# LIMIT CYCLES FOR SOME FAMILIES OF SMOOTH AND NON-SMOOTH PLANAR SYSTEMS

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Abstract. We apply the averaging method in a class of planar systems given by a linear center perturbed by a sum of continuous homogeneous vector fields, to study lower bounds for their number of limit cycles. Our results can be applied to models where the smoothness is lost on the set $\Sigma = \{xy = 0\}$. They also motivate to consider a variant of Hilbert 16th problem, where the goal is to bound the number of limit cycles in terms of the number of monomials of a family of polynomial vector fields, instead of doing this in terms of their degrees.

## 1. Introduction

A limit cycle is a periodic orbit of a differential system that is isolated in the set of all its periodic orbits. The investigation of the existence of limit cycles is relevant for its theoretical interest, because they are the $\alpha$ or $\omega$ limit set of many other trajectories, as well as for their importance in the study of many phenomena in applied sciences, see of instance [3, 11, 21, 29, 30]. One of the approaches to detect such objects is the averaging theory. We refer the books of Sanders and Verhulst [27] and Verhulst [31] for an introduction on this subject.

In this paper we will apply this theory to give a lower bound for the number of limit cycles of a special family of continuous planar differential equations. Motivated from our results on this family we will propose to approach the classical Hilbert 16th problem from a different point of view. Our results on both questions are described in more detail in next two subsections.

1.1. **Limit cycles for a sum of continuous homogeneous vector fields.** The piecewise smooth differential system theory has developed very quickly in recents years and has certainly become an important common frontier between Mathematics, Physics and Engineering, for example. Many studies on piecewise smooth differential system concern the case in which the set $\Sigma$, where the systems lose smoothness, is a regular manifold. Nevertheless, in the recent years the interested in the case where $\Sigma$ is the union of regular manifolds has increased, including the case when $\Sigma$ is not regular, but it is an algebraic manifold. See for instance Panazzolo and Da Silva in [23]. There are also studies that deal with the search of limit cycles of discontinuous systems with $\Sigma$ being an algebraic manifold, see for instance [18] and [22].

In this work we give some lower bounds for the number of limit cycles in some classes of continuous, not necessarily locally Lipschitz, piecewise smooth differential systems where $\Sigma = \{xy = 0\}$. The main technique is the averaging theory.

Some systems and problems that motivated this study are introduced in Section 2. They include models of capillary rise, population models and also some type of SIR models. All of them have in common that can be written as differential equations of the

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form

$$\dot{x} = f(x, y, \sqrt{x}, \sqrt{y}), \quad \dot{y} = g(x, y, \sqrt{x}, \sqrt{y}), \tag{1}$$

where $f$ and $g$ are smooth or polynomial functions and $x, y \geq 0$. We can consider the natural odd extension of the function $\sqrt{u}$ given by $\sqrt{u}^{\mathbb{R}} := \operatorname{sgn}(u)\sqrt{|u|}$ defined for any real number and then these systems can be considered in the full plane but they are non-smooth on the set $\Sigma = \{xy = 0\}$. They clearly belong to the category described above. Notice also that the corresponding vector fields are not Lipschitz functions on $\Sigma$.

In fact, these systems could also be treated by introducing new variables $u$ and $v$ such that $u^2 = x$ and $v^2 = y$ and changing the time, but as we will see, the approach presented in the paper can be applied directly to the original system and can also be applied to more general systems involving simultaneously several non-differentiable functions. For example, the functions $\sqrt[k]{x}$, for different values of $k$, also can be treated with our point of view.

Recall that a continuous vector field $X(x, y)$ is called homogeneous with degree of homogeneity $\alpha$, where $0 \leq \alpha \in \mathbb{R}$, if $X(rx, ry) = r^\alpha X(x, y)$ for all $(x, y) \in \mathbb{R}^2$ and all $0 \leq r \in \mathbb{R}$. For simplicity sake, we will denote $X(x, y) = (f(x, y), g(x, y))$ instead of the more usual notation $X(x, y) = f(x, y)\frac{\partial}{\partial x} + g(x, y)\frac{\partial}{\partial y}$. When $\alpha < 1$, this vector field is continuous but it is not Lipschitz. Its associated planar system of differential equations is $(\dot{x}, \dot{y}) = X(x, y)$, or equivalently, $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$. Our first result is the following:

**Theorem 1.1.** *Consider the class $\mathcal{F}_{\mathbf{a}}$ of planar vector fields*

$$X(x, y) = (-y, x) + \sum_{j=0}^{n} a_j X_j(x, y), \quad \mathbf{a} = (a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}, \tag{2}$$

*where for each $j$, $X_j = (f_j, g_j)$ is a fixed continuous homogeneous vector field with degree of homogeneity $0 \leq \alpha_j \in \mathbb{R}$ and $\alpha_0 < \alpha_1 < \cdots < \alpha_n$. There exist values of $\mathbf{a}$ such that the differential equation associated to $X$ has at least $m$ hyperbolic limit cycles, where $m + 1$ is the number of non-zero values among*

$$I_j = \int_0^{2\pi} \big(f_j(\cos\theta, \sin\theta)\cos\theta + g_j(\cos\theta, \sin\theta)\sin\theta\big)\, d\theta, \quad j = 0, 1, \ldots, n.$$

As we have already said, the proof of Theorem 1.1 is based on the averaging first order method. This theorem extends some of the results of [9] to the non-smooth case.

