THE SOLUTION OF THE SECOND PART OF THE 16TH HILBERT PROBLEM FOR NINE FAMILIES OF DISCONTINUOUS PIECEWISE DIFFERENTIAL SYSTEMS

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ABSTRACT. We provide the exact maximum number of limit cycles of some families of discontinuous piecewise differential systems formed by two differential systems separated by a straight line, when these differential systems are linear centers or three families of cubic isochronous centers. These maximum number of limit cycles vary from 0, 1, 2, 3, 5 and 7 depending of the chosen families.

1. Introduction and the statement of the main results

We consider a polynomial differential systems in $\mathbb{R}^2$ of the form

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{1}$$

where the degree of the systems is the maximum degree of the polynomials P and Q.

In 1900 David Hilbert [15] gave a talk at the International Congress of Mathematicians in Paris, where he provided a list of 23 problems. One of these problems which remain open up to now is the second part of the 16-th problem, in which Hilbert asked for an upper bound for the maximum number of limit cycles of all polynomial differential systems of a given degree, see also [19, 20].

In this paper we consider the discontinuous piecewise differential systems

$$X^{\pm} : (\dot{x}, \dot{y}) = (f^{\pm}(x,y), g^{\pm}(x,y)), \tag{2}$$

defined in the half-planes $\Sigma^{\pm} = \{(x,y) \in \mathbb{R}^2 : \pm x > 0\}$. On the straight line $\Sigma = \{x = 0\}$ the differential system is bivaluated. The straight line Σ is called the *straight line of discontinuity* when the two vector fields $X^{\pm}$ do not coincide on it. We use the Filippov conventions for defining the discontinuous piecewise differential system on Σ, see [9]. If $f^{+}(0,y)f^{-}(0,y) > 0$ at the point $(0,y) \in \Sigma$ we say that $(0,y)$ is a *crossing point*. If a periodic orbit of a discontinuous piecewise differential system (2) has exactly two crossing points we say that it is a *crossing periodic orbit*.

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A *limit cycle* (respectively *crossing limit cycle*) of system (1) (respectively (2)) is an isolated periodic orbit in the set of all periodic orbits (respectively crossing periodic orbits) of system (1) (respectively (2)).

The study of the piecewise linear differential systems goes back to Andronov, Vitt and Khaikin [1], and nowadays such systems still continue to receive the attention of many researchers. These differential systems are widely used to model processes appearing in electronics, mechanics, economy, etc., see for instance the books of Bernardo [3] and Simpson [32], the survey of Makarenkov and Lamb [31], as well as hundreds of references quoted in these last three works.

In recent years many authors have been widely interesting to solve the second part of the 16-th problem of Hilbert for discontinuous piecewise linear differential systems in $\mathbb{R}^2$ mainly separated by an straight line, see for instance [2, 4, 5, 6, 8, 10, 11, 12, 13, 14, 16, 17, 18, 21, 22, 23, 24, 25, 26, 27, 29, 30].

In this paper we shall work with the following four kinds of isochronous centers.

(I) A linear differential system having a center can be written as

$$\dot{x} = -bx - \frac{4b^2+\omega^2}{4a}y + d, \qquad \dot{y} = ax + by + c,$$

with $a > 0$ and $\omega > 0$, and the first integral

$$H_1(x,y) = 4(ax+by)^2 + 8a(cx-dy) + y^2\omega^2.$$

For a proof see Lemma 5 of [28].

(II) The cubic isochronous differential system

$$\dot{x} = y(2R_1x + 2R_2x^2 - 1), \qquad \dot{y} = R_1\left(y^2 - x^2\right) + 2R_2xy^2 + x,$$

with the first integral

$$\tilde{H}_2(x,y) = \frac{x^2+y^2}{1-2x(R_1+R_2x)}.$$

For a proof see page 42 of [7].

(III) The cubic isochronous differential system

$$\dot{x} = y\left(\frac{8x}{3} - \frac{32y^2}{9} - 1\right), \qquad \dot{y} = x - \frac{4y^2}{3},$$

with the first integral

$$\tilde{H}_3(x,y) = \left(3x - 4y^2\right)^2 + 9y^2.$$

For a proof see page 45 of [7].

(IV) The cubic isochronous differential system

$$\dot{x} = -y(1-x)(1-2x), \qquad \dot{y} = 2x^3 - 2x^2 + x + y^2,$$

with the first integral

$$\tilde{H}_4(x,y) = \frac{(x-1)^2\left(x^2+y^2\right)}{(2x-1)^2}.$$

For a proof see page 44 of [7].

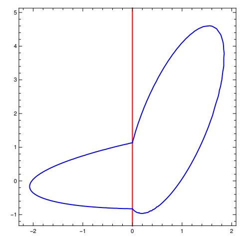

Figure 1. The unique crossing limit cycle of the discontinuous piecewise differential systems (7)–(8).

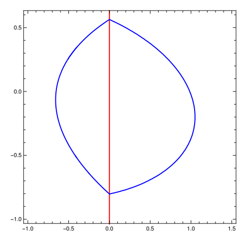

Figure 2. The unique crossing limit cycle of the discontinuous piecewise differential systems (10)–(11).

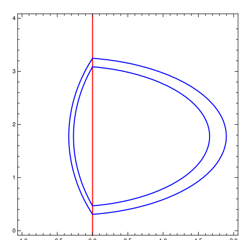

Figure 3. The two crossing limit cycles of the discontinuous piecewise differential systems (13)–(14).

Our first main goal is to provide the maximum number of crossing limit cycles of discontinuous piecewise differential systems in two half-planes separated by the straight line $x = 0$, such that in one half-plane there is an arbitrary linear differential center (I) and in the other there is either another arbitrary linear differential system (I) or one of the three cubic isochronous differential systems (II), (III) or (IV) after an arbitrary affine change of variables.

Theorem 1. *The maximum number of crossing limit cycles of discontinuous piecewise differential systems separated by the straight line $x = 0$ and formed by*

(a) *two arbitrary linear differential centers (I) in each half-plane is zero.*
(b) *the cubic isochronous system (II) after an arbitrary affine change of variables and a linear differential center is at most one. There are systems of this type with one limit cycle, see Figure* 1*;*

(c) *the cubic isochronous system (III) after an arbitrary affine change of variables and a linear differential center is at most one. There are systems of this type with one limit cycle, see Figure* 2*;*

(d) *the cubic isochronous system (IV) after an arbitrary affine change of variables and a linear differential center is at most two. There are systems of this type with two limit cycles, see Figure* 3*.*

Statement (a) of Theorem 1 has been proved in Theorem 4 of [23] and in Theorem 3 of [28]. Statements (b), (c) and (d) of Theorem 1 are proved in section 3.

