POLYNOMIAL AND RATIONAL FIRST INTEGRALS FOR NON–AUTONOMOUS POLYNOMIAL HAMILTONIAN SYSTEMS

AZUCENA CAICEDO[1], JAUME LLIBRE[2] AND CLAUDIO VIDAL[3]

Abstract. Known results on the existence of polynomial and rational first integrals for autonomous polynomial Hamiltonian systems are extended to non–autonomous polynomial Hamiltonian systems invariant under an involution.

The key tool for proving these results is the existence of Darboux polynomials for the non–autonomous polynomial Hamiltonian systems.

1. Introduction and statement of the main results

In this work we deal with the non–autonomous polynomial Hamiltonian systems

$$\dot{q}_i = \frac{\partial H(q,p,t)}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H(q,p,t)}{\partial q_i}, \quad i = 1, \ldots, m, \tag{1}$$

with Hamiltonian $H(q, p, t)$, where $q = (q_1, \ldots, q_m) \in \mathbb{C}^m$ and $p = (p_1, \ldots, p_m) \in \mathbb{C}^m$ are the generalized coordinates. The dot denotes derivative with respect to the time t. Usually q is called the *position vector* and p the *momenta vector*.

In this work we extend the results on the autonomous polynomial Hamiltonian differential systems obtained in [9] to the non–autonomous polynomial Hamiltonian systems (1). More precisely, using the existence of some involution under which the Hamiltonian system (1) remains invariant and the existence of Darboux polynomials, we provide sufficient conditions in order that the non–autonomous polynomial Hamiltonian systems (1) have a second polynomial or rational first integral independent of the Hamiltonian first integral $H(q, p, t)$. These first integrals are polynomial or rational in the variables q and p with coefficients C^1 functions in the time $t \in \mathbb{R}$.

We denote by X_H the associated Hamiltonian vector field in $\mathbb{C}^{2m}$ to the Hamiltonian system (1), i.e.,

$$X_H = \sum_{i=1}^{m} \frac{\partial H(q,p,t)}{\partial p_i}\frac{\partial}{\partial q_i} - \sum_{i=1}^{m} \frac{\partial H(q,p,t)}{\partial q_i}\frac{\partial}{\partial p_i} + \frac{\partial}{\partial t}.$$

Let U be an open subset of $\mathbb{C}^{2m}$. Then a non–locally constant function $I : U \to \mathbb{C}$ such that it is constant on the orbits of the Hamiltonian vector field X_H contained in U is called a *first integral* of X_H in U, i.e. $X_H I \equiv 0$ on U. Our first integrals depend on time, some authors called the first time-dependent integrals as invariants. See for example, [2], [4], [6], [11], [13].

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A non–constant polynomial $F \in \mathbb{C}(t)[q,p]$ in the variables q and p with coefficients C^1 functions in t is a *Darboux polynomial* of the polynomial Hamiltonian vector field X_H if there exists a polynomial $K \in \mathbb{C}(t)[q,p]$ called the cofactor of F, such that $X_H F = KF$. We say that F is a *proper Darboux polynomial* if its cofactor is not zero, i.e. if F is not a polynomial first integral of X_H.

The Darboux polynomials where introduced by Darboux [1] in 1878 for studying the existence of first integrals in the polynomial differential systems in $\mathbb{C}^m$. His original ideas have been developed by many authors; see the survey [7] and the paper [8] with the references therein on some recent results on the Darboux theory of integrability.

Usually a (smooth) involution is a (smooth) map τ such that $\tau \circ \tau = Id$, where Id is the identity.

Let τ be an involution given by the diffeomorphism $\tau : \mathbb{C}^{2m} \times \mathbb{R} \to \mathbb{C}^{2m} \times \mathbb{R}$. The vector field X_H on $\mathbb{C}^{2m} \times \mathbb{R}$ is *τ-time-reversible* if $\tau_*(X_H) = -X_H$ where τ_* is the push-forward associated to the diffeomorphism τ. We recall that the push-forward $\tau_*(X_H) = (D\tau X_H) \circ \tau^{-1}$. In our case $\tau = \tau^{-1}$ because τ is an involution.

Some examples of involutions are $\tau_1(q,p,t) = (q,-p,t)$, $\tau_2(q,p,t) = (q,-p,-t)$, $\tau_3(q,p,t) = (-q,p,t)$, $\tau_4(q,p,t) = (-q,p,-t)$, $\tau_5(q,p,t) = (-q,-p,t)$, $\tau_6(q,p,t) = (-q,-p,-t)$.