For instance, simple examples of non-smooth $X_j$ where our approach can be used are

$$X_j(x, y) = \big(a_j \operatorname{sgn}(x)|x|^{\alpha_j} + b_j \operatorname{sgn}(y)|y|^{\alpha_j}, c_j \operatorname{sgn}(x)|x|^{\alpha_j} + d_j \operatorname{sgn}(y)|y|^{\alpha_j}\big),$$

where $0 < \alpha_j < 1$. They clearly include our goal functions.

1.2. **A new Hilbert 16th type problem.** The second part of the paper deals with polynomial vector fields. Recall that the second part of the Hilbert's 16th problem asks about the maximum number of limit cycles for planar polynomial vector fields in terms of their degrees. Usually, the maximum number of limit cycles of vector fields of degree $n$, is denoted as $\mathcal{H}(n)$ (admitting in principle that this number could be infinity) and it is called Hilbert number. One of the most famous and difficult open problems in Mathematics is to prove its finiteness and to exhibit the number $\mathcal{H}(n)$, see [12, 28]. It is known that $\mathcal{H}(1) = 0$, $\mathcal{H}(2) \geq 4$, and $\mathcal{H}(3) \geq 13$, see [26] for more lower bounds for small $n$ and other related references. It is also known that there is a sequence of values $n$ going to infinity such that $\mathcal{H}(n) \geq M(n)$ where $M(n) = \big(\frac{n^2 \log(n)}{2 \log 2}\big)(1 + o(1))$, see for instance [2] and their references. To the best of our knowledge the first result proving the existence of a lower bound of type $\mathcal{O}(n^2 \log(n))$ for $\mathcal{H}(n)$ is due to Christopher and Lloyd [8].

From the statement of Theorem 1.1 we started to think on a different version of Hilbert's sixteenth problem facing the question from a different point of view. Instead of trying to bound the number of limit cycles in terms of the degrees of the vector fields we start wondering ourselves if it is not better to do this in terms of the number of homogeneous vector fields involved in a family. This formulation leads us essentially to the same problem, since polynomial vector fields of degree $n$ are the sum of $n+1$ homogeneous vector fields. In fact, it is even a worst point of view in the light of the following family of polynomials vector fields studied in [10],

$$\dot{z} = Az + Bz|z|^{2(k-2)} + \mathrm{i}c\bar{z}^{k-1},$$

where $z = x + \mathrm{i}y$, $A = a_1 + \mathrm{i}a_2$, $B = b_1 + \mathrm{i}b_2 \in \mathbb{C}$, $c \in \mathbb{R}$ and $k \geq 3$. It has at least $k$ limit cycles but it can also be written in real variables as

$$(\dot{x}, \dot{y}) = a_1X_1(x, y) + a_2X_2(x, y) + b_1X_3(x, y) + b_2X_4(x, y) + cX_5(x, y),$$

that is, involving only five homogeneous vector fields with only three different degrees.

However, this way of thinking the problem leads us to a new point of view that may be interesting to the reader: Why do not try to study the number of limit cycles in terms of the number of homogeneous vector fields formed by single monomials?

Somehow this point of view tries to mimic the role of Descartes theorem in the study of the number of real zeroes of a polynomial $P(x)$ of degree $n$, having $m$ non-zero monomials. Recall that while the maximum number of real roots is $n$, the actual maximum number of real roots is $2m - 1$ and this bound is independent of the degree of $P$. Indeed, $P$ has at most $m - 1$ positive roots, $m - 1$ negative roots, and eventually the root 0.

To state more clearly our point of view and our results, for each $m \in \mathbb{N}$ fixed, we consider the following family of polynomials differential equations:

- Family $\mathcal{M}_m$ given by

$$(\dot{x}, \dot{y}) = \sum_{j=1}^{m} a_jX_j(x, y), \quad \text{with} \quad X_j(x, y) = \begin{cases} \big(x^{n_j}y^{k_j}, 0\big), & \textit{or}, \\ \big(0, x^{n_j}y^{k_j}\big), \end{cases}$$

  where $\mathbf{a} \in \mathbb{R}^m$ and the pairs $(n_j, k_j) \in \mathbb{N}^2$ vary among all the possible values. Varying $m$, this family covers all polynomial differential equations. The letter $\mathcal{M}$ is chosen because the important point is to count the number of involved monomials.

We define $\mathcal{H}^M[m] \in \mathbb{N} \cup \{\infty\}$ to be the maximum number of limit cycles that systems of the family $\mathcal{M}_m$ can have.

The next theorem includes the results on lowers bounds for this Hilbert type number. The proof of the first part for $m \geq 4$ is a straightforward consequence of Theorem 1.1 and also a consequence of other known results on classical Liénard systems. The second part is a direct corollary of the recent paper [2] and uses results on generalized Liénard systems.

**Theorem 1.2.** *With the notation introduced above it holds that $\mathcal{H}^M[m] = 0$ for $m = 1, 2, 3$ and for $m \geq 4$, $\mathcal{H}^M[m] \geq m - 3$. Moreover, there exits a sequence of values of $m$ tending to infinity such that $\mathcal{H}^M[m] \geq N(m)$, where*

$$N(m) = \Big(\frac{(\frac{m-3}{2})\log(\frac{m-3}{2})}{\log 2}\Big)(1 + o(1)).$$

A similar result could be stated by using the lower bounds of $\mathcal{H}(n)$ of type $\mathcal{O}(n^2\log(n))$ because the systems of degree $n$ involve $m = (n+1)(n+2)$ monomials. These systems and the ones presented in [2] are relevant because for $m$ big enough they have more limit cycles than monomials.