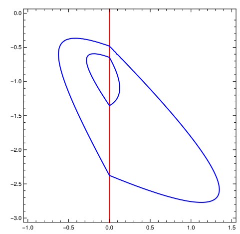

FIGURE 4. The two crossing limit cycles of the discontinuous piecewise differential systems (16)–(17).

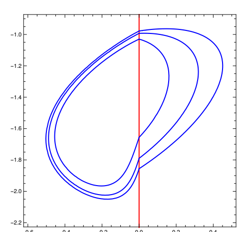

FIGURE 5. The three crossing limit cycles of the discontinuous piecewise differential systems (19)–(20).

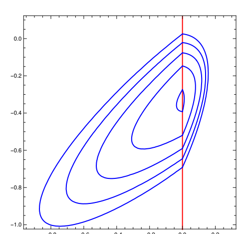

FIGURE 6. The five crossing limit cycles of the discontinuous piecewise differential systems (22)–(23).

The second main goal of this paper is to give the maximum number of crossing limit cycles of discontinuous piecewise differential systems in two half-planes separated by the straight line $x = 0$, such that in one half-plane there is one of the three cubic isochronous diferenial systems (II), (III) or (IV) after an arbitrary affine change of variables.

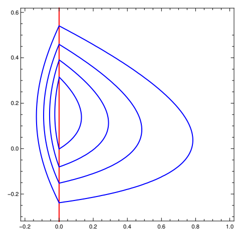

FIGURE 7. The four crossing limit cycles of the discontinuous piecewise differential systems (25)–(26).

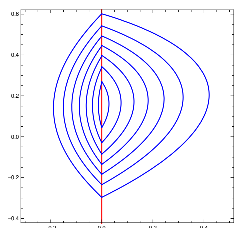

FIGURE 8. The seven crossing limit cycles of the discontinuous piecewise differential systems (28)–(29).

Theorem 2. *The maximum number of crossing limit cycles of discontinuous piecewise isochronous cubic differential systems* (II), (III) *or* (IV) *after an affine change of the variables, which are separated by the straight line* $x = 0$ *and formed by*

(a) *the cubic isochronous system* (II) *in each half-plane is at most two. There are systems of this type with exactly two limit cycles, see Figure* 4*;*
(b) *the cubic isochronous systems* (II) *and* (III) *is at most three. There are systems of this type with exactly three limit cycles, see Figure* 5*;*
(c) *the cubic isochronous systems* (II) *and* (IV) *is at most five. There are systems of this type with exactly five limit cycles, see Figure* 6*;*
(d) *the cubic isochronous system* (III) *in each half-plane is at most four. There are systems of this type with exactly four limit cycles, see Figure* 7*;*
(e) *the cubic isochronous system* (III) *and* (IV) *is at most seven. There are systems of this type with exactly seven limit cycle, see Figure* 8*;*
(f) *the cubic isochronous system* (IV) *in each half-plane is at most twelve.*

The proof of Theorem 2 is given in section 4. Note that under the assumptions of statement (f) we cannot find a discontinuous piecewise differential system realizing the upper bound of twelve of crossing limit cycles for the maximum number of such limit cycles.

2. The cubic isochronous differential systems (II), (III) and (IV) after an affine change of variables

In this section we give the expression of the cubic isochronous differential systems (II), (III) and (IV) and their first integrals after the general affine change of variables

$(x,y) \to (a_1x+b_1y+c_1, a_2x+b_2y+c_2)$. Thus after this affine change of variables the differential system (II) becomes

$$
(3)\qquad
\begin{aligned}
\dot{x} = \ & \frac{1}{(a_2b_1-a_1b_2)}\big(b_1c_1+b_2c_2-b_1c_1^2R_1-2b_2c_1c_2R_1+b_1c_2^2R_1\\
&-2b_2c_1^2c_2R_2+2b_1c_1c_2^2R_2-(-a_1b_1-a_2b_2+2a_1b_1c_1R_1+2a_2b_2c_1R_1\\
&-2a_2b_1c_2R_1+2a_1b_2c_2R_1+2a_2b_2c_1^2R_2-4a_2b_1c_1c_2R_2+4a_1b_2c_1c_2R_2\\
&-2a_1b_1c_2^2R_2)x-(-b_1^2-b_2^2+2b_1^2c_1R_1+2b_2^2c_1R_1+2b_2^2c_1^2R_2\\
&-2b_1^2c_2^2R_2)y-(a_1^2b_1R_1-a_2^2b_1R_1+2a_1a_2b_2R_1-2a_2^2b_1c_1R_2\\
&+4a_1a_2b_2c_1R_2-4a_1a_2b_1c_2R_2+2a_1^2b_2c_2R_2)x^2-2(a_1b_1^2R_1\\
&+a_1b_2^2R_1+2a_1b_2^2c_1R_2-2a_2b_1^2c_2R_2)xy-b_1(b_1^2R_1+b_2^2R_1+2b_2^2c_1R_2\\
&-2b_1b_2c_2R_2)y^2-2a_1a_2R_2(-a_2b_1+a_1b_2)x^3+2(a_2b_1-a_1b_2)(a_2b_1\\
&+a_1b_2)R_2x^2y+2b_1b_2R_2(a_2b_1-a_1b_2)xy^2\big),\\
\dot{y} = \ & \frac{1}{(a_2b_1-a_1b_2)}\big(-a_1c_1-a_2c_2+a_1c_1^2R_1+2a_2c_1c_2R_1-a_1c_2^2R_1\\
&+2a_2c_1^2c_2R_2-2a_1c_1c_2^2R_2+(-a_1^2-a_2^2+2a_1^2c_1R_1+2a_2^2c_1R_1\\
&+2a_2^2c_1^2R_2-2a_1^2c_2^2R_2)x+(-a_1b_1-a_2b_2+2a_1b_1c_1R_1+2a_2b_2c_1R_1\\
&+2a_2b_1c_2R_1-2a_1b_2c_2R_1+2a_2b_2c_1^2R_2+4a_2b_1c_1c_2R_2-4a_1b_2c_1c_2R_2\\
&-2a_1b_1c_2^2R_2)y+a_1(a_1^2R_1+a_2^2R_1+2a_2^2c_1R_2-2a_1a_2c_2R_2)x^2\\
&+2(a_1^2b_1R_1+a_2^2b_1R_1+2a_2^2b_1c_1R_2-2a_1^2b_2c_2R_2)xy+(a_1b_1^2R_1\\
&+2a_2b_1b_2R_1-a_1b_2^2R_1+4a_2b_1b_2c_1R_2-2a_1b_2^2c_1R_2+2a_2b_1^2c_2R_2\\
&-4a_1b_1b_2c_2R_2)y^2-2a_1a_2R_2(-a_2b_1+a_1b_2)x^2y+2R_2(a_2b_1-a_1b_2)\\
&(a_2b_1+a_1b_2)xy^2+2b_1b_2R_2(a_2b_1-a_1b_2)y^3\big),
\end{aligned}
$$