If we consider the involutions τ_2, τ_4 and τ_6, according to the definition of the push-forward, the condition $\tau_*(X_H) = -X_H$ means respectively that

$$\frac{\partial H(q,p,t)}{\partial p} = -\frac{\partial H(q,-p,-t)}{\partial p}, \quad \frac{\partial H(q,p,t)}{\partial q} = \frac{\partial H(q,-p,-t)}{\partial q},$$

$$\frac{\partial H(q,p,t)}{\partial p} = \frac{\partial H(-q,p,-t)}{\partial p}, \quad \frac{\partial H(q,p,t)}{\partial q} = -\frac{\partial H(-q,p,-t)}{\partial q},$$

and

$$\frac{\partial H(q,p,t)}{\partial p} = \frac{\partial H(-q,-p,-t)}{\partial p}, \quad \frac{\partial H(q,p,t)}{\partial q} = \frac{\partial H(-q,-p,-t)}{\partial q}.$$

Theorem 1. *Assume that the polynomial Hamiltonian system with Hamiltonian $H(q,p,t)$ is τ-time-reversible and that $F(q,p,t)$ is a proper Darboux polynomial of this system with cofactor $K(q,p,t)$.*

(a) *If $K = K \circ \tau$, then $F \cdot (F \circ \tau)$ is a polynomial first integral of X_H in the variables q and p with coefficients C^1 functions in the variable t.*
(b) *If $K = -(K \circ \tau)$, then $F/(F \circ \tau)$ is a rational first integral of X_H in the variables q and p with coefficients C^1 functions in the variable t.*

Now given an involution $\hat{\tau}$ defined by the diffeomorphism $\hat{\tau} : \mathbb{C}^{2m} \times \mathbb{R} \to \mathbb{C}^{2m} \times \mathbb{R}$, the vector field X_H on $\mathbb{C}^{2m} \times \mathbb{R}$ is *$\hat{\tau}$-time-direct* if $\hat{\tau}_*(X_H) = X_H$ where $\hat{\tau}_*$ is the push-forward associated to the diffeomorphism $\hat{\tau}$.

Considering for example the involutions defined above τ_1, τ_3 and τ_5, the condition $\hat{\tau}_*(X_H) = X_H$ means respectively that

$$\frac{\partial H(q,p,t)}{\partial p} = \frac{\partial H(q,-p,t)}{\partial p}, \quad \frac{\partial H(q,p,t)}{\partial q} = -\frac{\partial H(q,-p,t)}{\partial q},$$

$$\frac{\partial H(q,p,t)}{\partial p} = -\frac{\partial H(-q,p,t)}{\partial p}, \quad \frac{\partial H(q,p,t)}{\partial q} = \frac{\partial H(-q,p,t)}{\partial q},$$

and

$$\frac{\partial H(q,p,t)}{\partial p} = -\frac{\partial H(-q,-p,t)}{\partial p}, \quad \frac{\partial H(q,p,t)}{\partial q} = -\frac{\partial H(-q,-p,t)}{\partial q}.$$

Theorem 2. *Assume that the polynomial Hamiltonian system with Hamiltonian $H(q,p,t)$ is $\hat{\tau}$-time-direct and that $F(q,p,t)$ is a proper Darboux polynomial of this system with cofactor $K(q,p,t)$.*

(a) *If $K = -(K \circ \hat{\tau})$, then $F \cdot (F \circ \hat{\tau})$ is a polynomial first integral of X_H in the variables q and p with coefficients C^1 functions in the variable t.*
(b) *If $K = K \circ \hat{\tau}$, then $F/(F \circ \hat{\tau})$ is a rational first integral of X_H in the variables q and p with coefficients C^1 functions in the variable t.*

2. Proofs

To prove Theorem 1 we need the following result.