It is not difficult to see that all the results given in Theorem 1.2 also hold for the subclass of $\mathcal{M}_m$ of second order differential equations $\ddot{x} = P(x, \dot{x})$ where $P$ is a polynomial with $m-1$ monomials, since classical Liénard differential equations are written as $\ddot{x} = f(x) + g(x)\dot{x}$, or equivalently, like the system $(\dot{x}, \dot{y}) = (y, -f(x) - g(x)y)$.

It is also worth to mention that the celebrated examples of quadratic systems that prove that $\mathcal{H}(2) \geq 4$ are given by systems with $m = 8$ monomials, see for instance [7, 24], and so they have $m-4$ limit cycles. The cubic system given in [14] that proves that $\mathcal{H}(3) \geq 13$ has $m = 9$ monomials and at least $m+4$ limit cycles.

Under the light of the above results, a natural problem is to find the minimal $m$ such that there exists a system with $m$ monomials having at least $m+1$ limit cycles.

We remark that, although our results give a new point of view for counting the number of limit cycles of polynomials vector fields, they do not provide new lower bounds for the classical Hilbert numbers.

## 2. Some motivating models

In this section we shortly explain some models that motivate the class of equations (1) that can be treated with the tools introduced in this paper.

2.1. **Capillary rise.** A first example is given by the equation that models the capillary rise. The capillary action is a physical property that the fluids have in to go down or up in extremely thin tubes. Sometimes this action forces the liquid to go up against the force of gravity or even to induce a magnetic field. This ability to rise or fall results from the ability of the liquid to "wet" or not the pipe surface (glass, plastic, metal, etc.). For instance in the case of water in a glass beaker, we have tendency of water to adhere to the glass, bending upward near the wall, forming a concave meniscus and rising to a certain height above water level, here we have a capillary rise. In the case of mercury the opposite happens, the tendency of mercury is to move away from the wall, forming a convex meniscus and descending at a certain height from the mercury level, here we have a capillary depression.

This phenomenon is described in more detail in [25] and can be modeled in an adimensional way by the planar system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = 1 - ay - \sqrt{2x}, \quad x \geq 0, \end{cases}$$

where $a$ is a positive parameter.

2.2. **Some population models.** Following [1] we introduce the herd behavior. If $R$ represents the density of certain population, namely number of individuals per surface unit, with the herd occupying an area $A$, then the individuals who take the outermost positions in the herd are proportional to the perimeter of the region where the herd is located whose length depends on $\sqrt{A}$. They are therefore in number proportional to the square root of the density, that is to $\sqrt{R}$, with a proportionality constant that depend on the shape of the herd. Then, the interactions with the second population with density $Q$ occur only via these peripheral individuals, so that instead of the usual $QR$ that appears in most predator-prey systems, there is a term proportional to $Q\sqrt{R}$. In a dimensional-less set of variables these type of models write as

$$\begin{cases} \dot{x} = x(1-x) - y\sqrt{x}, \\ \dot{y} = -xy + cy\sqrt{x}, \quad x \geq 0, \end{cases}$$

for $c \in \mathbb{R}$, see also [4]. For other population models, involving also square roots, see [3, Sec. 4.10].

2.3. **A SIR type model.** In [20] the author proposes a variation of the classical SIR model. Recall that it is a mathematical model of the spread of infectious diseases that classifies the population in three categories: Susceptible, Infectious, or Recovered. This model relate these categories by the differential system

$$\dot{S} = -\beta\sqrt{SI}, \quad \dot{I} = \beta\sqrt{SI} - \gamma\sqrt{I}, \quad \dot{R} = \gamma\sqrt{I}, \text{ for } S, I, R \geq 0,$$

where $\alpha, \beta$ and $\gamma$ are real parameters. Notice that it can be studied via a planar system because $\dot{S} + \dot{I} + \dot{R} = 0$ and as a consequence $S(t) + I(t) + R(t) = S_0 + I_0 + R_0$.

## 3. Definitions and Preliminaries.

In this section we review some definitions and results that will be used in this paper. For the characterization of Chebyshev Systems in an open interval we will use the following results which can be found in [13] and [19].

**Definition 3.1.** *Let $u_0, \ldots, u_{n-1}, u_n$ be functions defined in an open interval $L$ of $\mathbb{R}$. The ordered set $(u_i)_{i=0}^n$ forms an extended complete Chebyshev system, for short $ECT$-system, on $L$ if any nontrivial linear combination $a_0u_0 + \cdots + a_ku_k$ has at most $k$ isolated roots in $L$ counting multiplicity, for every $k = 0, 1, \ldots, n$.*

The following result is a very useful characterization of smooth $ECT$-systems in terms of Wronskians.

**Proposition 3.2.** *The set of ordered $\mathcal{C}^n$-functions $(u_0, \ldots, u_n)$ forms an $ECT$-system on $L$ if, and only if, for every $k = 0, ..., n$,*

$$W(u_0, \ldots, u_k)(x) = \begin{vmatrix} u_0(x) & \cdots & u_k(x) \\ u_0'(x) & \cdots & u_k'(x) \\ \vdots & \ddots & \vdots \\ u_0^{(k)}(x) & \cdots & u_k^{(k)}(x) \end{vmatrix} \neq 0,$$

*for every $x \in L$.*

We will need the following lemma.

