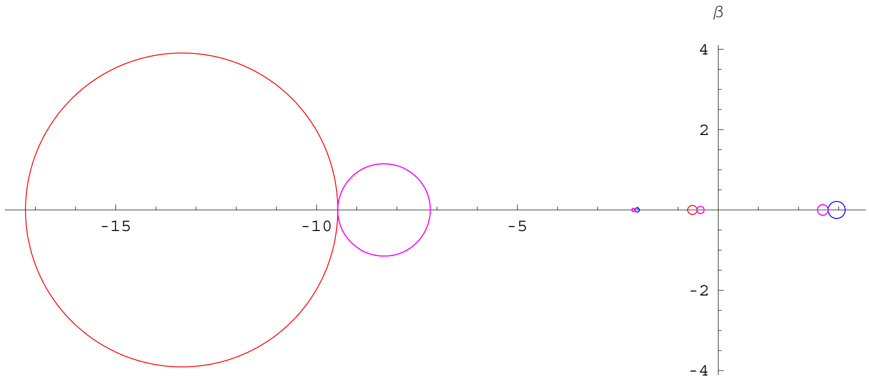


FIGURE 2. Regions of stability of the strange fixed points for the operator O_b.

where the magenta and blue curves coincide and where the magenta and red curves coincide, respectively.

Moreover, from Figure 2 we can easily identify the regions of the parameter plane (see Figure 1) for which the strange fixed points are attractors.

2.2. **Critical points.** As every attractor has a critical point in its basin of attraction, the iteration of free critical points tells us the existence of strange attractors. Hence, to be able to draw the parameter plane it is important to locate all critical points.

Critical points satisfy $O_b'(z) = 0$. The derivative of our operator is given by (2). Then, we obtain that the fixed points $z = 0$ and $z = \infty$ are also critical points of degree three and, consequently, these fixed points (that are associated to the roots of the quadratic family) are always superattractive points and have their own basin of attraction for any value of the parameter.

We obtain other two critical points, given by the roots of the degree two polynomial in the numerator of (2), that are *free critical points*; i.e. they are not tied to any fixed (or periodic) point. The dynamics of these free critical points depends on the parameter b. Their expression is

$$c_{\pm} = \frac{-51 - 42b - 4b^2 + 2b^3 - b^4 \pm 2\sqrt{(b-3)(b+1)(b+2)(b^2-2b-7)(b^2+14b+9)}}{(b-1)(b+3)(b^2-6b-11)} \tag{3}$$

for $b \neq 1, b \neq -3$ and $b \neq 3 \pm 2\sqrt{5}$.

The critical points satisfy $c_+ = 1/c_-$. Since the operator O_b is conjugate to itself by the map $I(z) = 1/z$ (compare Lemma 6), it follows that the orbits of both critical points have the same asymptotic behaviour. Hence, when drawing the parameter plane, it is enough to iterate one of the critical points.

2.3. **The parameter plane.** As mentioned before, we can draw the parameter plane of the operator by iterating one of the critical points and studying its asymptotic behaviour. In this paper, parameter planes are done as follows. We take a grid of 1501×1501 points (3001×1501 for Figure 1). Then we iterate the critical point c_+ up to 100 times. If before reaching 100 iterations the point w is close enough to $z = 0$ or $z = \infty$ ($|w| < 10^{-8}$ or $|w| > 10^8$), then we conclude that the critical orbit converges to one of the roots of the polynomial and plot the parameter using a scaling from pallid blue to green to yellow and to red depending on the number of iterates taken before escaping. If the critical orbit has not escaped to $z = 0$ or $z = \infty$ in less than 100 iterates, then we point the parameter in black. Black parameters are, precisely, those parameters for which the critical orbit may have accumulated on an strange attractor. Hence, black parameters are not good for the stability of the numerical method.

In Figure 1 we show the parameter plane of the operator O_b. Also, in Figure 3 we do a zoom in on Figure 1 so that the little regions corresponding to parameters for which the strange fixed points are attractors can be observed.

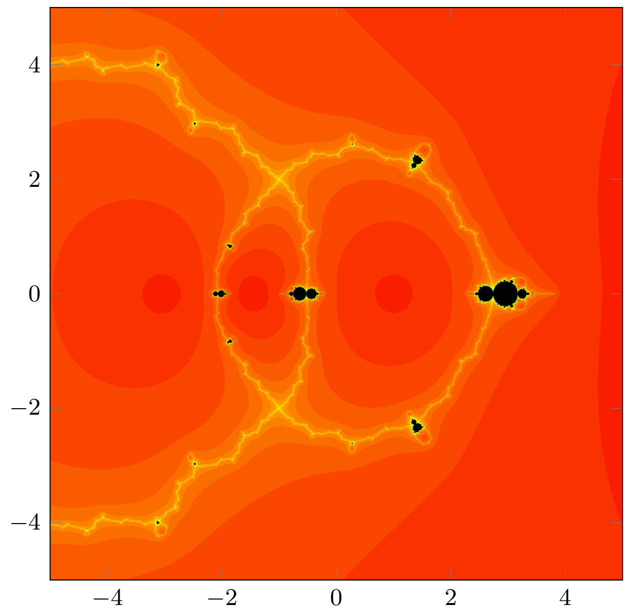


FIGURE 3. A zoom in on Figure 1.

Let us point out that the big black region in Figure 1 corresponds to values of the parameter b where $z = -1$ is attractive, the black region to its right corresponds to values of the parameter where $z_{\pm}$ are attractive; the big black region located on its left corresponds to values of the parameter where an attractive periodic orbit of period two appears. The bigger black disk in Figure 3 corresponds to a region of parameters for which $z = 1$ is attractive.

Another important feature to point out from Figure 1 is the unbounded set of 'bad' parameters which appears following the negative real axis. This set of parameters, that we shall call *antenna*, corresponds to parameters for which the orbits of the critical points do not converge to $z = 0$ nor $z = \infty$. In Proposition 12 we analyse why this antenna appears and prove that it is actually unbounded.

We have also noticed a duplicity in the dynamical information obtained in this section. This duplicity appears from the terms in b^2 in Equation (1). Because of this term, one can find two different parameters b_1 and b_2 which lead to the same operator ($O_{b_1} = O_{b_2}$). In order to avoid this phenomenon we introduce the parameter $a = a(b)$ given by

$$a = \frac{11 + 6b - b^2}{-3 + 2b + b^2}. \tag{4}$$

With this parameter, the operator (1) is expressed as

$$O_a(z) = z^4 \frac{z - a}{1 - az}. \tag{5}$$

Given that $(11 + 6b - b^2)/(-3 + 2b + b^2)$ is a rational map of degree two, for every parameter a there exist two parameters b_1 and b_2 such that $a = a(b_1) = a(b_2)$. In Figure 5 we show this (2-1)-correspondence for the case of real parameters.