Lemma 3. *Under the assumptions of Theorem* 1*, $(F \circ \tau)(q,p,t)$ is another proper Darboux polynomial of X_H with cofactor $-(K \circ \tau)(q,p,t)$.*

Proof. As $F(q,p,t)$ is a Darboux polynomial with cofactor $K(q,p,t)$ we have

$$X_H F(q,p,t) = K(q,p,t)F(q,p,t), \tag{2}$$

Applying to the equality (2) the transformation τ_*, we get

$$\tau_*(X_H F)(q,p,t) = \tau_*(KF)(q,p,t). \tag{3}$$

In the above relation the left side is

$$\begin{aligned} \tau_*(X_H F)(q,p,t) &= \tau_*(X_H)(\tau_*(F))(q,p,t) = -X_H\left(F \circ \tau^{-1}\right)(q,p,t) \\ &= -X_H(F \circ \tau)(q,p,t), \end{aligned} \tag{4}$$

where it has been used that $\tau^{-1} = \tau$. The right side from equation (3) is

$$\begin{aligned} \tau_*(KF)(q,p,t) &= \left((KF) \circ \tau^{-1}\right)(q,p,t) = ((KF) \circ \tau)(q,p,t) \\ &= (K \circ \tau)(F \circ \tau)(q,p,t). \end{aligned} \tag{5}$$

Since (4) is equal to (5) we get

$$X_H(F \circ \tau)(q,p,t) = -(K \circ \tau)(F \circ \tau)(q,p,t).$$

So $F \circ \tau$ is a Darboux polynomial of the Hamiltonian vector field X_H with cofactor $-(K \circ \tau) \neq 0$, because $K(q,p,t) \neq 0$ due to the fact that $F(q,p,t)$ is a proper Darboux polynomial. □

Proof of Theorem 1. Under the assumptions of Theorem 1, we have $X_H F(q,p,t) = K(q,p,t)F(q,p,t)$ with $K(q,p,t) \neq 0$. By Lemma 3 we get that $X_H(F \circ \tau)(q,p,t) = -(K \circ \tau)(F \circ \tau)(q,p,t)$.

We consider

$$\begin{aligned} X_H(F \cdot (F \circ \tau))(q,p,t) = &\ (X_H(F) \cdot (F \circ \tau) + F \cdot X_H(F \circ \tau))(q,p,t) \\ = &\ ((KF) \cdot (F \circ \tau) + F \cdot (-(K \circ \tau)))(q,p,t) \\ = &\ (K - (K \circ \tau))(F \cdot (F \circ \tau))(q,p,t). \end{aligned}$$

But under the assumption of statement (a) we have $K = K \circ \tau$, so

$$X_H \left(F \cdot (F \circ \tau)\right)(q, p, t) = 0$$

and consequently $F \cdot (F \circ \tau)$ is a polynomial first integral of the Hamiltonian vector field X_H in the variables q and p with coefficients C^1 functions in the variable t. Therefore statement (a) is proved.

On the other hand, we consider

$$\begin{aligned} X_H \left(\frac{F}{F \circ \tau}\right)(q, p, t) &= \left(\frac{X_H(F) \cdot (F \circ \tau) - F \cdot X_H(F \circ \tau)}{(F \circ \tau)^2}\right)(q, p, t) \\ &= \left(\frac{(KF) \cdot (F \circ \tau) - F \cdot (-(K \circ \tau))}{(F \circ \tau)^2}\right)(q, p, t) \\ &= \left(\frac{(K + (K \circ \tau))(F \cdot (F \circ \tau))}{(F \circ \tau)^2}\right)(q, p, t) \end{aligned}$$

But under the assumption of statement (b) we have $K = -(K \circ \tau)$, so

$$X_H \left(\frac{F}{F \circ \tau}\right)(q, p, t) = 0$$

and consequently $F/(F \circ \tau)$ is a rational first integral of the Hamiltonian vector field X_H in the variables q and p with coefficients C^1 functions in the variable t. Hence statement (b) is proved. □

To prove Theorem 2 we need the following result.

Lemma 4. *Under the assumptions of Theorem 2, $(F \circ \hat{\tau})(q, p, t)$ is another proper Darboux polynomial of X_H with cofactor $(K \circ \hat{\tau})(q, p, t)$.*

Proof. As $F(q, p, t)$ is a Darboux polynomial with cofactor $K(q, p, t)$ we have

$$X_H F(q, p, t) = K(q, p, t) F(q, p, t), \tag{6}$$

Applying to the equality (6) the transformation $\hat{\tau}_*$, we get

$$\hat{\tau}_*(X_H F)(q, p, t) = \hat{\tau}_*(KF)(q, p, t). \tag{7}$$

In the above relation the left side is

$$\begin{aligned} \hat{\tau}_* (X_H F)(q, p, t) &= \hat{\tau}_* (X_H)(\hat{\tau}_* (F))(q, p, t) = X_H \left(F \circ \hat{\tau}^{-1}\right)(q, p, t) \\ &= X_H (F \circ \hat{\tau})(q, p, t), \end{aligned} \tag{8}$$

where it has been used that $\hat{\tau}^{-1} = \hat{\tau}$. The right side from equation (7) is

$$\begin{aligned} \hat{\tau}_*(KF)(q, p, t) &= \left((KF) \circ \hat{\tau}^{-1}\right)(q, p, t) = ((KF) \circ \hat{\tau})(q, p, t) \\ &= (K \circ \hat{\tau})(F \circ \hat{\tau})(q, p, t). \end{aligned} \tag{9}$$