**Lemma 3.3.** *Consider $\beta_i \in \mathbb{R}$ such that $\beta_0 < \beta_1 < \cdots < \beta_m$. Then the functions $(x^{\beta_0}, \ldots, x^{\beta_m})$ form an $ECT$-system on $(0, \infty)$.*

*Proof.* We claim that

$$W = W(x^{\beta_0}, \ldots, x^{\beta_k}) = x^S \Big( \prod_{0 \leq i < j \leq k} (\beta_j - \beta_i) \Big), \quad \text{where} \quad S = \sum_{i=0}^{k} \beta_i - \frac{k(k+1)}{2}. \tag{3}$$

Then, each $W(x^{\beta_0}, \ldots, x^{\beta_k}) \neq 0$ in $(0, \infty)$, for $k = 0, \ldots, m$, and by Proposition 3.2 the functions $\left(x^{\beta_j}\right)_{j=0}^m$ form an $ECT$ on $(0, \infty)$ as we wanted to prove.

Let us prove the claim. For $1 \leq k \in \mathbb{N}$, set $(\beta)_k = \beta(\beta-1)(\beta-2)\cdots(\beta-k)$. Then,

$$W = \begin{vmatrix} x^{\beta_0} & \cdots & x^{\beta_k} \\ \beta_0 x^{\beta_0-1} & \cdots & \beta_k x^{\beta_k-1} \\ (\beta_0)_1 x^{\beta_0-2} & \cdots & (\beta_k)_1 x^{\beta_k-2} \\ \vdots & \ddots & \vdots \\ (\beta_0)_{k-1} x^{\beta_0-k} & \cdots & (\beta_k)_{k-1} x^{\beta_k-k} \end{vmatrix} = x^S \begin{vmatrix} 1 & \cdots & 1 \\ \beta_0 & \cdots & \beta_k \\ \beta_0(\beta_0-1) & \cdots & \beta_k(\beta_k-1) \\ \vdots & \ddots & \vdots \\ (\beta_0)_{k-1} & \cdots & (\beta_k)_{k-1} \end{vmatrix}$$

$$= x^S \begin{vmatrix} 1 & \cdots & 1 \\ \beta_0 & \cdots & \beta_k \\ \beta_0^2 & \cdots & \beta_k^2 \\ \beta_0(\beta_0-1)(\beta_0-2) & \cdots & \beta_k(\beta_k-1)(\beta_k-2) \\ \vdots & \ddots & \vdots \\ (\beta_0)_{k-1} & \cdots & (\beta_k)_{k-1} \end{vmatrix} = \cdots = x^S \begin{vmatrix} 1 & \cdots & 1 \\ \beta_0 & \cdots & \beta_k \\ \beta_0^2 & \cdots & \beta_k^2 \\ \beta_0^3 & \cdots & \beta_k^3 \\ \vdots & \ddots & \vdots \\ \beta_0^k & \cdots & \beta_k^k \end{vmatrix},$$

where this last determinant is the celebrated Vandermonde determinant and coincides with expression (3). Notice that in the first equality we have used that taking the product of $k+1$ elements of the determinant, being each one of them elements of different rows and columns, always appears $x^S$ as a factor. Moreover, in the first equality of the second line of equalities we have changed the third file by the sum of the second and third files of the previous determinant. Similarly, we change the fourth file of this new determinant by a suitable linear combinations of the second, third and fourth ones, and so on, until arriving to the final equality. So, the claim follows. □

A second key tool for proving Theorem 1.1 will be next classical result about averaging theory proved in [31] see also [16]. The hyperbolicity is guaranteed by the fundamental lemma of [17].

**Theorem 3.4.** (First order averaging). *Consider the system of differential equations*

$$x'(t) = \varepsilon H(t,x) + \varepsilon^2 K(t,x,\varepsilon), \tag{4}$$

*where $H : \mathbb{R} \times C \to \mathbb{R}^n$, $K : \mathbb{R} \times C \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $T$-periodic in the first variable ($T$ independent of $\varepsilon$), $H$ is twice differentiable in the second variable, $K$ is differentiable in the second variable, $C$ is a bounded domain of $\mathbb{R}^n$ and $(-\varepsilon_0, \varepsilon_0)$ is a neighborhood of $0 \in \mathbb{R}$. Define the averaged function, $h : C \to \mathbb{R}^n$ as*

$$h(z) = \frac{1}{T}\int_0^T H(t,z)\,dt. \tag{5}$$

*Suppose that the functions $H, K, \mathrm{D}_x H, \mathrm{D}_x^2 H$ and $\mathrm{D}_x K$ are continuous and bounded by a constant $M$ (independent of $\varepsilon$) in $[0,\infty) \times C$. If $a$ is a zero of $h$ such that $\det(\mathrm{D}_x h(a)) \neq 0$ then for each $|\varepsilon| > 0$ small enough, there is a $T$-periodic solution $x = \varphi(t,\varepsilon)$ of system (4) such that $\varphi(0,\varepsilon) \to a$ as $\varepsilon \to 0$.*

*Moreover, if all the eigenvalues of $\mathrm{D}_x h(a)$ have negative real parts, then the corresponding periodic solution $x = \varphi(t,\varepsilon)$ of equation (4) is hyperbolic and asymptotically stable for $\varepsilon$ sufficiently small. If one of these eigenvalues has positive real part then it is unstable.*

As we will see in the proof of Theorem 1.1, although the components the differential equation (2) are only continuous, their homogeneity allows to use the above classical result of averaging theory for smooth differential equations. The reason is that when one writes our planar system in polar coordinates it becomes smooth with respect to $r$, outside a neighborhood of the origin $r = 0$, and the limit cycles that we find lie in compact set contained in this domain. For more general continuous vector fields an extension of the averaging theory is developed by Buică and Llibre in [5], see also [6].