3. DYNAMICAL STUDY OF THE FAMILY $O_a(z)$

The goal of this section is to provide a brief study of the dynamics of the operator O_a (5) and to analyse the existence of antennas in the parameter plane.

The operator O_a has degree five except for $a = 1$ and $a = -1$, parameters for which the degree is four: $O_1(z) = -z^4$, $O_{-1}(z) = z^4$. Moreover, if $a = 0$ then $O_0(z) = z^5$.

For $a \notin \{1, -1\}$, the fixed points of operator (5) are 0 and ∞, that corresponds to the roots of the polynomial $x^2 + c$, and the strange fixed points $1, -1$, and $z_{\pm} = \frac{a \pm \sqrt{a^2-4}}{2}$.

The critical points are given by:

$$c_{\pm} = \frac{5 + 3a^2 \pm \sqrt{(a+1)(a-1)(3a-5)(3a+5)}}{8a}. \tag{6}$$

The following lemma will be used to prove that the orbits of both free critical points are tied.

Lemma 6. *Let $I(z) = 1/z$. Then, fixed any $a \in \mathbb{C}$ and $z \in \widehat{\mathbb{C}}$, $z \neq 0$, we have that*

$$O_a \circ I(z) = I \circ O_a(z).$$

Proof.

$$O_a \circ I(z) = \left(\frac{1}{z}\right)^4 \frac{\frac{1}{z} - a}{1 - a\frac{1}{z}} \cdot \frac{z}{z} = \frac{1}{z^4 \frac{z-a}{1-az}} = I \circ O_a(z).$$

□

The importance of this lemma comes from the fact that it ties the dynamics of all pairs of points v and w such that $v = 1/w$. Indeed, it follows from the lemma that $O_a(v) = O_a(I(w)) = I(O_a(w)) = 1/O_a(w)$. Iterating this relation we have that $O_a^n(v) = 1/O_a^n(w)$, for all $n \in \mathbb{N}$. We can apply this property to the critical points since $c_+ = 1/c_-$. In particular, we can conclude that if one critical orbit converges to $z = 0$ then the other one converges to $z = \infty$. This implies that it is enough to analyse the asymptotic behaviour of one of the critical orbits in order to study the existence of attractors different of the roots.

The following lemma shows a symmetry of the parameter plane (see Figure 4). More specifically, it shows that the operators O_a and O_{-a} are conjugate.

Lemma 7. *Let $h(z) = -z$. Then, fixed any $a \in \mathbb{C}$ and $z \in \widehat{\mathbb{C}}$, $z \neq 0$, we have that*

$$h^{-1} \circ O_{-a} \circ h = O_a.$$

Proof.

$$h^{-1} \circ O_{-a} \circ h(z) = h^{-1} \circ O_{-a}(-z) = h^{-1}\left(-O_{-a}(-z)\right) = -(-z)^4 \frac{-z+a}{1-az} = O_a(z).$$

□

In this parameter plane, the big black regions correspond to values of the parameter for which the strange fixed points are attractive, as we can see in the following propositions.

Proposition 8. *The strange fixed point $z = 1$ satisfies the following statements.*

a) The point $z = 1$ is attractive if $\left|a - \frac{7}{4}\right| < \frac{1}{4}$ and it is a superattractor for $a = \frac{5}{3}$.

b) If $\left|a - \frac{7}{4}\right| = \frac{1}{4}$, then $z = 1$ is an indifferent fixed point.

c) The point $z = 1$ is repulsive for any other value of the complex parameter a.

Proof. The point $z = 1$ is indifferent on the curve defined by

$$|O_a'(1)| = 1$$

that is,

$$\left|\frac{5 - 3a}{1 - a}\right| = 1$$

By writing $a = \alpha + i\beta$ and simplifying the previous expression we obtain

$$\left(\alpha - \frac{7}{4}\right)^2 + \beta^2 = \frac{1}{16}.$$

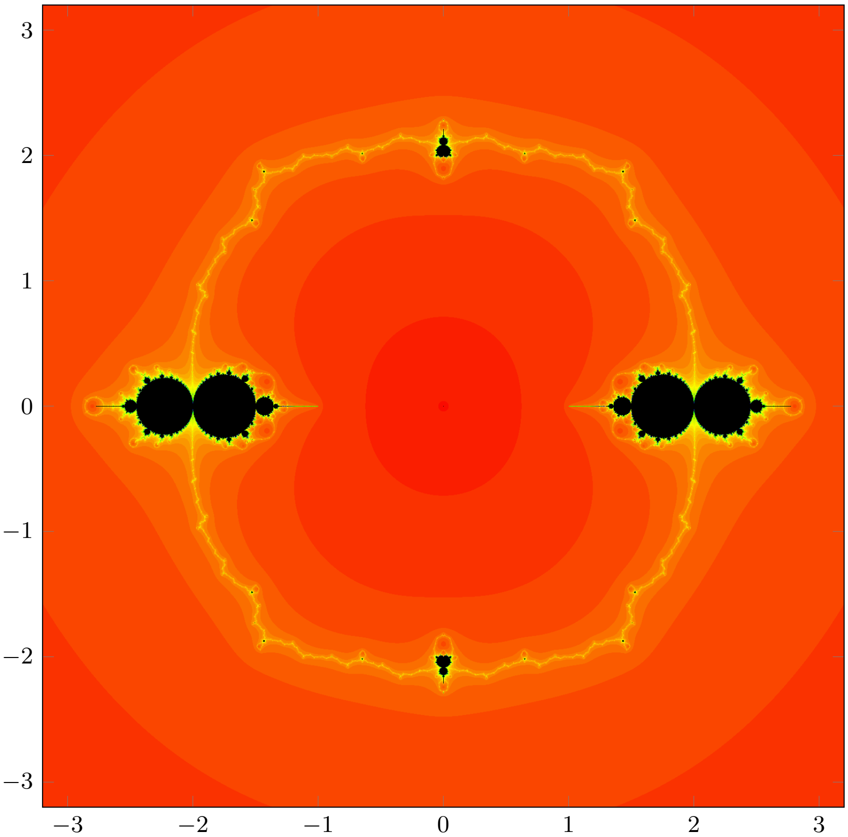


FIGURE 4. Parameter plane of the operator O_a.

For $a = \frac{5}{3}$ we have that $O_a'(1) = 0$ and, hence, $z = 1$ is superattractive. As this value is inside the defined circle, the fixed point $z = 1$ is attractive inside the circle and it is repulsive outside the circle and the above statements are proved.

□

From the conjugacy $h(z) = -z$ between O_a and O_{-a} described in Lemma 7, we obtain the region of parameters for which $z = -1$ is attractive.

Proposition 9. *The strange fixed point $z = -1$ satisfies the following statements.*

a) The point $z = -1$ is attractive if $\left|a + \frac{7}{4}\right| < \frac{1}{4}$ and it is a superattractor for $a = -\frac{5}{3}$.
b) If $\left|a + \frac{7}{4}\right| = \frac{1}{4}$, then $z = -1$ is an indifferent fixed point.
c) The point $z = -1$ is repulsive for any other value of the complex parameter a.