with the first integral

$$
\begin{aligned}
H_2(x,y) = \ & \frac{1}{1-2(c_1+a_1x+b_1y)(R_1+R_2(c_1+a_1x+b_1y))}\big(c_1^2+c_2^2\\
&+(a_1^2+2a_1c_1+2a_2c_2)x+a_2^2x^2+2(b_1c_1+b_2c_2)y\\
&+2(a_1b_1+a_2b_2)xy+(b_1^2+b_2^2)y^2\big).
\end{aligned}
$$

The differential system (III) becomes

$$
(4)\qquad
\begin{aligned}
\dot{x} = \ & \frac{1}{9(a_2b_1-a_1b_2)}\big(9b_1c_1+9b_2c_2-24b_2c_1c_2-12b_1c_2^2+32b_2c_2^3+3(3a_1b_1\\
&+3a_2b_2-8a_2b_2c_1-8a_2b_1c_2-8a_1b_2c_2+32a_2b_2c_2^2)x+3(3b_1^2+3b_2^2\\
&-8b_2^2c_1-16b_1b_2c_2+32b_2^2c_2^2)y-12a_2(a_2b_1+2a_1b_2-8a_2b_2c_2)x^2\\
&-24b_2(2a_2b_1+a_1b_2-8a_2b_2c_2)xy-12b_2^2(3b_1-8b_2c_2)y^2+32a_2^3b_2x^3\\
&+96a_2^2b_2^2x^2y+96a_2b_2^3xy^2+32b_2^4y^3\ big),\\
\dot{y} = \ & \frac{1}{9(a_2b_1-a_1b_2)}\big(-9a_1c_1-9a_2c_2+24a_2c_1c_2+12a_1c_2^2-32a_2c_2^3\\
&-3(3a_1^2+3a_2^2-8a_2^2c_1-16a_1a_2c_2+32a_2^2c_2^2)x-3(3a_1b_1+3a_2b_2\\
&-8a_2b_2c_1-8a_2b_1c_2-8a_1b_2c_2+32a_2b_2c_2^2)y+12a_2^2(3a_1-8a_2c_2)x^2\\
&+24a_2(a_2b_1+2a_1b_2-8a_2b_2c_2)xy+12b_2(2a_2b_1+a_1b_2-8a_2b_2c_2)y^2\\
&-32a_2^4x^3-96a_2^3b_2x^2y-96a_2^2b_2^2xy^2-32a_2b_2^3y^3\big),
\end{aligned}
$$

with the first integral

$$
\begin{aligned}
H_3(x,y) = \;& 9c_1^2+9c_2^2-24c_1c_2^2+16c_2^4+2(9a_1c_1+9a_2c_2-24a_2c_1c_2\\
&-12a_1c_2^2+32a_2c_2^3)x+2(9b_1c_1+9b_2c_2-24b_2c_1c_2-12b_1c_2^2\\
&+32b_2c_2^3)y+3(3a_1^2+3a_2^2-8a_2^2c_1-16a_1a_2c_2+32a_2^2c_2^2)x^2\\
&+6(3a_1b_1+3a_2b_2-8a_2b_2c_1-8a_2b_1c_2-8a_1b_2c_2+32a_2b_2c_2^2)\\
&xy+3(3b_1^2+3b_2^2-8b_2^2c_1-16b_1b_2c_2+32b_2^2c_2^2)y^2-8a_2^2(3a_1\\
&-8a_2c_2)x^3-24a_2(a_2b_1+2a_1b_2-8a_2b_2c_2)x^2y-24b_2(2a_2b_1\\
&+a_1b_2-8a_2b_2c_2)xy^2-8b_2^2(3b_1-8b_2c_2)y^3+16a_2^4x^4\\
&+64a_2^3b_2x^3y+96a_2^2b_2^2x^2y^2+64a_2b_2^3xy^3+16b_2^4y^4.
\end{aligned}
$$

The differential system (IV) becomes

$$
\begin{aligned}
\dot{x} = \;& \frac{1}{(a_2b_1-a_1b_2)}\big(b_1c_1-2b_1c_1^2+2b_1c_1^3+b_2c_2-3b_2c_1c_2+2b_2c_1^2c_2+b_1c_2^2\\
&+(a_1b_1+a_2b_2-4a_1b_1c_1-3a_2b_2c_1+6a_1b_1c_1^2+2a_2b_2c_1^2+2a_2b_1c_2\\
&-3a_1b_2c_2+4a_1b_2c_1c_2)x+(b_1^2+b_2^2-4b_1^2c_1-3b_2^2c_1+6b_1^2c_1^2+2b_2^2c_1^2\\
&-b_1b_2c_2+4b_1b_2c_1c_2)y+(-2a_1^2b_1+a_2^2b_1-3a_1a_2b_2+6a_1^2b_1c_1\\
&+4a_1a_2b_2c_1+2a_1^2b_2c_2)x^2+(-4a_1b_1^2-a_2b_1b_2-3a_1b_2^2+12a_1b_1^2c_1\\
&+4a_2b_1b_2c_1+4a_1b_2^2c_1+4a_1b_1b_2c_2)xy+2b_1(-b_1^2-b_2^2+3b_1^2c_1+2b_2^2c_1\\
&+b_1b_2c_2)y^2+2a_1^2(a_1b_1+a_2b_2)x^3+2a_1(3a_1b_1^2+2a_2b_1b_2+a_1b_2^2)x^2y\\
&+2b_1(3a_1b_1^2+a_2b_1b_2+2a_1b_2^2)xy^2+2b_1^2(b_1^2+b_2^2)y^3\big),\\
\dot{y} = \;& \frac{1}{(a_2b_1-a_1b_2)}\big(-a_1c_1+2a_1c_1^2-2a_1c_1^3-a_2c_2+3a_2c_1c_2-2a_2c_1^2c_2\\
&-a_1c_2^2-(a_1^2+a_2^2-4a_1^2c_1-3a_2^2c_1+6a_1^2c_1^2+2a_2^2c_1^2-a_1a_2c_2\\
&+4a_1a_2c_1c_2)x-(a_1b_1+a_2b_2-4a_1b_1c_1-3a_2b_2c_1+6a_1b_1c_1^2\\
&+2a_2b_2c_1^2-3a_2b_1c_2+2a_1b_2c_2+4a_2b_1c_1c_2)y-2a_1(-a_1^2-a_2^2\\
&+3a_1^2c_1+2a_2^2c_1+a_1a_2c_2)x^2-(-4a_1^2b_1-3a_2^2b_1-a_1a_2b_2\\
&+12a_1^2b_1c_1+4a_2^2b_1c_1+4a_1a_2b_2c_1+4a_1a_2b_1c_2)xy-(-2a_1b_1^2\\
&-3a_2b_1b_2+a_1b_2^2+6a_1b_1^2c_1+4a_2b_1b_2c_1+2a_2b_1^2c_2)y^2-2a_1^2(a_1^2\\
&+a_2^2)x^3-2a_1(3a_1^2b_1+2a_2^2b_1+a_1a_2b_2)x^2y-2b_1(3a_1^2b_1+a_2^2b_1\\
&+2a_1a_2b_2)xy^2-2b_1^2(a_1b_1+a_2b_2)y^3\big),
\end{aligned}
\tag{5}
$$