The key result for proving the second part of Theorem 1.2 will be the following theorem.

**Theorem 3.5.** ([2]) *There is a sequence of natural numbers $n$, tending to infinity, such that for these values of $n$ there exist generalized Liénard systems*

$$\begin{cases} \dot{x} = y - F(x), \\ \dot{y} = G(x), \end{cases}$$

*with $F$ and $G$ are polynomials of degree at most $n$, having at least $K(n)$ limit cycles, where*

$$K(n) = \Big(\frac{n \log n}{\log 2}\Big)(1 + o(1)).$$

We also will need next result about non-existence of limit cycles.

**Proposition 3.6.** *Systems*

$$(\dot{x}, \dot{y}) = \big(ax^p y^q, bx^i y^j + cx^k y^l\big),$$

*where $(a, b, c) \in \mathbb{R}^3$ and $(p, q, i, j, k, l) \in \mathbb{N}_0^6$, with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, have no limit cycle.*

*Proof.* We will use the following well-known properties for proving non-existence of limit cycles:

- $P_1$: Periodic orbits must surround some critical point. So systems without critical points have no periodic orbit.
- $P_2$: If a system has an invariant line passing by all its critical points, if any, then it has no periodic orbits. This is so by property $P_1$ if the system has no critical points, or, otherwise, by uniqueness of solutions, because an eventual periodic orbit would surround some of the critical points and as a consequence, must cut the line.
- $P_3$: If one of the two differential equations only involves ones of the variables (for instance $\dot{x} = f(x)$) then the system has no periodic orbits. This is so, because autonomous one dimensional ordinary differential equations have no non-constant periodic solution.
- $P_4$: If a planar system has a smooth first integral defined on an open set $\mathcal{U} \subset \mathbb{R}^2$, although it can have continua of periodic orbits, it can not have limit cycles entirely contained in $\mathcal{U}$.
- $P_5$: If the divergence of a planar system $(\dot{x}, \dot{y}) = (P(x, y), Q(x, y))$, $\operatorname{div}(P, Q) = \frac{\partial P(x,y)}{\partial x} + \frac{\partial Q(x,y)}{\partial y}$ does not change sign and vanishes only on a null Lebesgue measure set, then the system has not periodic orbits.
- $P_6$: Let $X$ be a planar vector field with a unique critical point, $(0, 0)$, and assume that it is reversible, that is, invariant by one of the two changes of variables and time:

  $$(x, y, t) \longrightarrow (-x, y, -t) \quad \text{or} \quad (x, y, t) \longrightarrow (x, -y, -t).$$

  If the system has a periodic orbit that crosses transversally the axes then it is in the interior of a continua of periodic orbits and it is not a limit cycle. This is so, because any of the described symmetries implies that if an orbit turns around the origin it is periodic. Sometimes this criterion is called *reversibility criterion of Poincaré*, because he was the first in using it for proving the existence of periodic orbits.

When $a = 0$, the system can not have periodic orbits because of property $P_3$. When $bc = 0$, we assume, for instance, that $c = 0$ and $b \neq 0$, because when $c \neq 0$ and $b = 0$

the situation is the same and the case $b = c = 0$ is trivial. Then, two situations may happen: either $x = 0$ or $y = 0$ are a continuum of critical points and no other critical points appear or it writes as $(\dot{x}, \dot{y}) = (ay^q, bx^i)$. In the first case the only critical points belong to an invariant line full of critical points, so the system can not have periodic orbits by property $P_2$. In the second case the system is integrable with $\mathcal{U} = \mathbb{R}^2$ and by property $P_4$ no limit cycle appears.

Hence, from now on, we will assume that $abc \neq 0$. Next step will use that the phase portraits of any two systems of the form

$$(\dot{x}, \dot{y}) = (P(x,y)R(x,y), Q(x,y)R(x,y)) \quad \text{and} \quad (\dot{x}, \dot{y}) = (P(x,y), Q(x,y)),$$

are the same (modulus a change of time orientation) in each connected component of $\mathbb{R}^2 \setminus \{R(x,y) = 0\}$. We will take $R$ as some suitable polynomial of one of the forms $x^n$ or $y^n$ to reduce our study to simpler vector fields. Taking $R(x,y) = x^s$ for $s$ to be the minimum of $p, i$ and $k$ we can reduce the situation to one of the next two differential systems:

$$(\dot{x}, \dot{y}) = \left(ay^q, bx^iy^j + cx^ky^l\right),\ i \geq 1 \quad \text{or} \quad (\dot{x}, \dot{y}) = \left(ax^py^q, by^j + cx^ky^l\right), \tag{6}$$

where for simplicity we keep the same notation for the new exponents and without loss of generality we have assumed that $i \leq k$. Next we take $R(x,y) = y^u$ with $u$ being the minimum of $q, j$ and $l$. Finally, we only need to study the following five cases:

(*i*) $(\dot{x}, \dot{y}) = \left(a, bx^iy^j + cx^ky^l\right),\ i \geq 1,$ (*ii*) $(\dot{x}, \dot{y}) = \left(ay^q, bx^i + cx^ky^l\right),\ i \geq 1,\ q \geq 1,$

(*iii*) $(\dot{x}, \dot{y}) = \left(ax^p, by^j + cx^ky^l\right),$ (*iv*) $(\dot{x}, \dot{y}) = \left(ax^py^q, b + cx^ky^l\right),$

(*v*) $(\dot{x}, \dot{y}) = \left(ax^py^q, by^j + cx^k\right),$

where we also keep the old notation for the new exponents. Notice that (*i*) and (*ii*) come from the first differential equations of (6) and the other three cases from the second one.