Since (8) is equal to (9) we get

$$X_H (F \circ \hat{\tau})(q, p, t) = (K \circ \hat{\tau})(F \circ \hat{\tau})(q, p, t).$$

So $F \circ \hat{\tau}$ is a Darboux polynomial of the Hamiltonian vector field X_H with cofactor $K \circ \hat{\tau} \neq 0$, because $K(q, p, t) \neq 0$ due to the fact that $F(q, p, t)$ is a proper Darboux polynomial. □

Proof of Theorem 2. Under the assumptions of Theorem 2, we have $X_H F(q,p,t) = K(q,p,t)F(q,p,t)$ with $K(q,p,t) \neq 0$. By Lemma 4 we get that $X_H\left(F \circ \hat{\tau}\right)(q,p,t) = \left(K \circ \hat{\tau}\right)\left(F \circ \hat{\tau}\right)(q,p,t)$.

We consider

$$\begin{aligned} X_H\left(F \cdot \left(F \circ \hat{\tau}\right)\right)(q,p,t) = &\ \left(X_H\left(F\right) \cdot \left(F \circ \hat{\tau}\right) + F \cdot X_H\left(F \circ \hat{\tau}\right)\right)(q,p,t) \\ = &\ \left(\left(KF\right) \cdot \left(F \circ \hat{\tau}\right) + F \cdot \left(K \circ \hat{\tau}\right)\right)(q,p,t) \\ = &\ \left(K + \left(K \circ \hat{\tau}\right)\right)\left(F \cdot \left(F \circ \hat{\tau}\right)\right)(q,p,t). \end{aligned}$$

But under the assumption of statement (a) we have $K = -\left(K \circ \hat{\tau}\right)$, so

$$X_H\left(F \cdot \left(F \circ \hat{\tau}\right)\right)(q,p,t) = 0$$

and consequently $F \cdot \left(F \circ \hat{\tau}\right)$ is a polynomial first integral of the Hamiltonian vector field X_H in the variables q and p with coefficients C^1 functions in the variable t. Therefore statement (a) is proved.

On the other hand, we consider

$$\begin{aligned} X_H\left(\frac{F}{F \circ \hat{\tau}}\right)(q,p,t) = &\ \left(\frac{X_H(F) \cdot \left(F \circ \hat{\tau}\right) - F \cdot X_H(F \circ \hat{\tau})}{\left(F \circ \hat{\tau}\right)^2}\right)(q,p,t) \\ = &\ \left(\frac{\left(KF\right) \cdot \left(F \circ \hat{\tau}\right) - F \cdot \left(K \circ \hat{\tau}\right)}{\left(F \circ \hat{\tau}\right)^2}\right)(q,p,t) \\ = &\ \left(\frac{\left(K - \left(K \circ \hat{\tau}\right)\right)\left(F \cdot \left(F \circ \hat{\tau}\right)\right)}{\left(F \circ \hat{\tau}\right)^2}\right)(q,p,t) \end{aligned}$$

But under the assumption of statement (b) we have $K = K \circ \hat{\tau}$, so

$$X_H\left(\frac{F}{F \circ \hat{\tau}}\right)(q,p,t) = 0$$

and consequently $F/\left(F \circ \hat{\tau}\right)$ is a rational first integral of the Hamiltonian vector field X_H in the variables q and p with coefficients C^1 functions in the variable t. Hence statement (b) is proved. □

Example 5. *Consider the non-autonomous Hamiltonian polynomial vector field in the variables $(q,p) \in \mathbb{R}^2$ with coefficients C^1 functions in the variable t given by*

