THE LIMIT CYCLES OF PIECEWISE LINEAR DIFFERENTIAL SYSTEMS FORMED BY CENTERS AND SEPARATED BY IRREDUCIBLE CUBIC CURVES

REBIHA BENTERKI[1] AND JAUME LLIBRE[2]

Abstract. In the qualitative theory of the planar discontinuous piecewise linear differential systems one of the main problems is the study of the number of crossing limit cycles that these systems can have. We study the number of crossing limit cycles of discontinuous piecewise linear differential systems formed by centers and separated by an irreducible algebraic cubic curve. We prove that these differential systems only can exhibit 0, 1, 2 or 3 crossing limit cycles having two intersection points with the cubic of separation.

1. Introduction

One of the first works studying the discontinuous piecewise linear differential systems in the plane is due to Andronov, Vitt and Khaikin in [1]. Later on these systems became a topic of great interest in the mathematical community due to their applications for modeling real phenomena, see for instance the books [3, 19] and references there quoted.

To determine the non–existence, the existence of limit cycles and their number is one of the big problems in the qualitative theory of the planar differential systems, and in particular of the planar discontinuous piecewise linear differential systems separated by a curve Σ. In this work we are considering that a *crossing limit cycle* is a periodic orbit isolated in the set of all periodic orbits of the system which has exactly two points on the discontinuity curve Σ.

The problem of finding the best upper bound for the maximum number of limit cycles that a family of piecewise linear differential systems in the plane separated by a straight line can have, has been studied by many authors recently, see for instance [2, 5, 7, 18]. Lum and Chua[16, 17] in 1990 conjectured that the continuous (but non–smooth) piecewise linear systems in the plane separated by one straight line have at most one limit cycle. This conjecture was proved by Freire et al [6] in 1998, for a shorter proof see [11]. Han and Zhang [8] in 2010 conjectured that discontinuous piecewise linear differential systems in the plane separated by a straight line have at most two crossing limit cycles. Huan and Yang [9] in 2012 provided a negative answer to this conjecture exhibiting a numerical example with three crossing limit cycles. Llibre and Ponce in [12] proved the existence of these three limit cycles analytically. Nowadays it remains as an open problem to know

2010 *Mathematics Subject Classification.* Primary 34C29, 34C25, 47H11.
Key words and phrases. limit cycles, discontinuous piecewise linear differential systems, linear differential centers, irreducible cubic curves.

if three is the maximum number of crossing limit cycles that this class of systems can have.

In the paper [10] the authors considered the problem of Lum and Chua restricted to the class of discontinuous piecewise linear differential centers in the plane separated by a straight line, and they proved that those systems has no crossing limit cycles. But in [14, 15] were studied planar discontinuous piecewise linear differential centers with a curve of discontinuity different from a straight line, and then those systems can exhibit crossing limit cycles. For this reason it is interesting to study the role which plays the shape of the discontinuity curve in the number of crossing limit cycles that planar discontinuous piecewise linear differential centers can have.

The objective of this paper is to study the maximum number $\mathcal{N}$ of crossing limit cycles of the discontinuous piecewise linear differential centers in $\mathbb{R}^2$ separated by an irreducible algebraic cubic curve.

1.1. Classification of the irreducible cubic polynomials.

An *algebraic cubic curve* or simple a *cubic curve* is the set of points $(x, y) \in \mathbb{R}^2$ satisfying $P(x, y) = 0$ for some polynomial $P(x, y)$ of degree three. This real cubic is *irreducible* (respectively *reducible*) if the polynomial $P(x, y)$ is irreducible (respectively reducible) in the ring of all real polynomials in the variables x and y.

A point (x_0, y_0) of a cubic $P(x, y) = 0$ is *singular* if $P_x(x_0, y_0) = 0$ and $P_y(x_0, y_0) = 0$. A cubic curve is *singular* if it has some singular point.

A *flex* of an algebraic curve C is a point p of C such that C is nonsingular at p and the tangent at p of the curve C intersects C at least three times.