The case (*i*) has no critical point, so it has no periodic orbit by property $P_1$.

In case (*ii*), when $l = 0$ we can apply property $P_4$ with $\mathcal{U} = \mathbb{R}^2$ because the system has a polynomial first integral.

When $l \neq 0$ the system has a unique critical point $(0,0)$ and it writes as

$$(\dot{x}, \dot{y}) = (ay^q, bx^i + cx^ky^l), \quad i \geq 1,\ q \geq 1,\ l \geq 1. \tag{7}$$

Notice that studying the vector field on the axes we get

$$\dot{x}\big|_{x=0} = ay^q \quad \text{and} \quad \dot{y}\big|_{y=0} = bx^i.$$

Since a periodic orbit must surround the origin, the above conditions imply that this is only possible when $q$ and $i$ are both odd numbers and $ab < 0$. So, in this case we will assume that these conditions hold because otherwise the system has not periodic orbits.

If $l$ is even the system is invariant by the change $(x,y,t) \longrightarrow (x,-y,-t)$ and by property $P_6$ the system has no limit cycle and we are done. If $k$ is odd, then the system is invariant by the change $(x,y,t) \longrightarrow (-x,y,-t)$ and again by property $P_6$ we are done. Hence it only remains to consider the case $l$ odd and $k$ even. Notice that

$$\operatorname{div}(X) = clx^ky^{l-1},$$

and then it does not change sign and only vanishes on $\{xy = 0\}$, provided that $l \neq 1$, or on $\{x = 0\}$ provided that $l = 1$. Hence by property $P_5$ the system has no periodic orbit.

In case (*iii*), we use property $P_3$.

In case (*iv*) when $pqkl \neq 0$ we can apply property $P_1$. Also, when $p = q = 0$ we can apply property $P_1$. Next we split the study according one of the variables $p, q, k$ or $l$ vanishes and taking into account that $p^2 + q^2 \neq 0$.

Assume that $p = 0$. Then $q \neq 0$. When $l \neq 0$ we can apply again property $P_1$. When $l = 0$ we can apply property $P_4$ because the system has a polynomial first integral.

Assume that $q = 0$. Then the first equation of the system is $\dot{x} = ax^p$ and we can apply property $P_3$.

Assume that $k = 0$. Then the second equation of the system is $\dot{y} = b + cy^l$ and we can apply again property $P_3$.

Finally, assume that $l = 0$. When $p = 0$ the system has a polynomial first integral and we can apply property $P_4$ with $\mathcal{U} = \mathbb{R}^2$. When $p \neq 0$, the system has the invariant line $\mathcal{L} = \{x = 0\}$, and it can be integrated by separating the variables, giving an smooth first integral in $\mathbb{R}^2 \setminus \mathcal{L}$. Then we can apply again property $P_4$ to each of the connected components of $\mathbb{R}^2 \setminus \mathcal{L}$ and prove the non-existence of limit cycles because $\mathcal{L}$ is also invariant and eventual limit cycles can not cut it.

Finally we study case $(v)$. When $q = 0$ by property $P_3$ no periodic orbit appears. We consider four diferent subcases that cover all the situations.

When $q \neq 0$, $p = 0$ and $j = 0$ the system has a polynomial first integral and by property $P_4$ we are done.

When $q \neq 0$, $p = 0$ and $j \neq 0$ the system writes as

$$(\dot{x}, \dot{y}) = (ay^q, by^j + cx^k), \quad q \geq 1,\ j \geq 1. \tag{8}$$

If $k = 0$ in (8) then we use property $P_3$. The case $k \neq 0$ in (8) we notice that, by changing the names of same of the parameters it coincides with the system (7) studied in case $(ii)$ taking in that system $k = 0$. Hence, again this system has no limit cycle.

When $q \neq 0$, $p \neq 0$ and $j = 0$ the system has once more the invariant line $\mathcal{L} = \{x = 0\}$, and it can be integrated by separating the variables, giving an smooth first integral in $\mathbb{R}^2 \setminus \mathcal{L}$. As in the similar previous situation, we can prove that it has no limit cycles by using property $P_4$.

In the remaining case $q \neq 0$, $p \neq 0$ and $j \neq 0$. Then the $(0, 0)$ is its unique critical point and $x = 0$ is an invariant line. By property $P_2$ it has no periodic orbit.

Hence we have proved that although sometimes the system has continua of periodic orbits it has not limit cycles, as is stated in the lemma. $\square$

## 4. Proof of Theorem 1.1

To find $m$ hyperbolic limit cycles for the continuous planar differential system

$$(\dot{x}, \dot{y}) = (-y, x) + \varepsilon \sum_{j=0}^{n} b_j X_j(x, y),$$

we will apply the averaging method given in the Theorem 3.4. Notice that we have taken $\mathbf{a} = \varepsilon \mathbf{b}$ in the expression (2) and $\varepsilon$ is a small parameter. As usual, we write the system in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, see for instance [5]. We get

$$\begin{aligned}
\dot{r} =& \varepsilon \sum_{j=0}^{n} b_j \big(x f_j(x, y) + y g_j(x, y)\big) = \varepsilon \sum_{j=0}^{n} b_j F_j(\theta) r^{\alpha_j},\\
\dot{\theta} =& 1 + \varepsilon \sum_{j=0}^{n} b_j \big(x g_j(x, y) - y f_j(x, y)\big) = 1 + \varepsilon \sum_{j=0}^{n} b_j G_j(\theta) r^{\alpha_j - 1},
\end{aligned}$$

where

$$\begin{aligned}
F_j(\theta) =& f_j(\cos\theta, \sin\theta)\cos\theta + g_j(\cos\theta, \sin\theta)\sin\theta,\\
G_j(\theta) =& g_j(\cos\theta, \sin\theta)\cos\theta - f_j(\cos\theta, \sin\theta)\sin\theta.
\end{aligned}$$