For the other strange fixed points, we have the following results.

Proposition 10. *The strange fixed points $z_\pm = \frac{a \pm \sqrt{a^2-4}}{2}$ satisfy the following statements.*

a) The points $z = \frac{a \pm \sqrt{a^2-4}}{2}$ are attractive if

$$-\sqrt{6} < \alpha < -2 \quad \textit{and} \quad -\sqrt{-5-\alpha^2+\sqrt{1+20\alpha^2}} < \beta < \sqrt{-5-\alpha^2+\sqrt{1+20\alpha^2}}$$
$$2 < \alpha < \sqrt{6} \quad \textit{and} \quad -\sqrt{-5-\alpha^2+\sqrt{1+20\alpha^2}} < \beta < \sqrt{-5-\alpha^2+\sqrt{1+20\alpha^2}}$$

and they are superattractors for $a = \pm\sqrt{5}$.

b) The points $z = \frac{a \pm \sqrt{a^2-4}}{2}$ *are indifferent fixed points if*

$$-\sqrt{6} < \alpha < -2 \quad \textit{and} \quad \beta = \pm\sqrt{-5-\alpha^2+\sqrt{1+20\alpha^2}}$$
$$2 < \alpha < \sqrt{6} \quad \textit{and} \quad \beta = \pm\sqrt{-5-\alpha^2+\sqrt{1+20\alpha^2}}$$

c) The points $z = \frac{a \pm \sqrt{a^2-4}}{2}$ *are repulsive for any other value of the complex parameter* a.

Proof. The fixed points $z = \frac{a \pm \sqrt{a^2-4}}{2}$ are indifferent on the curve defined by

$$\left| O_a' \left(\frac{a \pm \sqrt{a^2-4}}{2} \right) \right| = 1$$

that is,

$$\left|5 - a^2\right| = 1.$$

Writing $a = \alpha + i\beta$ and simplifying the previous expression we obtain:

$$24 + \alpha^4 + 10\beta^2 + \beta^4 + 2\alpha^2(-5+\beta^2) = 0.$$

For $a = \pm\sqrt{5}$ we have that $O_a'\left(\frac{a \pm \sqrt{a^2-4}}{2}\right) = 0$ and the strange fixed points are superattractors. Then, they are attractive inside the curves defined by the previous expression and they are repulsive outside the curves, and the above statements are proved. □

In Figure 4, the two big black regions inside the collar correspond to values of the parameter where $z = 1$ and $z = -1$ are attractive; and the two big black regions outside the collar correspond to values of the parameter where the strange fixed points $z_\pm = \frac{a \pm \sqrt{a^2-4}}{2}$ are attractive.

From the previous results it is easy to check, by undoing the change of parameters, that the regions of attraction of the strange fixed points obtained in Propositions 8, 9 and 10 correspond to the attraction regions obtained in Propositions 3, 4 and 5. Let us highlight that the number of these regions has been halved.

3.1. **The antennas on the real line.** We study now the antennas on the real line in the parameter planes. We analyse how the change of parameter explains the duplicity in the parameter plane of the original family and the existence of an infinite antenna for the operator O_b.

First, we prove that the operator O_a (5) maps the unit circle into itself if $a \in \mathbb{R}$.

Lemma 11. *If* $a \in \mathbb{R}$*, then the operator* O_a *leaves the unit circle* $\mathbb{S}^1$ *invariant.*

Proof. Let $z = e^{i\theta} \in \mathbb{S}^1$ be a point of the unit circle. Then,

$$O_a\left(e^{i\theta}\right) = e^{4i\theta}\frac{e^{i\theta}-a}{1-ae^{i\theta}} = e^{4i\theta}\frac{e^{i\theta}(1-ae^{-i\theta})}{1-ae^{i\theta}}.$$

If $a \in \mathbb{R}$, the image of this points is also on the unit circle:

$$\left|O_a\left(e^{i\theta}\right)\right| = \left|e^{4i\theta}\right|\left|e^{i\theta}\right|\left|\frac{1-ae^{-i\theta}}{1-ae^{i\theta}}\right| = 1$$

being that the numbers $1 - ae^{i\theta}$ and $1 - ae^{-i\theta}$ are complex conjugate when a is real. □

Proposition 12. *There is an infinite antenna on the real line in the parameter plane of operator* $O_b(z)$*, that corresponds to a finite antenna located in the interval* $\left(-\frac{5}{3}, -1\right)$ *in the parameter plane of operator* $O_a(z)$*.*

Proof. For a real, the critical points

$$c_\pm = \frac{5 + 3a^2 \pm \sqrt{(a^2-1)(9a^2-25)}}{8a}$$

are real if $a \in \left(-\infty, -\frac{5}{3}\right) \cup (-1,1) \cup \left(\frac{5}{3}, \infty\right)$.

For $a \in \left(-\frac{5}{3}, -1\right) \cup \left(1, \frac{5}{3}\right)$ these critical points are complex conjugate and, by using Lemma 2.1, we conclude that they are in the unit circle; that is

$$c_+ = \bar{c}_- = \frac{1}{c_-} \Rightarrow |c_+| = |c_-| = 1.$$

As O_a leaves the unit circle invariant for $a \in \mathbb{R}$, the orbits of the critical points remain in the unit circle for $a \in \left(-\frac{5}{3}, -1\right) \cup \left(1, \frac{5}{3}\right)$. This fact explains the existence of bounded antennas of parameters for which the critical points cannot be in the basins of attraction of 0 nor ∞ for the operator O_a (see Figure 4).

Recall that for $a = \pm 1$, the operator O_a degenerates to a degree 4 map. On the other hand, for $a = \frac{5}{3}$ (resp. $a = -\frac{5}{3}$) the strange fixed point $z = 1$ (resp. $z = -1$) is superattractive.

We can use the relation between the parameter a and the parameter b given by Equation (4) (Figure 5 illustrates this relation for real parameters) to obtain the analogous antennas in the parameter space of O_b. More specifically:

- The parameters $a \in \left(-\frac{5}{3}, -1\right)$ correspond to $b \in \left(-\infty, -7 - 2\sqrt{10}\right) \cup \left(-1, -7 + 2\sqrt{10}\right)$. The unbounded interval is obtained from the fact that if $b \to -\infty$ then $a \to -1$.
- If $a \in \left(1, \frac{5}{3}\right)$ then $b \in \left(-2, 1 - 2\sqrt{2}\right) \cup \left(3, 1 + 2\sqrt{2}\right)$.