with the first integral

$$
\begin{aligned}
H_4(x,y) = \;& \frac{(c_1-1+a_1x+b_1y)^2}{2c_1(-1+2a_1x+2b_1y)^2}\big(c_1^2+c_2^2+2(a_1c_1+a_2c_2)x+2(b_1c_1\\
&+b_2c_2)y+(a_1^2+a_2^2)x^2+2(a_1b_1+a_2b_2)xy+(b_1^2+b_2^2)y^2\big).
\end{aligned}
$$

3. Proof of Theorem 1

We recall that statement (a) of Theorem 1 has already been proved in Theorem 4 of [23] and in Theorem 3 of [28].

Proof of statement (b) of Theorem 1. We consider the cubic polynomial differential system (3) with its first integral $H_2(x,y)$ in the half-plane $R_1=\{(x,y): x>0\}$, and the planar linear differential center (I) with its first integral $H_1(x,y)$ in the half-plane $R_2=\{(x,y): x<0\}$. The crossing limit cycles of this discontinuous piecewise differential systems (I)-(3) intersect the line of discontinuity $x=0$ in two different points $(0, y)$ and $(0, Y)$. Clearly these two points must satisfy the system of equations

$$
\begin{aligned}
H_1(0,y) - H_1(0,Y) = &\quad (Y-y)(8ad - 4b^2y - 4b^2Y - \omega^2 y - \omega^2 Y) = 0, \\
= &\quad (Y-y)Q_1(y,Y), \\
H_2(0,y) - H_2(0,Y) = &\quad P_2(y,Y) = 0,
\end{aligned} \tag{6}
$$

where $Q_1(y, Y)$ and $P_2(y, Y)$ are polynomials of degrees two and one, respectively.

From $Q_1(x, y) = 0$ we obtain $Y = f(y)$. Substituting the expression of Y in $P_2(y, Y) = 0$, we obtain a quadratic equation in the variable y. This equation has at most two real solutions y_1 and y_2. Therefore system (6) has at most two real solutions of the form $(y_k, f(y_k))$ for $k = 1, 2$. We can easily show that the two solutions are symmetric in the following sense $(y_1, f(y_1)) = (f(y_2), y_2)$. So both solutions provides the same limit cycle for the discontinuous piecewise differential system (I)-(3). In summary we have proved that the discontinuous piecewise differential system (I)-(3) can have at most one limit cycle.

Now we give an example of a discontinuous piecewise differential system (I)-(3) having one limit cycle. In the half-plane R_1 we consider the cubic isochronous differential center

$$
\begin{aligned}
\dot{x} = &-3.2..x^3 + x^2(6.64..y + 12.5403..) + x((-1.2..y - 4.17612..)y \\
&-5.77612..) + y(-1.70989..y - 3.29478..) - 1.10075.., \\
\dot{y} = &\; x^2(-3.2..y - 5.89851..) + x(y(6.64..y + 25.3194..) + 20.1194..) \\
&+y((-1.2..y - 8.31194..)y - 16.2239..) - 9.25373..,
\end{aligned} \tag{7}
$$

of type (3) with the first integral

$$
H_2(x,y) = \frac{-3.625..x^2 + x(3.125..y + 5.625..) + y(-1.88281..y - 5.3125..) - 3.90625..}{x^2 + x(-3.75..y - 6.875..) + y(3.51563..y + 12.8906..) + 10.1563..}.
$$

In the half-plane R_2 we consider the linear differential center

$$
\dot{x} = 2 + 2x - 13y, \quad \dot{y} = 2 + x - 2y, \tag{8}
$$

with the first integral

$$
H_1(x,y) = 4(x-2y)^2 + 8(-2x-2y) + 36y^2.
$$

Without loss of generality we can restrict our attention to the solutions of system (6) such that $y < Y$, then we obtain the following unique solution of system (6) is $\left(2\left(947 - \sqrt{55178849}\right)/12311, 2\left(\sqrt{55178849} + 947\right)/2311\right)$, that provides the crossing limit cycle of the discontinuous piecewise differential system (7)–(8) shown in Figure 1. □

Proof of statement (c) of Theorem 1. In the half-plane R_1 we consider the cubic differential system (4) with its first integral $H_3(x, y)$. In the half-plane R_2 we consider the linear differential center (I) with its first integral $H_1(x, y)$. The crossing limit cycles of this discontinuous piecewise differential systems (I)-(4) intersect the line of discontinuity $x = 0$ in two different points $(0, y)$ and $(0, Y)$, and these two points must satisfy the system of equations

$$(9)\qquad \begin{aligned} H_1(0,y)-H_1(0,Y) = &\quad (Y-y)(8ad-4b^2y-4b^2Y-\omega^2y-\omega^2Y)=0,\\ = &\quad (Y-y)Q_1(y,Y),\\ H_3(0,y)-H_3(0,Y) = &\quad (Y-y)P_3(y,Y)=0, \end{aligned}$$

where $Q_1(y,Y)$ and $P_3(y,Y)$ are polynomials of degrees three and one, respectively.