Theorem 1. *The following statements classify all the irreducible cubic algebraic curves.*

(a) *A cubic curve is nonsingular and irreducible if and only if it can be transformed with affine transformations into one of the following two curves;*

$$c_1(x, y) = y^2 - x(x^2 + bx + 1) = 0 \quad \text{with } b \in (-2, 2), \text{ or}$$
$$c_2(x, y) = y^2 - x(x - 1)(x - r) = 0 \quad \text{with } r > 1.$$

(b) *A cubic curve is singular and irreducible if and only if it can be transformed with affine transformations into one of the following three curves:*

$$c_3(x, y) = y^2 - x^3 = 0, \qquad \text{or}$$
$$c_4(x, y) = y^2 - x^2(x - 1) = 0, \quad \text{or}$$
$$c_5(x, y) = y^2 - x^2(x + 1) = 0.$$

See the graphics of the irreducible cubic curves $c_i = 0$ for $i = 1, \ldots, 5$ in Figures 1 to 5, respectively.

Statement (a) of Theorem 1 is proved in Theorem 8.3 of the book [4] under the additional assumption that the cubic has a flex, but in section 12 of that book it is shown that every nonsingular irreducible cubic curve has a flex. While statement (b) of Theorem 1 follows directly from Theorem 8.4 of [4].

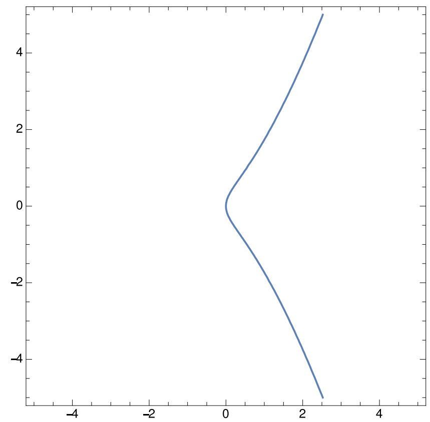

FIGURE 1. $c_1(x, y) = 0$.

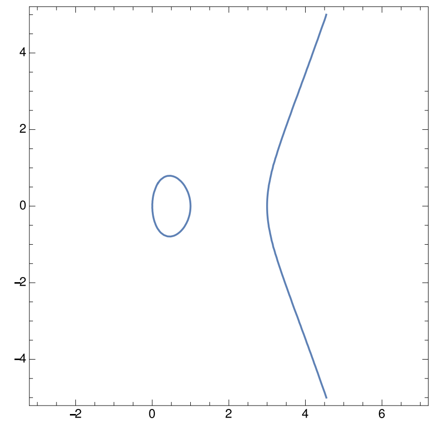

FIGURE 2. $c_2(x, y) = 0$.

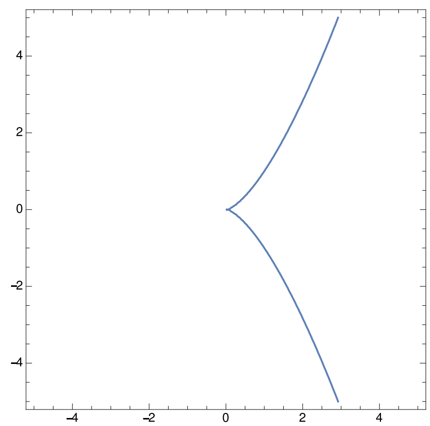

FIGURE 3. $c_3(x, y) = 0$.

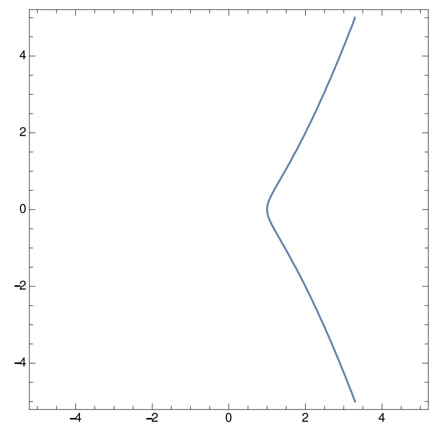

FIGURE 4. $c_4(x, y) = 0$.