$\square$

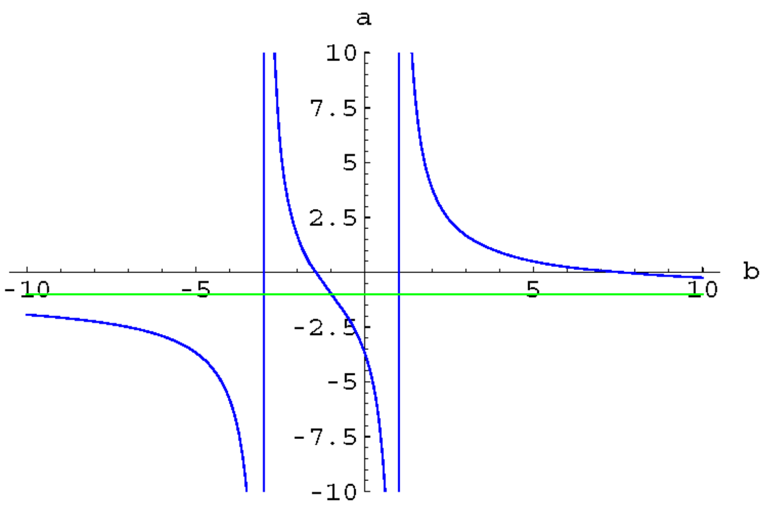


FIGURE 5. Relation between the parameters a and b for b real.

It follows from the proof of Proposition 12 that there is an infinite antenna in the parameter plane of operator O_b, as seen in the Figure 1, that corresponds to the finite antenna located at $\left(-\frac{5}{3}, -1\right)$ in the parameter plane of operator O_a.

4. Dynamical study of the generalised family

In this section we study a generalization of the family of operators O_a. More specifically, we consider the family of operators $O_{a,n,k}$ given by

$$O_{a,n,k}(z) = z^n \left(\frac{z-a}{1-az}\right)^k, \tag{7}$$

where $a \in \mathbb{C}$, $n \geq 2$ and $k \geq 1$. We study this generalised family given that it can be obtained, for different parameters n and k, from several root-finding families of numerical methods. The operator $O_{a,4,1}$ coincides with the map O_a, which we studied in the previous section. Notice that this case was studied first in [1]. Also, the operator of the Chebyshev-Halley family of numerical methods acting on the polynomials z^2+c can be transformed into $O_{a,n,k}$ for $n=3$ and $k=1$ (see [10]). Another example of appearance of the family $O_{a,n,k}$ is obtained [2], where the case $n=6$ and $k=2$ is studied .

We start de dynamical study of the operator $O_{a,n,k}$ by analysing the fixed points which are superattractive independently of the parameter. From the term z^n of the operator $O_{a,n,k}$ and the fact that $(z-a)/(1-az)\neq 1/z$ for all $a\in\mathbb{C}$, we conclude that the points $z=0$ and $z=\infty$ are superattractive fixed points of local degree, at least, n. Actually, the idea behind the generalised operator $O_{a,n,k}$ is that, when obtained from a numerical method, the points $z=0$ and $z=\infty$ would correspond to the solutions of a quadratic equation. In that case, the method would have order of convergence n to the solutions.

In order to be able to draw the parameter planes of the family $O_{a,n,k}$ it is important to know the expressions of the critical points. They correspond to points at which the operator is not injective or, equivalently, points at which the derivative of the operator vanishes. The derivative of the operator (7) is

$$O'_{a,n,k}(z)=z^{n-1}\frac{(z-a)^{k-1}\left(-anz^2+((n+k)+a^2(n-k))z-an\right)}{(1-az)^{k+1}}. \tag{8}$$

As all degree $n+k$ rational maps, the operator $O_{a,n,k}$ has $2(n+k)-2$ critical points. Since the points $z=0$ and $z=\infty$ are superattracting fixed points of local degree n, they are critical points of multiplicity $n-1$. The points $z=a$ and $z=1/a$ are mapped with degree k to $z=0$ and $z=\infty$, respectively. Hence, they are critical points of multiplicity $k-1$ (they are not critical points if $k=1$). Up to now we have counted $2(n+k)-4$ critical points. To find the remaining 2 critical points we have to look for points at which $O'_{a,n,k}$ vanishes. We obtain two critical points $c_\pm$ given by

$$c_\pm=\frac{(n+k)+(n-k)a^2\pm\sqrt{(a^2-1)((n-k)^2a^2-(n+k)^2)}}{2na}.$$

We shall call $c_\pm$ *free critical points* since they are the only critical points whose dynamics may vary. Indeed, $z=0$ and $z=\infty$ are superattractive fixed points (they have their own basins of attraction) while $z=a$ and $z=1/a$ are preimages of $z=0$ and $z=\infty$, respectively.

The critical points $c_\pm$ satisfy $c_+=1/c_-$. Using this property, it follows from the next lemma that they have related asymptotic behaviour.

Lemma 13. *Let $I(z)=1/z$. Then, fixed any $a\in\mathbb{C}$ and $z\in\widehat{\mathbb{C}}$, $z\neq 0$, we have that*

$$O_{a,n,k}(z)\circ I(z)=I\circ O_{a,n,k}(z).$$

Proof.

$$O_{a,n,k}(z)\circ I(z)=\left(\frac{1}{z}\right)^n\left(\frac{\frac{1}{z}-a}{1-a\frac{1}{z}}\right)^k.=\frac{1}{z^n\left(\frac{z-a}{1-az}\right)^k}=I\circ O_{a,n,k}(z)$$

□

By Lemma 13, since $c_+=1/c_-$, we can relate the orbits of c_+ and c_-. Indeed, we have that $O^m_{a,n,k}(c_+)=1/O^m_{a,n,k}(c_-)$ for all $m>0$. In particular, if one critical orbit converges to $z=0$ then the other one converges to $z=\infty$. This implies that it is enough to analyse the asymptotic behaviour of one of the critical orbits to study the existence of any attractor other than the basins of attraction of 0 and ∞.

In Figure 6 and 7 we show the parameter planes of the operator $O_{a,n,k}$ for different values of n and k. In Figure 6 we fix $k=2$ and let n vary. In Figure 7 we fix $n=3$ and let k vary. This drawings are done by iterating a critical point as explained in Section 2.3. We observe in black the regions of parameters for which the critical points do not converge to the roots. In these figures we can observe how, when $|n-k|\leq 1$, we obtain unbounded black regions. These unbounded regions can be understood by analysing the stability of the points $z=1$ and $z=-1$. Indeed, it follows from the expression of the operator $O_{a,n,k}$ that the point $z=1$ is a fixed points for all values of the parameters. Moreover, $O_{a,n,k}(-1) = (-1)^{n+k}$, so $z = -1$ is a fixed point if $n+k$ is odd and a pre-fixed point if $n+k$ is even. In the following lemmas we study the stability of the points $z = 1$ and $z = -1$. We start with a lemma which analyses the stability of $z = 1$ for $|n-k| \neq 0, 1$.