By solving $Q_1(y,Y)=0$ we obtain $Y=f(y)$. We replace Y in $P_3(y,Y)=0$, and we obtain a quadratic equation in the variable y, the coefficient of y^3 vanishes. This equation has at most two real solutions again symmetric in the sense of the proof of statement (b). Therefore system (9) has at most one real solution (y,Y) satisfying $y<Y$. Hence the discontinuous piecewise differential system (I)–(4) has at most one limit cycle.

In what follows we provide a discontinuous piecewise differential system (I)–(4) with one limit cycle. In the half-plane R_1 we consider the cubic isochronous differential center

$$(10)\qquad \begin{aligned} \dot{x} = &\quad (1/38961)(9(-432(24x+35)y^2+144x(223-6x)y+x(3x(927\\ &\quad -8x)+5108)-41472y^3)+97119y+13264),\\ \dot{y} = &\quad (1/116883)(9(216(12x-223)y^2+54x(4x-309)y+x(6(x-176)x\\ &\quad +41051)+10368y^3)-137916y+146756), \end{aligned}$$

of type (4) with the first integral

$$\begin{aligned} H_3(x,y) = &\quad 18(24x+35)y^3+\frac{9}{16}(16x(6x-223)-1199)y^2+\frac{1}{8}x(3x(8x-927)\\ &\quad -5108)y+\frac{1}{432}x(9x(x(3x-704)+41051)+293512)+1296y^4\\ &\quad -\frac{1658y}{9}+\frac{10897}{81}. \end{aligned}$$

In the half-plane R_2 we consider the linear differential center

$$(11)\qquad \dot{x}=-\frac{x}{2}-\frac{13y}{2}-0.776878.., \quad \dot{y}=x+\frac{y}{2}+1,$$

with the first integral

$$H_1(x,y)=4\left(x+\frac{y}{2}\right)^2+8(x+0.776878..y)+25y^2.$$

The unique solution of system (9) is $(-0.802603..,0.563563..)$. The crossing limit cycle of the discontinuous piecewise differential system (10)–(11) associated to this solution is shown in Figure 2. □

Proof of statement (d) of Theorem 1. In the half-plane R_1 we consider the cubic differential system (5) with its first integral $H_4(x,y)$. In the half-plane R_2 we consider the linear differential center (I) with its first integral $H_1(x,y)$.

If the discontinuous piecwise differential system (I)–(5) has a crossing limit cycle intersecting the line of discontinuity $x=0$ in the points $(0,y)$ and $(0,Y)$, these points satisfy the system of equations

$$(12)\qquad \begin{aligned} H_1(0,y)-H_1(0,Y) = &\quad (Y-y)(8ad-4b^2y-4b^2Y-\omega^2y-\omega^2Y)=0,\\ = &\quad (Y-y)Q_1(y,Y),\\ H_4(0,y)-H_4(0,Y) = &\quad \frac{(y-Y)P_5(y,Y)}{(2c_1-1+2b_1y)^2(2c_1-1+2b_1Y)^2}=0, \end{aligned}$$

where $Q_1(y,Y)$ and $P_5(y,Y)$ are polynomials of degrees four and one, respectively.

By solving $Q_1(y,Y)=0$ we obtain $Y=f(y)$. We replace Y in $P_5(y,Y)=0$, and we obtain a quartic equation in the variable y, the coefficient of y^5 vanishes. This equation has at most four real solutions again symmetric in the sense of the proof of statement (b). Therefore system (12) has at most two real solutions (y,Y) satisfying $y<Y$. Hence the discontinuous piecewise differential system (I)–(5) has at most two limit cycles.

Now we give a discontinuous piecewise differential system (I)–(5) having two limit cycles. In the half-plane R_1 we consider the cubic isochronous differential center

$$\begin{aligned}\dot{x}= & -\frac{1}{3}(3x+5.68138..)(6x+12.3628..)(2x+3y-5.33333..),\\ \dot{y}= & \ 26(x^3+x^2(0.461538..y+4.84226..)+x(1.97889..y+6.9949..)\\ & +(0.115385..y+1.3907..)y+3.75173..),\end{aligned} \tag{13}$$

of type (5) with the first integral

$$\begin{aligned}H_1(x,y)= & \frac{1}{(x+2.06046..)^2}(3.25(x+1.89379..)^2(x^2+x(0.923077..y+1.44269..)\\ & +(0.692308..y-2.46154..)y+5.62194..)).\end{aligned}$$

In the half-plane R_2 we consider the linear differential center

$$\dot{x}=4-\frac{9y}{4},\quad \dot{y}=\frac{1}{3}+x, \tag{14}$$

with the first integral

$$H_2(x,y)=4x^2+\frac{8x}{3}+y(9y-32).$$

The two real solutions of system (12) are $\left(\left(16-5\sqrt{7}\right)/9,(5\sqrt{7}+16)/9\right)$ and $\left(\left(16-\sqrt{139})/9\right),(\sqrt{139}+16)/9\right)$. Then the two crossing limit cycles of systems (I)–(5) associated to these two solutions are shown in Figure 3. □

In short, the proof of Theorem 1 is done.

4. Proof of Theorem 2

Proof of statement (a) of Theorem 2. In the half-plane R_1 we consider an isochronous cubic differential system (3) with its first integral $H_2(x,y)$. By changing the parameters $(a_1,a_2,b_1,b_2,c_1,c_2,R_1,R_2)$ by the parameters $(\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2,S_1,S_2)$ in system (3) and in its first integral, we get a second isochronous cubic differential system of type (3) with the first integral $\tilde{H}_2(x,y)$.

If a crossing limit cycle of this discontinuous piecewise differential system (3)–(3) has two intersecting points $(0,y)$ and $(0,Y)$ with the line of discontinuity $x=0$, the coordinates y and Y must satisfy the system

$$\begin{aligned}H_2(0,y)-H_2(0,Y)= & \ Q_2(y,Y)=0,\\ \tilde{H}_2(0,y)-\tilde{H}_2(0,Y)= & \ P_2(y,Y)=0,\end{aligned} \tag{15}$$

where $Q_2(y,Y)$ and $P_2(y,Y)$ are polynomials of degrees two.