1.2. **Statement of the main results.** For $k = 1, \ldots, 5$ let C_k be the five classes of planar discontinuous piecewise linear differential systems formed by centers and separated by the irreducible cubic curve $c_k(x, y) = 0$.

We recall that a *limit cycle* is a periodic solution of the discontinuous piecewise linear differential system isolated in the set of all periodic solutions of the system.

In this paper we study the *crossing limit cycles* having exactly two points on the discontinuous cubic curve $c_k(x, y) = 0$.

Our first result provides the number of crossing limit cycles of the discontinuous piecewise linear differential systems formed by centers and separated by the irreducible cubic curves c_1, c_3 and c_4. We note that such piecewise systems are formed by two pieces in each one there is a linear differential center.

Theorem 2. *The following statements hold.*

(a) *There are systems of the classes C_1, C_3 and C_4 without crossing limit cycles.*

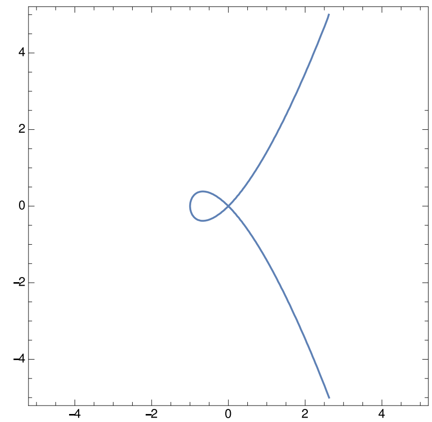

FIGURE 5. $c_5(x, y) = 0$.

(b) *There are systems of the classes C_1, C_3 and C_4 exhibiting exactly one crossing limit cycle, see Figures* 6, 7 *and* 8, *respectively.*
(c) *There are systems of the classes C_1, C_3 and C_4 exhibiting exactly two crossing limit cycles, see Figures* 9, 10 *and* 11, *respectively. These classes can't exhibit the configurations* 12, 13 *and* 14, *respectively, because the ellipses of every center are concentric.*
(d) *There are systems of the classes C_1, C_3 and C_4 exhibiting exactly three crossing limit cycles, see Figures* 15, 16 *and* 17, *respectively.*
(e) *Every system of the class C_k for $k = 1, 3, 5$ can exhibit at most three crossing limit cycles.*

Theorem 2 is proved in section 2.

Now we give our second main result which provides the number of crossing limit cycles of the discontinuous piecewise linear differential systems formed by centers and separated by the irreducible cubic curves c_2 and c_5. We note that such piecewise systems are formed by three pieces in each one there is a linear differential center.

Theorem 3. *The following statements hold.*

(a) *There are systems of the classes C_2 and C_5 without crossing limit cycles.*
(b) *There are systems of classes C_2 and C_5 exhibiting exactly one crossing limit cycle. Each class has three possible different configurations for the limit cycle, see Figures* 18, 19 *and* 20 *for the class C_2, and Figures* 21, 22 *and* 23 *for the class C_5.*
(c) *There are systems of classes C_2 and C_5 exhibiting exactly two crossing limit cycles. We get six possible configurations for the class C_2, see Figures* 24, 25, 26, 27, 28 *and* 29. *And six possible configurations for the class C_5, see Figures* 30, 31, 32, 33, 34 *and* 35. *The class C_2 can't exhibit the configurations* 36 *and* 37, *and the class C_5 can't exhibit the configurations* 38 *and* 39, *because the ellipses of every center are concentric.*
(d) *There are systems of the classes C_2 and C_5 exhibiting exactly three crossing limit cycles, see Figures* 40 *and* 41, *respectively.*

(e) *The systems of the classes C_2 and C_5 can exhibit at most three crossing limit cycles.*

Theorem 3 is proved in section 3.