Proposition 14. *For $|n-k| \neq 0, 1$, the fixed point $z = 1$ satisfies the following statements.*

(1) The point $z = 1$ is attractive if $\left|a - \frac{n^2-k^2-1}{(n-k)^2-1}\right| < \frac{2k}{(n-k)^2-1}$ and it is a superattractor for $a = \frac{n+k}{n-k}$.

(2) If $\left|a - \frac{n^2-k^2-1}{(n-k)^2-1}\right| = \frac{2k}{(n-k)^2-1}$ then $z = 1$ is an indifferent fixed point.

(3) The point $z = 1$ is repulsive for any other value of the complex parameter a.

Proof.

$$\left|O'_{a,n,k}(1)\right| = 1 \Rightarrow \left|\frac{a(k-n)+k+n}{1-a}\right| = 1.$$

Let $a = \alpha + i\beta$; then, after simplifying the previous expression we obtain

$$\left(\alpha - \frac{n^2-k^2-1}{(n-k)^2-1}\right)^2 + \beta^2 = \frac{4k^2}{\left((n-k)^2-1\right)^2}$$

that satisfies the previous items. □

For $n+k$ odd, the point $z = -1$ is a fixed point. In this case, the set of parameters where it is attractive can be directly obtained from the following lemma, which provides a symmetry of the parameter plane for $n+k$ odd (compare with Figures 6 and 7).

Lemma 15. *Let $h(z) = -z$. Then, for $n+k$ odd, fixed any $a \in \mathbb{C}$ and $z \in \widehat{\mathbb{C}}$, $z \neq 0$, we have that*

$$h^{-1} \circ O_{-a,n,k} \circ h = O_{a,n,k}.$$

Proof.

$$h^{-1} \circ O_{-a,n,k} \circ h(z) = h^{-1} \circ O_{-a,n,k}(-z) = h^{-1} = -O_{-a,n,k}(-z) = -(-z)^n \left(\frac{-z+a}{1-az}\right)^k = O_{a,n,k}(z).$$

□

Using Lemma 15, we can now provide the corresponding proposition for $z = -1$.

Proposition 16. *For $n+k$ odd and $|n-k| \neq 1$, the fixed point $z = -1$ satisfies the following statements.*

(1) The point $z = -1$ is attractive if $\left|a + \frac{n^2-k^2-1}{(n-k)^2-1}\right| < \frac{2k}{(n-k)^2-1}$ and it is a superattractor for $a = -\frac{n+k}{n-k}$.

(2) If $\left|a + \frac{n^2-k^2-1}{(n-k)^2-1}\right| = \frac{2k}{(n-k)^2-1}$ then $z = -1$ is an indifferent fixed point.

(3) The point $z = -1$ is repulsive for any other value of the complex parameter a.

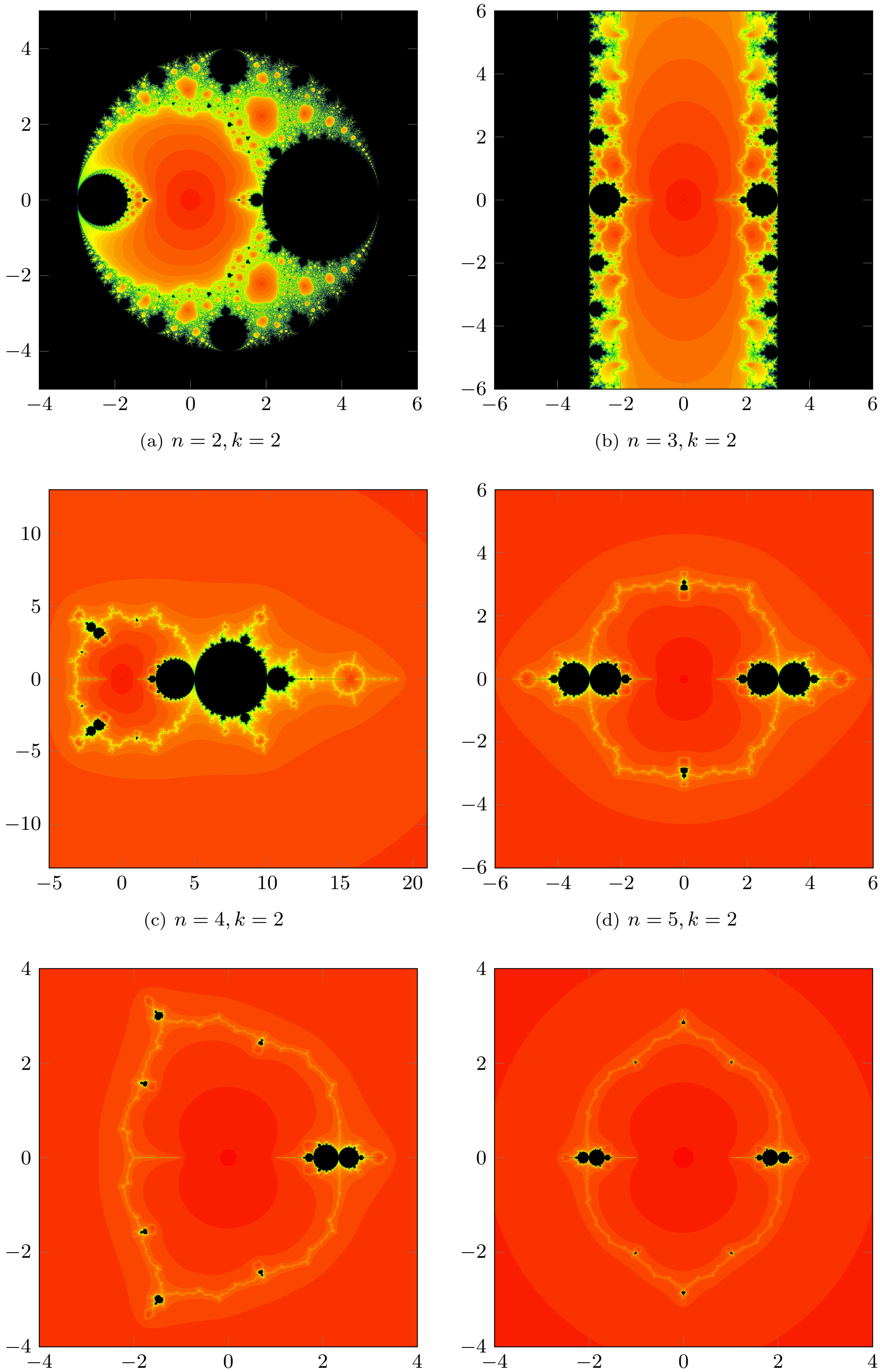
Propositions 14 and 16 describe the sets of parameters for which the hyperbolic regions in the parameter planes corresponding to parameters for which $z = 1$ or $z = -1$ are attractive are bounded. The next two lemmas describe the degeneracy cases for which these regions are unbounded. The proofs are analogous to those of Propositions 14 and 16.