By Bezout Theorem (see for instance [33]) system (15) has at most four real solutions (y, Y), which again are symmetric in the sense of the proof of statement (b) of Theorem 1, therefore these solutions at most provide two limit cycles of the discontinuous piecewise differential system (3)–(3).

Now we provide an example of discontinuous piecewise differential system (3)–(3) having two limit cycles. In the half-plane R_1 we consider the cubic isochronous differential center

$$\begin{aligned} \dot{x} &= -4 + 10y + 4((-1+x)x + x(7+x)y + (4+x)y^2), \\ \dot{y} &= 1 - 6x^2 + 2(-2+y)y(3+2y) + 4x(-1+(-4+y)y), \end{aligned} \tag{16}$$

of type (3) with the first integral

$$H_2(x,y) = \frac{-4\left(xy + x + y^2\right) - 1}{2x^2 + 4x(y+2) + 2y(y+4) + 3}.$$

In the half-plane R_2 we consider the cubic isochronous differential center

$$\begin{aligned} \dot{x} = {} & \frac{1}{64}(-8x^2(-16y + \sqrt{601} - 5) + x(16y(8y + \sqrt{57} + 5) + 5\sqrt{57} \\ & -\sqrt{34257} - 32) + (\sqrt{57} + \sqrt{601})(y(8y+5) - 2)), \\ \dot{y} = {} & \frac{(16y(8y - \sqrt{601} + 5) - 5\sqrt{601} + 57)(16x + 16y + \sqrt{57} + 5)}{1024}, \end{aligned} \tag{17}$$

of type (3) with the first integral

$$\begin{aligned} \tilde{H}_2(x,y) = {} & \frac{-1}{2(\frac{y}{2} + \frac{1}{32}(5 - \sqrt{601}))^2 - 1}(4(2x + \frac{1}{8}(\sqrt{57}+5))y + x(4x + \frac{1}{2}(\sqrt{57} \\ & +5)) + \frac{17y^2}{4} + \frac{1}{32}(5 - \sqrt{601})y + \frac{1}{64}(\sqrt{57}+5)^2 + \frac{(5-\sqrt{601})^2}{1024}). \end{aligned}$$

The real solutions of system (15) are $\left(\frac{1}{4}\left(-\sqrt{2}-4\right), \frac{1}{4}\left(\sqrt{2}-4\right)\right)$ and $\left(\frac{2}{7}\left(-\sqrt{11}-5\right), \frac{2}{7}\left(\sqrt{11}-5\right)\right)$. Then the two crossing limit cycles of systems (16)–(17) provided by these two solutions are shown in Figure 4. □

Proof of statement (b) of Theorem 2. In the half-plane R_1 we consider an isochronous cubic differential system (3) with its first integral $H_2(x, y)$. In the half-plane R_2 we consider an isochronous cubic differential system (4) with its first integral $H_3(x, y)$. If a crossing limit cycle of the discontinuous piecewise differential system (3)–(4) has the two intersecting points $(0, y)$ and $(0, Y)$ with the line of discontinuity $x = 0$, the coordinates y and Y must satisfy the system

$$\begin{aligned} H_2(0,y) - H_2(0,Y) = {} & Q_2(y,Y) = 0, \\ H_3(0,y) - H_3(0,Y) = {} & (y - Y)P_3(y,Y) = 0, \end{aligned} \tag{18}$$

where $Q_2(y, Y)$ and $P_3(y, Y)$ are polynomials of degree two and three, respectively.

From the Bézout Theorem we obtain that the number of real solutions (y, Y) of system (18) is at most six. Again these solutions are symmetric in the usual sense. Therefore the number of real solutions (y, Y) with $y < Y$ of system (18) is at most three. So the maximum number of crossing limit cycles that the discontinuous piecewise differential systems (3)–(4) can exhibit is at most three.

In what follows we give a discontinuous piecewise differential system of type (3)–(4) with exactly three crossing limit cycles. In the half-plane R_1 we consider the cubic isochronous differential center

$$
(19) \qquad
\begin{aligned}
\dot{x} = & \frac{1}{301}((7052x^2+6296x-2187)y+8(43x-185)y^2+7x(12x \\
& (99-43x)+1093)-413), \\
\dot{y} = & \frac{1}{301}(-28x^2(129y+217)+2x(2y(1763y+5474)+7105) \\
& +y(4y(86y+675)+6797)+4459),
\end{aligned}
$$

of type (3) with the first integral

$$H_1(x,y)=\frac{-49\left(9x^2+3x+5\right)+14(11x-29)y-197y^2}{49\left(4x^2-8x(2y+3)+16y(y+3)+31\right)}.$$

In the half-plane R_2 we consider the cubic isochronous differential center

$$
(20) \qquad
\begin{aligned}
\dot{x} = & \ 0.00235656..x^3+x^2(0.063627..y-0.187745..)+x((0.572643..y \\
& -0.712743..)y-2.89186..)+y(y(1.71793..y+8.79266..) \\
& +14.9187..)+8.27703.., \\
\dot{y} = & \ x^2(0.0373215..-0.00706967..y)+x(y(0.37549..-0.063627..y) \\
& -1.24286..)+y(y(0.356372..-0.190881..y)+2.89186..) \\
& -0.00026184..x^3+2.52245..,
\end{aligned}
$$

of type (4) with the first integral

$$
\begin{aligned}
H_2(x,y) = & \ 126.1054..-93.9715..x+23.1508..x^2-0.4634..x^3+0.0024..x^4 \\
& +308.3535..y-107.7336..xy-6.9942..x^2y+0.0877..x^3y \\
& +277.8910..y^2-13.2763..xy^2+1.1851..x^2y^2+109.1876..y^3 \\
& +7.1111..xy^3+16y^4.
\end{aligned}
$$

The three real solutions of system (18) are $(-1.85583..,-0.977752..)$, $(-1.78964..,-0.992802..)$ and $(-1.65615..,-1.02976..)$. Then the three crossing limit cycles of the discontinuous piecewise differential system (19)–(20) associated to these three solutions are shown in Figure 5. This completes the proof of statement (b). □

Proof of statement (c) of Theorem 2. In the half-plane R_2 we consider the isochronous cubic differential system (3) with its first integral $H_2(x,y)$ with the parameters $(\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2,S_1,S_2)$ instead of the parameters $(a_1,a_2,b_1,b_2,c_1,c_2,R_1,R_2)$. In the half-plane R_1 we consider the isochronous cubic differential system (5) and its first integral $H_4(x,y)$. For a crossing limit cycle of the differential system (3)–(5) which intersects the line of discontinuity $x=0$ at the points $(0,y)$ and $(0,Y)$, the coordinates y and Y must satisfy the system

$$
(21) \qquad
\begin{aligned}
H_2(0,y)-H_2(0,Y)=Q_2(y,Y)=0, \\
H_4(0,y)-H_4(0,Y)=\frac{(y-Y)P_5(y,Y)}{(2c_1-1+2b_1y)^2(2c_1-1+2b_1Y)^2}=0,
\end{aligned}
$$

where $Q_2(y,Y)$ and $P_5(y,Y)$ are polynomials of degree two and five, respectively.