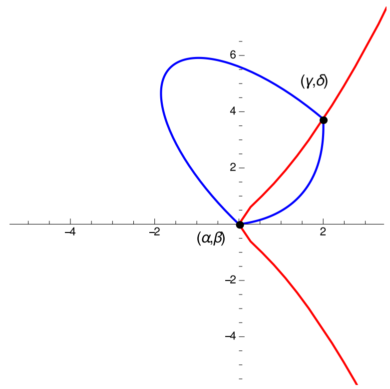

FIGURE 6. The unique limit cycle of the discontinuous piecewise linear differential system (1)–(2).

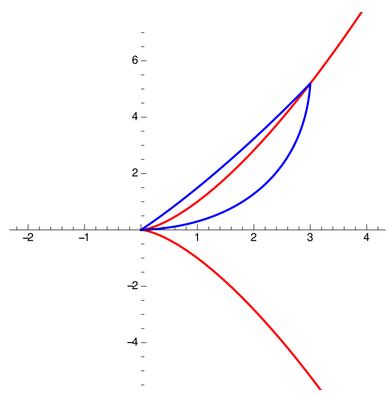

FIGURE 7. The unique limit cycle of the discontinuous piecewise linear differential system (4)–(5).

2. Proof of Theorem 2

We separate the proof of Theorem 2 by its statements. We remark that since the proofs of statements (a) and (e) of Theorems 2 and 3 are the same, we provide them in this section.

Proof of statement (*a*) *of Theorems* 2 *and* 3. It is sufficient to take in each piece the same linear differential center, for instance the center $\dot{x} = -y$, $\dot{y} = x$. Then the discontinuous piecewise linear differential system (in this case continuous but separated by the cubic $c_k(x, y) = 0$) has a continuum of periodic orbits and no limit cycles. □

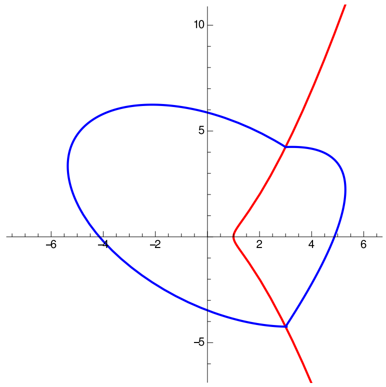

FIGURE 8. The unique limit cycle of the discontinuous piecewise linear differential system (7)–(8).

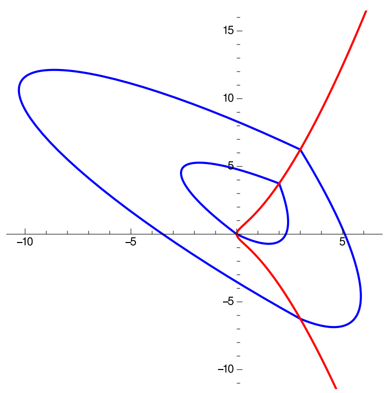

FIGURE 9. Two limit cycles of the discontinuous piecewise linear differential system (10)–(11).

Proof of statement (b) of Theorem 2. First we prove the statement for class C_1 when $b = 1$. We consider the first linear differential center

$$\dot{x} = -\frac{x}{2} - \frac{y}{2} + 1 + \frac{3}{2\sqrt{14}}, \quad \dot{y} = x + \frac{y}{2} - 2, \tag{1}$$

in the region $c_1(x, y) > 0$. This system has the first integral

$$H_1(x,y) = 4\left(x + \frac{y}{2}\right)^2 + 8\left(-2x - \frac{\left(6 + 4\sqrt{14}\right) y}{4\sqrt{14}}\right) + y^2.$$

Now we consider the second linear differential center

$$\dot{x} = -\frac{x}{2} - \frac{y}{2} + 1 + \frac{13}{2\sqrt{14}}, \quad \dot{y} = x + \frac{y}{2} + \frac{1}{2}, \tag{2}$$

in the region $c_1(x, y) < 0$. This differential system has the first integral

$$H_2(x,y) = 4(x + \frac{y}{2})^2 + 8(\frac{x}{2} - (1 + \frac{13}{2\sqrt{14}})y) + y^2.$$

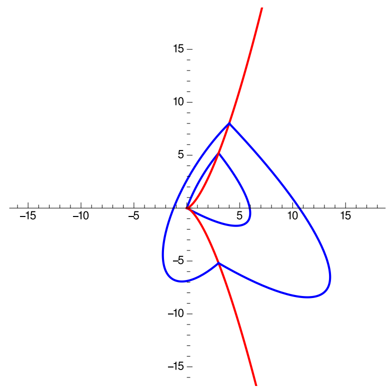

FIGURE 10. Two limit cycles of the discontinuous piecewise linear differential system (12)–(13).