Proposition 17. *Let $a = \alpha + i\beta$. Then, for $|n-k| = 1$ the following statements hold.*

(1) If $k-n = 1$, the fixed point $z = 1$ is an attractor if $\alpha < -n$ and $z = -1$ is an attractor if $\alpha > n$;

(2) If $n-k = 1$, the fixed point $z = 1$ is an attractor if $\alpha > n$ and $z = -1$ is an attractor if $\alpha < -n$.

Proposition 18. *For $n = k$ the fixed point $z = 1$ is an attractor if $|a-1| > 2k$, it is an indifferent point if $|a-1| = 2k$ and it is a repulsor if $|a-1| < 2k$.*

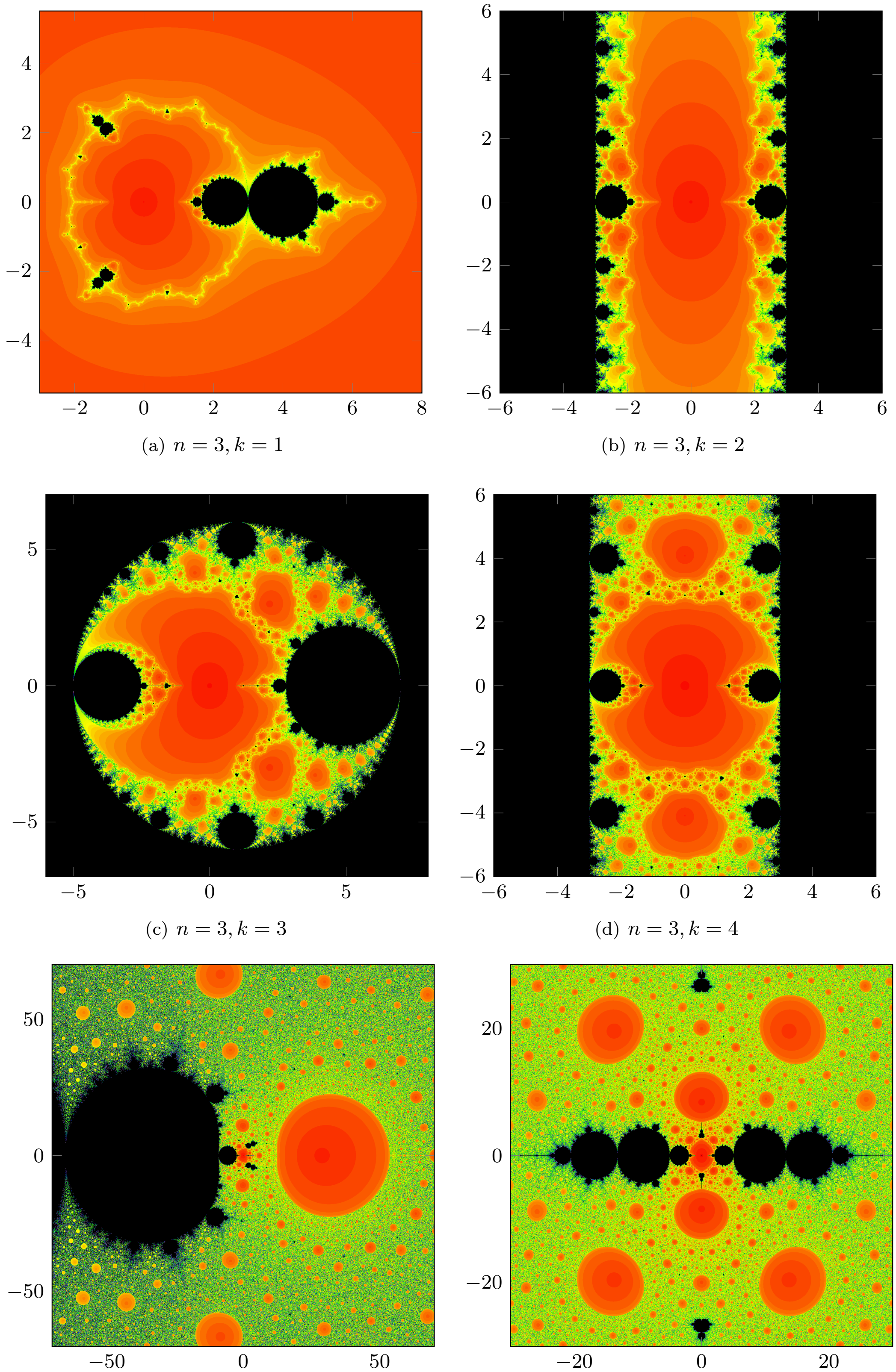


(a) $n = 2, k = 2$ (b) $n = 3, k = 2$

(c) $n = 4, k = 2$ (d) $n = 5, k = 2$

(e) $n = 6, k = 2$ (f) $n = 7, k = 2$

FIGURE 6. Parameter planes of $O_{a,n,k}(z)$ for $k = 2$.



(a) $n = 3, k = 1$

(b) $n = 3, k = 2$

(c) $n = 3, k = 3$

(d) $n = 3, k = 4$

(e) $n = 3, k = 5$

(f) $n = 3, k = 6$

FIGURE 7. Parameter planes of $O_{a,n,k}(z)$ for $n = 3$.

After analysing the stability of the points $z = 1$ and $z = -1$, we focus on the antennas in the parameter planes. Similarly to what happens for the operator O_a, the maps $O_{a,n,k}$ posses antennas. This antennas correspond to real sets of parameters for which the orbits of the critical points cannot converge to $z = 0$ and $z = \infty$. We prove this phenomenon in the next two results. First we show that, if $a \in \mathbb{R}$, then $O_{a,n,k}$ leaves the unit circle invariant.

Lemma 19. *If $a \in \mathbb{R}$, the operator $O_{a,n,k}$ leaves the unit circle $\mathbb{S}^1$ invariant.*

Proof. Let $z = e^{i\theta} \in \mathbb{S}^1$ and $a \in \mathbb{R}$. Then,

$$\left|O_{a,n,k}\left(e^{i\theta}\right)\right| = \left|e^{in\theta}\right| \left|\frac{e^{i\theta} - a}{1 - ae^{i\theta}}\right|^k = \left|e^{in\theta}\right| \left|\frac{e^{i\theta}(1 - ae^{-i\theta})}{1 - ae^{i\theta}}\right|^k = 1.$$

Hence, the image of a point of the unit circle is also in the unit circle. Notice that in the last equality we have used that the numbers $1 - ae^{i\theta}$ and $1 - ae^{-i\theta}$ are complex conjugate when a is real. $\square$

We can now describe the antennas. They are described in the next proposition as a real set of parameters (actually, the union of two intervals) for which $c_\pm \in \mathbb{S}^1$. In that case, by Lemma 19, we know that the orbits of the free critical points cannot exit the unit circle and, hence, cannot converge to $z = 0$ or $z = \infty$. These intervals are obtained by analysing for which real parameters the radical of the quadratic equation that provides $c_\pm$ is not positive. In that case, the free critical points are complex conjugate and lie in $\mathbb{S}^1$.