By using Bézout Theorem we obtain that the number of solutions of the system $Q_2(y,Y)=0$, $P_5(y,Y)=0$ is bounded by ten. Due to the usual symmetry of these solutions we obtain that the number of solutions (y,Y) with $y<Y$ is at most five, which is an upper bound for the maximum number of crossing limit cycles that a discontinuous piecewise differential system (3)–(5) can have.

Next we provide a discontinuous piecewise differential system (3)–(5) having five crossing limit cycles. In the half-plane R_1 we consider the cubic isochronous differential center

$$
\begin{aligned}
\dot{x} = & \quad -(1/2)(-2+2x)(-3+4x)(2-4x+6y), \\
\dot{y} = & \quad 4x(5-4x)y + \frac{5}{3}x(8(x-2)x+11) + 6y^2 - 8y - \frac{25}{6},
\end{aligned}
\tag{22}
$$

with the first integral

$$
H_4(x,y) = \frac{4(x-1)^2(5-20x+20x^2+24y-48xy+36y^2)}{(3-4x)^2}.
$$

In the half-plane R_2 we consider the cubic differential center

$$
\begin{aligned}
\dot{x} &= x(-221.895..x + 238.283..y + 132.807..) - 57.3212..y - 19.1071..,\\
\dot{y} &= x(74.0647.. - 153.943..x) + y(119.142..y + 26.0487..) - 7.48257..,
\end{aligned}
\tag{23}
$$

its first integral is

$$
\begin{aligned}
H_2(x,y) = & \quad \frac{1}{(x-0.240559..)}(0.643804..x^2 + x(-1.3827..y - 0.301274..) + (0.74241..y \\
& +0.49494..)y + 0.101736..).
\end{aligned}
$$

The five solutions of system (21) are $\Big((-\sqrt{74}-8)/24, (\sqrt{74}-8)/24\Big)$, $\Big((-\sqrt{14}-4)/12, (\sqrt{14}-4)/12\Big)$, $\Big((-\sqrt{38}-8)/24, (\sqrt{38}-8)/24\Big)$, $\Big((-\sqrt{5}-4)/12, (\sqrt{5}-4)/12\Big)$ and $\Big((-\sqrt{2}-8)/24, (\sqrt{2}-8)/24\Big)$. Then the five crossing limit cycles of the discontinuous piecewise differential systems (22)–(23) associated to these five solutions are shown in Figure 6. □

Proof of statement (d) of Theorem 2. In the half-plane R_1 we consider an isochronous cubic differential system (4) with its first integral $H_3(x,y)$. By changing the parameters $(a_1, a_2, b_1, b_2, c_1, c_2, R_1, R_2)$ by the parameters $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, S_1, S_2)$ in system (4) and in its first integral, we get a second isochronous cubic differential system of type (4) with the first integral $\tilde{H}_3(x,y)$.

For a crossing limit cycle of this discontinuous piecewise differential system (4)–(4) which has two intersecting points $(0,y)$ and $(0,Y)$ with the line of discontinuity $x=0$, its coordinates y and Y must satisfy the system

$$
\begin{aligned}
H_3(0,y) - H_3(0,Y) = & \quad (y-Y)Q_3(y,Y) = 0,\\
\tilde{H}_3(0,y) - \tilde{H}_3(0,Y) = & \quad (y-Y)P_3(y,Y) = 0,
\end{aligned}
\tag{24}
$$

where $Q_3(y,Y)$ and $P_3(y,Y)$ are polynomials of degree three.

By using Bézout Theorem we obtain that the number of solutions of system $Q_3(y,Y)=0$, $P_3(y,Y)=0$, is at most nine. Due to the usual symmetries we get that the number of solutions (y,Y) with $y<Y$ of system (24) is at most four, which is an upper bound for the maximum number of crossing limit cycles that can have the discontinuous piecewise differential systems (4)–(4).

In what follows we give a discontinuous piecewise differential systems (4)–(4) with four crossing limit cycles. In the half-plane R_1 we consider the cubic isochronous differential center

$$
(25) \qquad \begin{aligned}
\dot{x} = & \ (1/1437)(408 - 8x^3 + x^2(913 + 288y) + 3y(-655 + 144y(-15 \\
& +32y)) - 8x(-245 + 6y(217 + 72y))), \\
\dot{y} = & \ (1/4311)(-3730 - 2x^3 + 12x^2(29 + 6y) + 24y(-245 + 3y(217 \\
& +48y)) - 3x(4467 + 2y(913 + 144y))),
\end{aligned}
$$

of type (4) with the first integral

$$
\begin{aligned}
H_3(x, y) = & \ (1/16)(1060 + x^4 - 8x^3(29 + 6y) + 3x^2(4467 + 2y(913 + 144y)) \\
& +9y(272 + y(-655 + 288y(-5 + 8y))) + 4x(1865 - 12y(-245 \\
& +3y(217 + 48y)))).
\end{aligned}
$$

In the half-plane R_2 we consider the cubic differential center

$$
(26) \qquad \begin{aligned}
\dot{x} = & \ 0.0908181..x^3 + x^2(1.8829..y + 0.67943..) + x(y(13.0125..y \\
& +3.24792..) + 3.17466..) + y(y(29.9759..y - 10.0038..) \\
& +26.6624..) - 4.08603.., \\
\dot{y} = & \ -0.0131413..x^3 + x^2(-0.272454..y - 0.162624..) + x((-1.8829..y \\
& -1.35886..)y - 0.989936..) + y((-4.3375..y - 1.62396..)y \\
& -3.17466..) - 1.90192..,
\end{aligned}
$$

with the first integral

$$
\begin{aligned}
\tilde{H}_3(x, y) = & \ 0.197531..(x^4 + x^3(27.6435..y + 16.5.) + x^2(y(286.562..y + 206.807..) \\
& +150.66..) + x(y(y(1320.26..y + 494.305..) + 966.314..) + 578.914..) \\
& +y(y(y(2281.04..y - 1015..) + 4057.79..) - 1243.72..) + 1685.17..).
\end{aligned}
$$