Finally, we have the differential equation

$$\frac{dr}{d\theta} = r' = \frac{\varepsilon \sum_{j=0}^{n} b_j F_j(\theta) r^{\alpha_j}}{1 + \varepsilon \sum_{j=0}^{n} b_j G_j(\theta) r^{\alpha_j - 1}} = \varepsilon \sum_{j=0}^{n} b_j F_j(\theta) r^{\alpha_j} + \mathcal{O}(\varepsilon^2). \tag{9}$$

It is under the hypotheses of Theorem 3.4 for $(\theta, r) \in \mathbb{R} \times [R_0, R_1]$ for $R_1 > R_0 > 0$,with $T = 2\pi$ and $\varepsilon$ small enough. We can easily compute the averaged function $h$,

$$h(z) = \frac{1}{2\pi} \int_0^{2\pi} \sum_{j=0}^{n} b_j F_j(\theta) z^{\alpha_j} \, d\theta = \sum_{j=0}^{n} \frac{b_j}{2\pi} \Big( \int_0^{2\pi} F_j(\theta) \, d\theta \Big) z^{\alpha_j} = \sum_{j=0}^{n} \frac{b_j I_j}{2\pi} z^{\alpha_j}.$$

Since, from all $I_j, j = 0, 1, \ldots, n$, only $m + 1$ values are non-zero, we rename the corresponding ordered $\alpha_j$ as $\beta_0, \beta_1, \ldots, \beta_m$ and then

$$h(z) = \sum_{j=0}^{m} c_j z^{\beta_j},$$

with all $c_j$ arbitrary real constants and $\beta_0 < \beta_1 < \cdots < \beta_m$. By Lemma 3.3 the functions $z^{\beta_j}$ form an $ECT$-system on $(0, \infty)$, and consequently also on $(R_0, R_1)$. In particular, the maximum number of positive zeroes of $h$ is $m$ and there exist $c_0, c_1, \ldots, c_m$ such that the associated $h$ has exactly $m$ simple zeroes in $(R_0, R_1)$. Notice that the upper bound of $m$ zeroes for $h$ is also a straightforward consequence of Descarte's rule of signs, which also works for this family of functions. Taking the corresponding values of $\mathbf{b}$ we obtain a system with $|\varepsilon|$ small enough and at least $m$ periodic orbits. Since these positives zeros of $h$ are simple roots, $h'$ does not vanish on them. Then, again by Theorem 3.4, they are hyperbolic limit cycles. Hence, the theorem is proved.

4.1. **Examples of application.** As a first application we prove that the simple differential system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} + \begin{pmatrix} q_{1,1} & q_{1,2} \\ q_{2,1} & q_{2,2} \end{pmatrix} \begin{pmatrix} \sqrt{x}^{\,\mathbb{R}} \\ \sqrt{y}^{\,\mathbb{R}} \end{pmatrix} + \begin{pmatrix} p_{1,1} & p_{1,2} \\ p_{2,1} & p_{2,2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{10}$$

where recall that $\sqrt{z}^{\,\mathbb{R}} = \operatorname{sgn}(z)\sqrt{|z|}$, has for some values of the parameters a limit cycle crossing $\Sigma = \{xy = 0\}$. This family includes for instance the one given in Subsection 2.1.

In the notation of the theorem, all systems of the form (10) can be written as

$$(\dot{x}, \dot{y}) = \sum_{j=0}^{2} a_j X_j(x, y),$$

with $X_0(x, y) = (s_1, s_2)$, $X_1(x, y) = (q_{1,1}\sqrt{x}^{\,\mathbb{R}} + q_{1,2}\sqrt{y}^{\,\mathbb{R}}, q_{2,1}\sqrt{x}^{\,\mathbb{R}} + q_{2,2}\sqrt{y}^{\,\mathbb{R}})$ and $X_2(x, y) = (p_{1,1}x + p_{1,2}y, p_{2,1}x + p_{2,2}y)$. Moreover $(\alpha_0, \alpha_1, \alpha_2) = (0, 1/2, 1)$. Notice that for simplicity we keep the same names for the constants although they have varied. Clearly,

$$\begin{aligned}
I_0 &= \int_0^{2\pi} (s_1 \cos\theta + s_2 \sin\theta) \, d\theta = 0, \\
I_2 &= \int_0^{2\pi} (p_{1,1} \cos^2\theta + (p_{1,2} + p_{2,1}) \sin\theta \cos\theta + p_{2,2} \sin^2\theta) \, d\theta = (p_{1,1} + p_{2,2})\pi, \\
I_1 &= 4(q_{1,1} + q_{2,2}) \int_0^{\pi/2} \cos^{3/2}\theta \, d\theta,
\end{aligned}$$

where in the last equality we have used that

$$\int_0^{2\pi} \sqrt{\cos\theta}^{\,\mathbb{R}} \cos\theta \, d\theta = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{3/2}\theta \, d\theta = 4\int_0^{\frac{\pi}{2}} \cos^{3/2}\theta \, d\theta > 0,$$

$$\int_0^{2\pi} \sqrt{\sin\theta}^{\,\mathbb{R}} \sin\theta \, d\theta = 4\int_0^{\frac{\pi}{2}} \sin^{3/2}\theta \, d\theta = 4\int_0^{\frac{\pi}{2}} \cos^{3/2}\theta \, d\theta,$$

and by symmetry,

$$\int_0^{2\pi} \sqrt{\sin\theta}^{\,\mathbb{R}} \cos\theta \, d\theta = \int_0^{2\pi} \sqrt{\cos\theta}^{\,\mathbb{R}} \sin\theta \, d\theta = 0.$$