(10)

$$X_H = q^5\left(p + \frac{1}{8}\left(t + t^{-1}\right)\right)\frac{\partial}{\partial q}\bigg|_{(q,p,t)} - \frac{5}{2}q^4\left(p + \frac{1}{8}\left(t + t^{-1}\right)\right)^2\frac{\partial}{\partial p}\bigg|_{(q,p,t)} + \frac{\partial}{\partial t}\bigg|_{(q,p,t)},$$

where the Hamiltonian function defined by

$$H(q,p,t) = \frac{1}{2}q^5\left(p + \frac{1}{8}\left(t + t^{-1}\right)\right)^2, \tag{11}$$

is associated with the equation from conformal Weyl gravity [5]*. The Hamiltonian vector field (10) is $\hat{\tau}$-time-direct, i.e., $\hat{\tau}_*(X_H) = X_H$, where $\hat{\tau}_*$ is the pushforward associated with the involution $\hat{\tau}(q,p,t) = (-q,p,t)$. In fact, we have that the differential of $\hat{\tau}$ in $(q,p,t) \in \mathbb{R}^3$ is represented by its Jacobian matrix*

$$D\hat{\tau}\,(q,p,t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and so $D\hat{\tau}(\hat{\tau}^{-1}(q,p,t)) = D\hat{\tau}\,(-q,p,t)$ is represented by the same matrix.

For any (q,p,t) *in* $\mathbb{R}^3$ *we have that*

$$\begin{aligned} X_{H,\hat{\tau}^{-1}(q,p,t)} &= -q^5\left(p+\frac{1}{8}\left(t+t^{-1}\right)\right)\frac{\partial}{\partial q}\bigg|_{\hat{\tau}^{-1}(q,p,t)} \\ &-\frac{5}{2}q^4\left(p+\frac{1}{8}\left(t+t^{-1}\right)\right)^2\frac{\partial}{\partial p}\bigg|_{\hat{\tau}^{-1}(q,p,t)}+\frac{\partial}{\partial t}\bigg|_{\hat{\tau}^{-1}(q,p,t)}. \end{aligned}$$

Then

$$\begin{aligned} \hat{\tau}_{*,(q,p,t)}(X_{H,(q,p,t)}) &= D\hat{\tau}(\hat{\tau}^{-1}(q,p,t))\left(X_{H,\hat{\tau}^{-1}(q,p,t)}\right) \\ &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -q^5\left(p+\frac{1}{8}\left(t+t^{-1}\right)\right) \\ -\frac{5}{2}q^4\left(p+\frac{1}{8}\left(t+t^{-1}\right)\right)^2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} q^5\left(p+\frac{1}{8}\left(t+t^{-1}\right)\right) \\ -\frac{5}{2}q^4\left(p+\frac{1}{8}\left(t+t^{-1}\right)\right)^2 \\ 1 \end{bmatrix}, \end{aligned}$$

i.e., in terms of the coordinates of the base for the tangent space $T_{(q,p,t)}\mathbb{R}^3$*, we have that* $\hat{\tau}_* X_H$ *is expressed as*

$$\begin{aligned} \hat{\tau}_{*,(q,p,t)}(X_{H,(q,p,t)}) &= q^5\left(p+\frac{1}{8}\left(t+t^{-1}\right)\right)\frac{\partial}{\partial q}\bigg|_{(q,p,t)} \\ &-\frac{5}{2}q^4\left(p+\frac{1}{8}\left(t+t^{-1}\right)\right)^2\frac{\partial}{\partial p}\bigg|_{(q,p,t)}+\frac{\partial}{\partial t}\bigg|_{(q,p,t)} \\ &= X_H. \end{aligned}$$

We now affirm that $F(q,p,t)=q^5$ *is a Darboux proper polynomial of the system associated to the Hamiltonian vector field (10) with cofactor*

$$\Lambda(q,p,t)=5q^4\left(p+\frac{1}{8}\left(t+t^{-1}\right)\right).$$

As the Hamiltonian vector field is $\hat{\tau}$*-time-direct and it has a Darboux proper polynomial by the Lemma 4 we have that* $F\circ\hat{\tau}=-q^5$ *is another Darboux proper polynomial with cofactor* $K\circ\hat{\tau}$.