Proposition 20. *If $n \neq k$ and $a \in \left(-\left|\frac{n+k}{n-k}\right|, -1\right) \cup \left(1, \left|\frac{n+k}{n-k}\right|\right)$, then the critical points $c_\pm$ of $O_{a,n,k}$ satisfy $c_\pm \in \mathbb{S}^1$. Furthermore, the following statements hold.*

(1) If $a = \pm 1$ the operator $O_{a,n,k}$ decreases its degree by k. Moreover, $O_{1,n,k} = z^n$ and $O_{-1,n,k} = (-1)^k z^n$.

(2) If $n \neq k$ and $a = \frac{n+k}{n-k}$ then $c_+ = c_- = 1$ and $z = 1$ is a superattractive fixed point.

(3) If $n \neq k$ and $a = -\frac{n+k}{n-k}$ then $c_+ = c_- = -1$. Moreover, if $n + k$ is odd then $z = -1$ is a superattracive fixed point.

Let us notice that, for $|n - k| > 1$ both antennas are inside the collar-like sets that appear in the parameter planes (see Figure 6 and 7).

To finish this section we demonstrate the non-existence of Herman rings for the operator $O_{a,n,k}$. Recall that Herman rings are doubly connected sets of points where the map is conjugate to a rigid rotation. Herman rings are not related to fixed points, so their existence is more difficult to determine. However, if there would be a Herman ring for an operator coming from a numerical method, it would provide a positive measure set of initial conditions for which the method fails. Hence, it is important to determine if Herman rings do exist.

Theorem 21. *The operator $O_{a,n,k}$ has no Herman rings for $a \in \mathbb{C}$, $n \geq 2$, and $k \geq 1$.*

Proof. The idea of the proof is to semiconjugate the operator $O_{a,n,k}$ with a rational map $S_{a,n,k}$ of the same degree with a single free critical orbit. Since Herman rings have, at least, two free critical orbits accumulating on their boundaries (see [18]), it follows that $S_{a,n,k}$ cannot exhibit Herman rings and neither can $O_{a,n,k}$.

The first step is to conjugate $O_{a,n,k}$ with a rational map $R_{a,n,k}$ using the Möbius transformation $h(z) = (z+1)/(z-1)$. This Möbius transformation sends ∞ to 1, 0 to -1, and 1 to ∞. Moreover, $h^{-1}(z) = h(z)$. The map $R_{a,n,k} = h^{-1} \circ O_{a,n,k} \circ h$ is given by

$$R_{a,n,k}(z) = \frac{(z+1)^n \left(z(1-a) + 1 + a\right)^k + (z-1)^n \left(z(1-a) - 1 - a\right)^k}{(z+1)^n \left(z(1-a) + 1 + a\right)^k - (z-1)^n \left(z(1-a) - 1 - a\right)^k}. \tag{9}$$

Given that h sends ∞ to 1 and 0 to -1, it follows that 1 and -1 are superattractive fixed points of local degree n. A simple computation shows that $R_{a,n,k}(z) = -R_{a,n,k}(-z)$:

$$-R_{a,n,k}(-z) = -\frac{(-z+1)^n \left(-z(1-a) + 1 + a\right)^k + (-z-1)^n \left(-z(1-a) - 1 - a\right)^k}{(-z+1)^n \left(-z(1-a) + 1 + a\right)^k - (-z-1)^n \left(-z(1-a) - 1 - a\right)^k} =$$

$$= \frac{(z+1)^n \left(z(1-a)+1+a\right)^k + (z-1)^n \left(z(1-a)-1-a\right)^k}{(z+1)^n \left(z(1-a)+1+a\right)^k - (z-1)^n \left(z(1-a)-1-a\right)^k} = R_{a,n,k}(z).$$

We conclude that $R(z)$ is odd function. Therefore, $(R_{a,n,k}(z))^2$ is an even function and there exists a map $S_{a,n,k}$ of the same degree than $R_{a,n,k}$ such that $S_{a,n,k}(z^2) = (R_{a,n,k}(z))^2$. In order to find the critical points of $S_{a,n,k}$ we need to derivate the previous equation. We obtain

$$S'_{a,n,k}(z^2) \cdot 2z = 2R_{a,n,k}(z) \cdot R'_{a,n,k}(z).$$

Hence, the critical points of $S_{a,n,k}$ are given by:

- The points c^2 where c is a critical point of $R_{a,n,k}$;
- The points z_0^2, where z_0 is a zero or a pole of $R_{a,n,k}$.

The critical points which come from zeros and poles of $R_{a,n,k}(z)$ are not free since they are preperiodic. Indeed, the conjugacy $h(z) = (z+1)/(z-1)$ sends the fixed point 1 of $O_{a,n,k}$ to ∞. Hence, ∞ is a fixed point of $R_{a,n,k}(z)$. On the other hand, h sends the point -1 to 0. Since -1 is either a preimage of ∞ or a fixed point under $O_{a,n,k}$, it follows that 0 is either a preimage of ∞ or a fixed point.

Hence, we need to analyse the critical points of the form c^2 where c is a critical point of $R_{a,n,k}$. The free critical points of $R_{a,n,k}$, denoted by $\tilde{c}_\pm$, are the image under h of the free critical points $c_\pm$ of $O_{a,n,k}$. Recall that $c_\pm$ satisfy $c_+ \cdot c_- = 1$. We obtain that $\tilde{c}_\pm = (c_\pm + 1)/(c_\pm - 1)$ satisfy $\tilde{c}_+ = -\tilde{c}_-$. Indeed,

$$\frac{c_+ + 1}{c_+ - 1} = -\frac{c_- + 1}{c_- - 1} \Leftrightarrow (c_+ + 1)(c_- - 1) = -(c_- + 1)(c_+ - 1) \Leftrightarrow$$

$$\Leftrightarrow c_+ \cdot c_- - c_+ + c_- - 1 = -\left(c_+ \cdot c_- + c_+ - c_- - 1\right) \Leftrightarrow c_+ \cdot c_- - 1 = -c_+ \cdot c_- + 1$$

and the result follows since $c_+ \cdot c_- = 1$.

We conclude that $S_{a,n,k}$ has a single free critical point given by $\tilde{c}_+^2 = \tilde{c}_-^2$. Since a Herman ring requires, at least, two different critical orbits that accumulate on its boundary (see [18]), it follows that $S_{a,n,k}$ cannot have Herman rings, and neither can $O_{a,n,k}$. □

5. Conclusions

The operator O_b (1), as stated in [1], presents two unwanted features that can difficult the study of its dynamics. The first one is the duplicity of the information in the parameter plane: we can find two different parameters b_1 and b_2 such that $O_{b_1} = O_{b_2}$. The other unwanted feature is the unboundedness of 'bad' parameters: there is an 'antenna' of parameters that spreads through the negative real axis for which the corresponding critical orbits do not converge to the solutions of the equations. This property hinders the study of problematic dynamics.