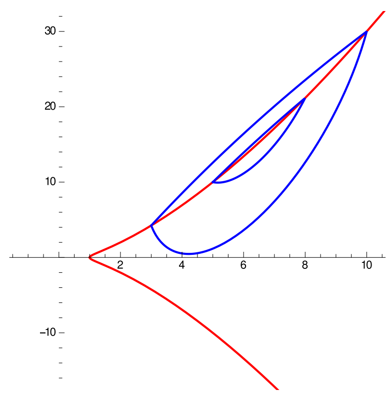

FIGURE 11. Two limit cycles of the discontinuous piecewise linear differential system (14)–(15).

The discontinuous piecewise linear differential system formed by the linear differential centers (1)–(2) has exactly one crossing limit cycle, because the system of equations

$$
\begin{aligned}
&H_l(\alpha,\beta) - H_l(\gamma,\delta) = 0,\\
&H_k(\alpha,\beta) - H_k(\gamma,\delta) = 0,\\
&\beta^2 - \alpha(\alpha^2+\alpha+1) = 0,\\
&\delta^2 - \gamma(\gamma^2+\gamma+1) = 0,
\end{aligned}
\tag{3}
$$

when $l = 1$ and $k = 2$, has a unique real solution $(\alpha, \beta, \gamma, \delta) = (0, 0, 2, \sqrt{14})$.

We prove the statement for the class of discontinuous piecewise linear differential systems C_3. We define two different linear differential centers, the first one is

$$
\dot{x} = \frac{1}{288}\left(-72x - 90y + 295\sqrt{3} + 216\right), \quad \dot{y} = x + \frac{y}{4} + \frac{1}{6}, \tag{4}
$$

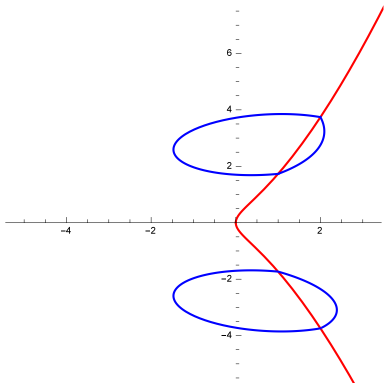

FIGURE 12

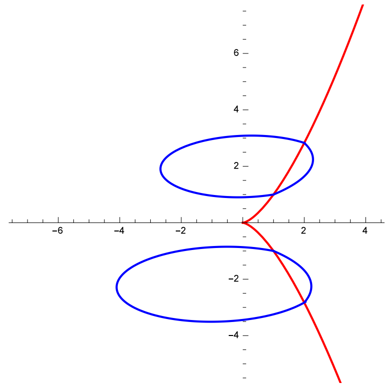

FIGURE 13

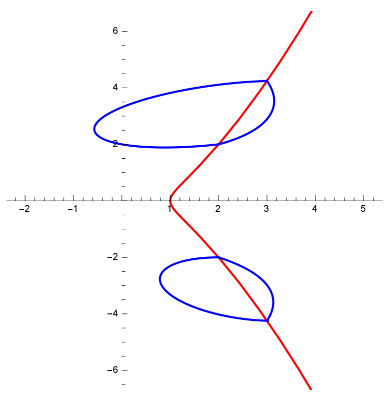

FIGURE 14

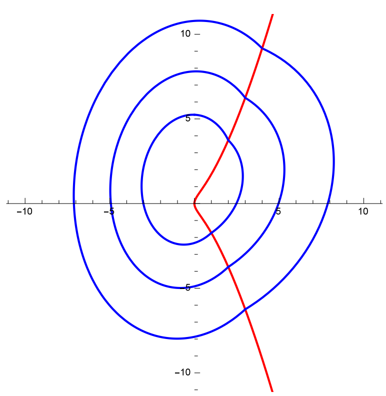

FIGURE 15. Three limit cycles of the discontinuous piecewise linear differential system (16)–(17).