Example 6. *Consider the non-autonomous Hamiltonian polynomial vector field in the variables* $(q,p)\in\mathbb{R}^4$ *with coefficients* C^1 *functions in the variable* t *given by*

(12)
$$X_H = p_1t^2\frac{\partial}{\partial q_1}\bigg|_{(q,p,t)}+p_2t^2\frac{\partial}{\partial q_2}\bigg|_{(q,p,t)}-4q_1^3t^2\frac{\partial}{\partial p_1}\bigg|_{(q,p,t)}-4q_2^3t^2\frac{\partial}{\partial p_2}\bigg|_{(q,p,t)}+\frac{\partial}{\partial t}\bigg|_{(q,p,t)},$$

with de Hamiltonian function

$$H(q,p,t)=\frac{1}{2}\left(t^2p_1^2+t^2p_2^2\right)+q_1^4t^2+q_2^4t^2. \tag{13}$$

The vector field (12) is τ-time-reversible, i.e., $\tau_*(X_H) = -X_H$, where τ_* is the pushforward associated with the involution $\tau(q,p) = (q,-p,-t)$. In fact, we have that the differential of τ in $(q,p,t) \in \mathbb{R}^5$ is represented by the Jacobian matrix

$$D\tau\,(q,p,t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

and so $D\tau(\tau^{-1}(q,p,t)) = D\tau(q,-p,-t)$ is represented by the matrix.

For any (q,p,t) in $\mathbb{R}^5$,

$$\begin{aligned} X_{H,\tau^{-1}(q,p,t)} &= -p_1t^2\frac{\partial}{\partial q_1}\Big|_{\tau^{-1}(q,p,t)} - p_2t^2\frac{\partial}{\partial q_2}\Big|_{\tau^{-1}(q,p,t)} - 4q_1^3t^2\frac{\partial}{\partial p_1}\Big|_{\tau^{-1}(q,p,t)} \\ &\quad -4q_2^3t^2\frac{\partial}{\partial p_2}\Big|_{\tau^{-1}(q,p,t)} + \frac{\partial}{\partial t}\Big|_{(q,p,t)}. \end{aligned}$$

Then

$$\begin{aligned} \tau_{*,(q,p,t)}(X_{H,(q,p,t)}) &= D\tau(\tau^{-1}(q,p,t))\left(X_{H,\tau^{-1}(q,p,t)}\right) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -p_1t^2 \\ -p_2t^2 \\ -q_1^3t^2 \\ -q_2^3t^2 \\ 1 \end{bmatrix} = \begin{bmatrix} -p_1t^2 \\ -p_2t^2 \\ q_1^3t^2 \\ q_2^3t^2 \\ -1 \end{bmatrix}, \end{aligned}$$

i.e., in terms of the coordinates of the base for the tangent space $T_{(q,p,t)}\mathbb{R}^5$, we have that τ_*X_H is expressed as

$$\begin{aligned} \tau_{*,(q,p,t)}(X_{H,(q,p,t)}) &= -p_1t^2\frac{\partial}{\partial q_1}\Big|_{(q,p,t)} - p_2t^2\frac{\partial}{\partial q_2}\Big|_{(q,p,t)} + q_1^3t^2\frac{\partial}{\partial p_1}\Big|_{(q,p,t)} \\ &\quad +q_2^3t^2\frac{\partial}{\partial p_2}\Big|_{(q,p,t)} - \frac{\partial}{\partial t}\Big|_{(q,p,t)} \\ &= -X_H. \end{aligned}$$

On the other hand, we now affirm the Hamiltonian system associated with (12) has a Darboux proper polynomial in the form

$$F(q,p) = ip_2 + \sqrt{2}q_2^2,$$

with $i^2 = -1$ and cofactor $K(q,p,t) = -2i\sqrt{2}q_2t^2$. Since $K(q,p,t) = K(q,-p,-t)$ by the Theorem 1 we have that

$$\begin{aligned} G_1(q,p) = F(q,p)F(q,-p) &= \left(ip_2 + \sqrt{2}q_2^2\right)\left(-ip_2 + \sqrt{2}q_2^2\right) \\ &= p_2^2 + 2q_2^4, \end{aligned}$$

is an additional polynomial first integral of the Hamiltonian vector field (12).

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[1] Departamento de Matemática, Facultad de Ciencias, Universidad del Bío-Bío, Casilla 5-C, Concepción, VIII-Región, Chile

Email address: `azu_bac@hotmail.com`

[2] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

Email address: `jllibre@mat.uab.cat`

[3] Grupo de Investigación en Sistemas Dinámicos y Aplicaciones-GISDA, Departamento de Matemática, Facultad de Ciencias, Universidad del Bío-Bío, Casilla 5-C, Concepción, VIII-Región, Chile

Email address: `clvidal@ubiobio.cl`