We avoid these two facts by re-defining the parameter, obtaining the operator O_a (5). This operator belongs to a more general family of operators $O_{a,n,k}$ (7) given by

$$O_{a,n,k}(z) = z^n \left(\frac{z-a}{1-az}\right)^k, \quad a \in \mathbb{C}.$$

This family also includes operators coming from other root finding algorithms applied to quadratic polynomials. In the case where $O_{a,n,k}$ actually comes from a root finding algorithm, the number n corresponds to the order of convergence to the roots. Hence, n is to be considered at least 2.

If $n > k+1$, the region of parameters for which there may be strange attractors seems to be bounded (see Figure 6 and 7). In Figure 8 (a) we can observe the dynamical plane of $O_{a,4,1}$ for a parameter a in the 'unbounded' hyperbolic region. We observe that the basin of attraction of $z = 0$ (in blue) has some holes in red corresponding to points which converge to $z = \infty$. These holes come from the pole $z = 1/a$. We want to point out that this is not very bad news since these holes seem to decrease its diameter very fast when $a \to \infty$ if $n > k+1$. In Figure 8 (b) we also show the dynamical plane of $O_{5/3,4,1}$. For this operator $z = 1$ is superattractive. Despite that, the basins of attraction of $z = 0$ and $z = \infty$ remain big.

On the other hand, if $n \leq k+1$ unwanted dynamical features appear. For instance, if $|n-k| \leq 1$ there are unbounded regions in the parameter plane corresponding to parameters for which the

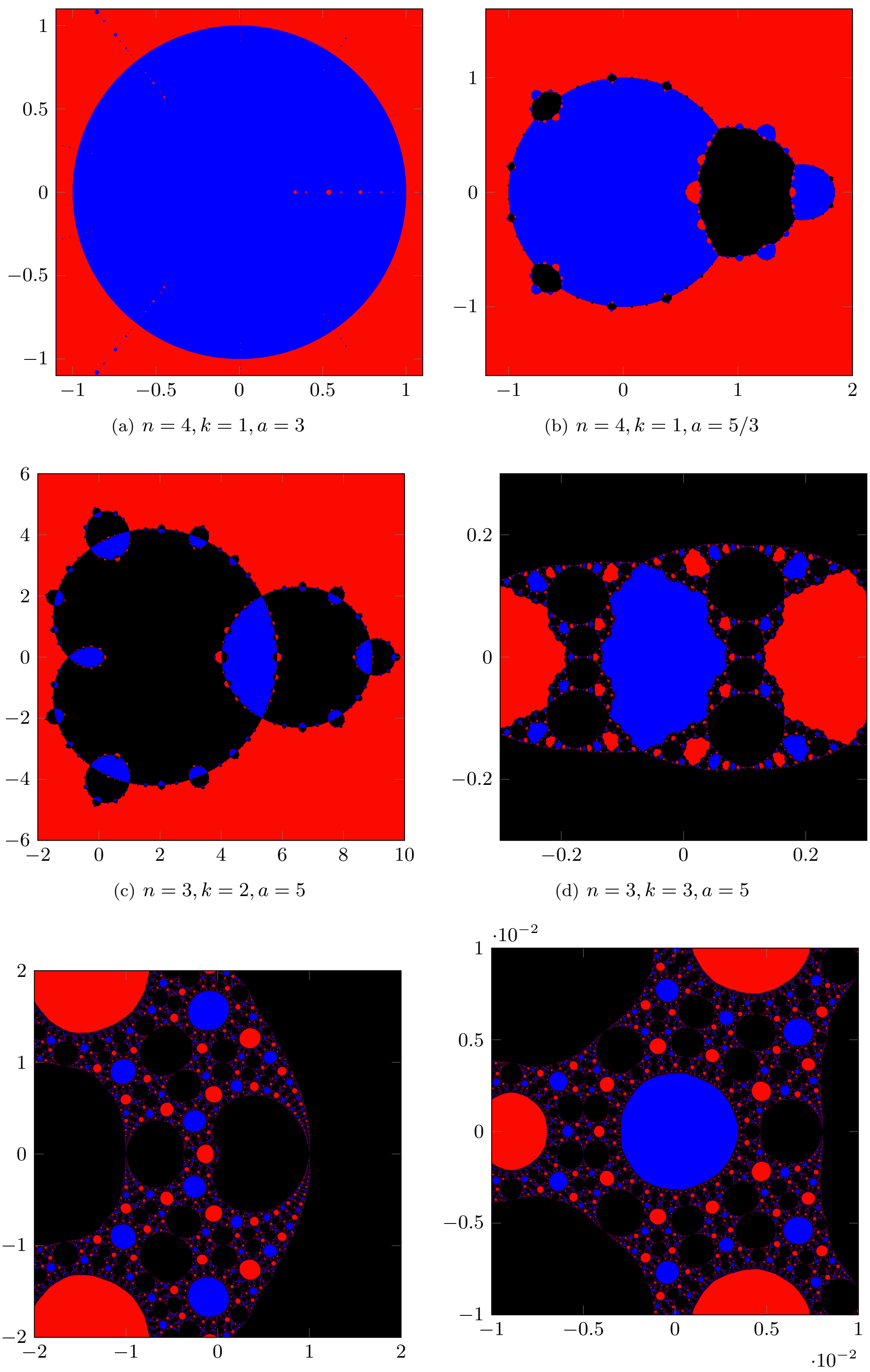


FIGURE 8. Dynamical planes of $O_{a,n,k}(z)$ for different values of n, k and a. Blue points converge under iteration to $z = 0$, red points converge to $z = \infty$ and black points converge to strange attractors.

operator has strange attractors. Also, if $n < k - 1$ quite complicated structures appear in the parameter planes (see Figure 7 (e) and (f)). Furthermore, if $n \leq k + 1$ the size of the immediate basin of attraction of $z = 0$ (the connected component of the basin of attraction that contains $z = 0$) can become quite small. Numerical experiments show that the bigger is k compared to n the smaller is this immediate basin (see Figure 8 (c), (d), (e), and (f)). For instance, for $n = 3$, $k = 5$ and $a = -10$ the diameter of his component is smaller than 10^{-2}. From the dynamical point of view, this would be a terrible feature if the operator actually came from a numerical method. We would like to remark that the red component which appears very close to $z = 0$ in Figure 8 (e) comes from the the pole $1/a = -1/10$. Notice that this red component of points which converge to $z = \infty$ is much bigger than the immediate basin of attraction of $z = 0$.

We can conclude that if the operator is obtained from a numerical method, then n and k should satisfy $n > k + 1$. Also, since $k \geq 1$, n should be at least 3. For such parameters, even if there may be strange attractors, the dynamics of the operator would be suitable from the point of view of numerical methods.

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