The real solutions of system (24) are $(-0.238368.., 0.540639..)$, $(-0.152094.., 0.459255..)$, $(-0.0806181.., 0.389055..)$ and $(-0.00162763.., 0.310719..)$. Therefore the four crossing limit cycles of the discontinuous piecewise differential system (25)–(26) associated to these four solutions are shown in Figure 7. □

Proof of statement (e) of Theorem 2. In the half-plane R_2 we consider the isochronous cubic differential system (4) with its first integral $H_3(x, y)$. In the half-plane R_1 we consider the isochronous cubic differential system (5) with its first integral $H_4(x, y)$. If a crossing limit cycle of the discontinuous piecewise differential system (4)–(5) has two intersecting points $(0, y)$ and $(0, Y)$ with the line of discontinuity $x = 0$, then y and Y must satisfy the system

$$
(27) \qquad \begin{aligned}
H_3(0, y) - H_3(0, Y) = & \ (y - Y)Q_3(y, Y) = 0, \\
H_4(0, y) - H_4(0, Y) = & \ \frac{(y - Y)P_5(y, Y)}{(2c_1 - 1 + 2b_1y)^2(2c_1 - 1 + 2b_1Y)^2} = 0,
\end{aligned}
$$

where $Q_3(y, Y)$ and $P_5(y, Y)$ are polynomials of degree three and five, respectively.

By using Bézout Theorem we obtain that the number of solutions of system (27) is bounded by fifteen. Due to the usual symmetry the maximum number of crossing limit cycles that can have a discontinuous piecewise differential system (4)–(5) is seven.

Now we give a discontinuous piecewise differential system (4)–(5) with seven limit cycles. In the half-plane R_2 we consider the cubic isochronous differential center

$$(28)\quad \begin{aligned} \dot{x} =& \ -0.00284919..(y(y(1582.03.. - 3375y) + 732.861..) + x^3 + x^2(-45..y \\ & -180.156..) + x(y(675y + 2596.88..) - 487.578..) - 139.16..), \\ \dot{y} =& \ -0.000189946..(x^3 + x^2(-45y - 273.75..) + x(y(675y + 5404.69..) \\ & +16593.3..) + y(y(-3375y - 19476.6..) + 7313.67..) + 6119.92..), \end{aligned}$$

of type (4) with the first integral

$$\begin{aligned} H_3(x,y) =& \ 0.0256..(x^4 + x^3(-60y - 365) + x^2(y(1350y + 10809.4..) + 33186.6..) \\ & +x(y(y(-13500y - 77906.3..) + 29254.7..) + 24479.7..) + y(y(y(50625y \\ & -31640.6..) - 21985.8..) + 8349.61..) + 4560.16..). \end{aligned}$$

In the half-plane R_1 we consider the cubic differential center

$$(29)\quad \begin{aligned} \dot{x} =& \ -0.0348443..(16y(54x^2 + 12x - 37.6..(-9x - 1) - 19.1327.. \\ & (-9.56636..y - 1.95208..)y + 531.16..) + 4(-54x^3 - 525.6..x^2 \\ & +183.03..(-6x - 19.8..)y^2 + 18.6743..(-12x - 38.6..)y - 1706.28..x \\ & -1843.48..) + 128(-9x - 29.2..)y^2 - 9.56636..(-3x - 10.4..)(-6x \\ & -19.8..)(-9.56636..y - 1.95208..) + 512y^3), \\ \dot{y} =& \ 0.0348443..(162x^3 - 54x^2(12y - 29.2..) + 9x(96y^2 - 16y - 37.6.(12y \\ & -1) + 531.16..) - 3(128y^3 - 842.885..y^2 + 2312.39..y - 1843.48..))y, \end{aligned}$$

of type (5) with the first integral

$$\begin{aligned} H_4(x,y) =& \ \frac{1}{(x - 1.33333..y + 3.3..)^2}(2.25..(x - 1.33333..y + 3.46667..)^2(x(6.26667.. \\ & -2.66667..y) + x^2 + y(11.9461..y - 4.2057..) + 10.2412..)). \end{aligned}$$

The seven real solutions of system (27) are $(-0.296563.., 0.601718..)$, $(-0.234605.., 0.542436..)$, $(-0.183628.., 0.492431..)$, $(-0.13516.., 0.444505..)$, $(-0.0852419.., 0.394945..)$, $(-0.0293624.., 0.339323..)$ and $(0.044055.., 0.266104..)$. Then the seven crossing limit cycles of the discontinuous piecewise differential systems (28)–(29) associated to the seven solutions are shown in Figure 8. □

Proof of statement (f) of Theorem 2. In the half-plane R_1 we consider an isochronous cubic differential system (5) with its first integral $H_4(x,y)$. By changing the parameters $(a_1, a_2, b_1, b_2, c_1, c_2, R_1, R_2)$ by the parameters $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, S_1, S_2)$ in system (5) and in its first integral, we get a second isochronous cubic differential system of type (5) with the first integral $\tilde{H}_4(x,y)$.

For a crossing limit cycle of this discontinuous piecewise differential system (5)–(5) which has two intersecting points $(0,y)$ and $(0,Y)$ with the line of discontinuity $x = 0$, its coordinates y and Y must satisfy the system

$$(30)\quad \begin{aligned} H_4(0,y) - H_4(0,Y) =& \ \frac{(y-Y)Q_5(y,Y)}{(2c_1 - 1 + 2b_1 y)^2(2c_1 - 1 + 2b_1 Y)^2} = 0, \\ \tilde{H}_4(0,y) - \tilde{H}_4(0,Y) =& \ \frac{(y-Y)P_5(y,Y)}{(2\gamma_1 - 1 + 2\beta_1 y)^2(2\gamma_1 - 1 + 2\beta_1 Y)^2} = 0, \end{aligned}$$

where $Q_5(y,Y)$ and $P_5(y,Y)$ are polynomials of degree five.

By using Bézout Theorem we obtain that the number of solutions of system $Q_5(y,Y) = 0$, $P_5(y,Y) = 0$, is at most twenty five. Due to the usual symmetries we get that the number of solutions (y,Y) with $y < Y$ of system (30) is at most twelve, which is an upper bound for the maximum number of crossing limit cycles that can have the discontinuous piecewise differential systems (5)–(5). □

This completes the proof of Theorem 2.

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