Thus when $(p_{1,1}+p_{2,2})(q_{1,1}+q_{2,2}) \neq 0$, the number of non-zero values in the list $I_0, I_1, I_2$ is 2 and by Theorem 1.1 we have a system of the form (10) with 1 hyperbolic limit cycle.

As a second example of application consider

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} + \begin{pmatrix} p_{1,1} & p_{1,2} \\ p_{2,1} & p_{2,2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} q_{1,1} & q_{1,2} \\ q_{2,1} & q_{2,2} \end{pmatrix} \begin{pmatrix} \sqrt[3]{x} \\ \sqrt{y}^{\,\mathbb{R}} \end{pmatrix}. \tag{11}$$

We will prove that it has at least 2 limit cycles crossing $\Sigma = \{xy = 0\}$ for same values of the parameters.

Writing it in the notation of Theorem 1.1 we get

$$(\dot{x}, \dot{y}) = \sum_{j=0}^{3} a_j X_j(x, y),$$

where $X_0(x, y) = (s_1, s_2)$, $X_1(x, y) = (q_{1,1}\sqrt[3]{x}, q_{2,1}\sqrt[3]{x})$, $X_2(x, y) = (q_{1,2}\sqrt{y}^{\,\mathbb{R}}, q_{2,2}\sqrt{y}^{\,\mathbb{R}})$ and $X_3(x, y) = (p_{1,1}x + p_{1,2}y, p_{2,1}x + p_{2,2}y)$. Moreover, $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (0, 1/3, 1/2, 1)$. Notice that again, for simplicity, we keep the same names for the constants although they have varied. In this case,

$$I_0 = 0, \quad I_1 = 4q_{1,1}\int_0^{\frac{\pi}{2}} \cos^{4/3}\theta \, d\theta, \quad I_2 = 4q_{2,2}\int_0^{\frac{\pi}{2}} \sin^{3/2}\theta \, d\theta, \quad I_3 = (p_{1,1} + p_{2,2})\pi.$$

where $I_0, I_2$ and $I_3$ are obtained similarly that in the previous case and to get $I_1$ we have used that

$$\int_0^{2\pi} \sqrt[3]{\cos\theta}\cos\theta \, d\theta = 4\int_0^{\frac{\pi}{2}} \cos^{4/3}\theta \, d\theta > 0 \quad \text{and} \quad \int_0^{2\pi} \sqrt[3]{\cos\theta}\sin\theta \, d\theta = 0.$$

Hence, when $q_{1,1}q_{2,2}(p_{1,1} + p_{2,2}) \neq 0$, the number of non-zero values in the list $I_0, I_1, I_2, I_3$ is 3 and by Theorem 1.1 we have an example of system (11) with at least 2 hyperbolic limit cycles.

## 5. Proof of Theorem 1.2

The statement $\mathcal{H}^M[j] = 0$, for $j = 1, 2, 3$, is a straightforward consequence of Proposition 3.6. Notice that this proposition covers all cases except the trivial ones, where either $\dot{x} = 0$ or $\dot{y} = 0$, and the right-hand side of the other equation has $j$ monomials.

Let us prove that for $m \geq 4$, $\mathcal{H}^M[m] \geq m - 3$. Consider the Liénard classic system in class $\mathcal{M}_m$,

$$(\dot{x}, \dot{y}) = (y, -x + a_0 y + a_1 y^3 + \cdots + a_{m-3}y^{2m-5}).$$

With the notation of Theorem 1.1 we get that for all $j = 0, 1, \ldots, m - 3$,

$$I_j = \int_0^{2\pi} \sin^{2j+2}\theta \, d\theta > 0,$$

and as a consequence we get examples with $m-3$ limit cycles. In fact, this system includes the celebrated van der Pol system when $m=4$ and coincides with the example of classical Liénard system studied in [15], where the author, with another notation, already proved the existence of $m-3$ limit cycles.

Notice that there are many different families in $\mathcal{M}_m$ with at least $m-3$ limit cycles. For instance it suffices to consider systems of the form

$$(\dot{x},\dot{y}) = (y, -x + a_0 x^{2n_0} y^{2k_0+1} + a_1 x^{2n_1} y^{2k_1+1} + \cdots + a_{m-3} x^{2n_{m-3}} y^{2k_{m-3}+1}),$$

with $n_j, k_j \in \mathbb{N}_0$ and all $2(n_j+k_j)$, $j=0,1,\ldots m-3$, taking different values. Also similar terms could be added in the first differential equation, removing some other ones from the second one.

To prove that $\mathcal{H}^M[m] \geq N(m)$ we will use Theorem 3.5. For a sequence of values of $n$ tending to infinity, the number of monomials of these generalized Liénard systems is $m = 2n+3$ while their number of limit cycles is at least $K(n)$. Hence these systems are in $\mathcal{M}_m$ and have at least $N(m) = K((m-3)/2)$ limit cycles. This function is the one that appears in the statement of the theorem.

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