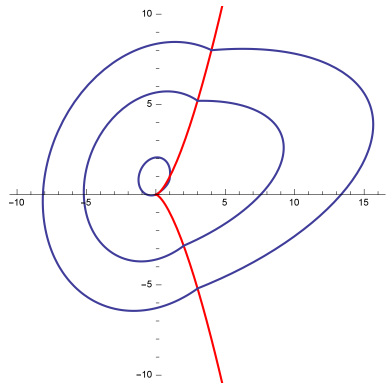

FIGURE 16. Three limit cycles of the discontinuous piecewise linear differential system (18)–(19).

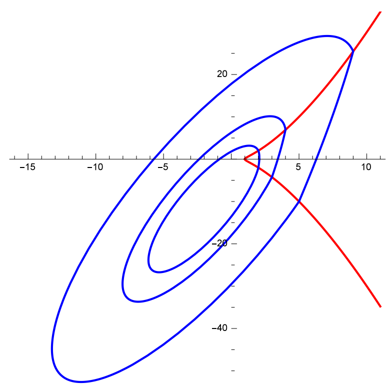

FIGURE 17. Three limit cycles of the discontinuous piecewise linear differential system (20)–(21).

in the region $c_3(x,y) > 0$. It has the first integral

$$H_1(x,y) = 4\left(x - \frac{y}{2}\right)^2 + 8\left(\frac{x}{2} + \frac{\left(36\sqrt{3} - 78\right)y}{24\sqrt{3}}\right) + \frac{y^2}{9}.$$

The second linear differential center is

$$\dot{x} = \frac{1}{288}\left(-72x - 90y + 295\sqrt{3} + 216\right), \quad \dot{y} = x + \frac{y}{4} + \frac{1}{6}, \tag{5}$$

in the region $c_3(x,y) < 0$. It has the first integral

$$H_2(x,y) = 4\left(x + \frac{y}{4}\right)^2 + 8\left(\frac{x}{6} + \frac{1}{24\sqrt{3}}\left(-\frac{295}{4} - 18\sqrt{3}\right)y\right) + y^2.$$

For the discontinuous piecewise linear differential system (4)–(5) the unique solution of the system of equations

$$
\begin{array}{l}
H_l(\alpha,\beta)-H_l(\gamma,\delta)=0,\\
H_k(\alpha,\beta)-H_k(\gamma,\delta)=0,\\
\beta^2-\alpha^3=0,\\
\delta^2-\gamma^3=0,
\end{array} \tag{6}
$$

when $l=1$ and $k=2$, is $(\alpha,\beta,\gamma,\delta)=(0,0,3,3\sqrt{3})$. This proves the statement for the class C_3.

Finally we prove the statement for the class C_4 and we define two different linear differential centers, the first one is

$$
\dot{x}=-3+x-(37y)/36,\quad \dot{y}=1+x-y \tag{7}
$$

in the region $c_4(x,y)<0$. It has the first integral

$$
H_1(x,y)=4(x-y)^2+8(x+3y)+\frac{y^2}{9}.
$$

The second linear differential center is

$$
\dot{x}=-\frac{x}{2}-\frac{5y}{4}+\frac{3}{2},\quad \dot{y}=x+\frac{y}{2}-1, \tag{8}
$$

in the region $c_4(x,y)>0$. It has the first integral

$$
H_2(x,y)=4\left(x+\frac{y}{2}\right)^2+8\left(-x-\frac{3y}{2}\right)+4y^2.
$$

For the piecewise linear differential system (7)–(8) the unique real solution of the system of equations

$$
\begin{array}{l}
H_l(\alpha,\beta)-H_l(\gamma,\delta)=0,\\
H_k(\alpha,\beta)-H_k(\gamma,\delta)=0,\\
\beta^2-\alpha^2(\alpha-1)=0,\\
\delta^2-\gamma^2(\gamma-1)=0,
\end{array} \tag{9}
$$

when $l=1$ and $k=2$, is $(\alpha,\beta,\gamma,\delta)=(3,3\sqrt{2},3,-3\sqrt{2})$. This completes the proof of the statement (b) of Theorem 2. □

Proof of statement (*c*) *of Theorem* 2. For the class C_1 we consider the linear differential center

$$
\dot{x}=-x-\frac{5y}{4}+3,\quad \dot{y}=x+y+\frac{1}{16}\left(8\sqrt{14}-86\right), \tag{10}
$$

in the region $c_1(x,y)>0$. This differential system has the first integral

$$
H_3(x,y)=4(x+y)^2+8\left(\frac{1}{16}\left(8\sqrt{14}-86\right)x-3y\right)+y^2.
$$

The other linear differential center is

$$
\dot{x}=-\frac{x}{2}-\frac{5y}{16}+\frac{3}{2},\quad \dot{y}=x+\frac{y}{2}+\frac{1}{16}\left(4\sqrt{14}-\frac{67}{2}\right), \tag{11}
$$

in the region $c_1(x,y)<0$. It has the first integral

$$
H_4(x,y)=4\left(x+\frac{y}{2}\right)^2+8\left(\frac{1}{16}\left(4\sqrt{14}-\frac{67}{2}\right)x-\frac{3y}{2}\right)+\frac{y^2}{4}.
$$

This discontinuous piecewise differential system formed by the linear differential centers (10)–(11) has exactly two crossing limit cycles, because the real solutions of the system (3), when $l = 3$ and $k = 4$ are $(3, -\sqrt{39}, 3, \sqrt{39})$ and $(0, 0, 2, \sqrt{14})$.

For the class C_3 we define the linear differential center

$$\dot{x} = \frac{1}{52}\left(-52x - 65y - 55\sqrt{3} + 376\right), \quad \dot{y} = x + y + \frac{55\sqrt{3}}{13} - \frac{1071}{104}, \tag{12}$$

in the region $c_3(x, y) < 0$, which has the first integral

$$H_3(x, y) = 4(x + y)^2 + 8\left(\frac{1}{104}\left(440\sqrt{3} - 1071\right)x + \frac{1}{52}\left(55\sqrt{3} - 376\right)y\right) + y^2.$$

The second linear differential center is

$$\dot{x} = \frac{1}{26}\left(13x - 13y - 5\sqrt{3} - 19\right), \quad \dot{y} = x - \frac{y}{2} + \frac{5}{52}\left(8\sqrt{3} - 45\right), \tag{13}$$

in the region $c_3(x, y) > 0$, which has the first integral

$$H_4(x, y) = 4\left(x - \frac{y}{2}\right)^2 + 8\left(\frac{5}{52}\left(8\sqrt{3} - 45\right)x + \frac{1}{26}\left(19 + 5\sqrt{3}\right)y\right) + y^2.$$

This discontinuous piecewise differential system formed by the linear differential centers (12)–(13) has exactly two crossing limit cycles, because the system of equations (6), when $l = 3$ and $k = 4$ has exactly two real different solutions $(0, 0, 3, 3\sqrt{3})$ and $(3, -3\sqrt{3}, 4, 8)$. Hence, the statement holds for this class.

Now we prove the statement for the class C_4.

We define the linear differential center

$$\begin{aligned} \dot{x} &= \frac{x}{10} - \frac{41y}{1600} + \frac{-72096 + 6009\sqrt{2} + 17224\sqrt{7} + 3072\sqrt{14}}{150400}, \\ \dot{y} &= x - \frac{y}{10} + \frac{-444982 + 18657\sqrt{2} + 35592\sqrt{7} + 2892\sqrt{14}}{75200}, \end{aligned} \tag{14}$$

in the region $c_4(x, y) < 0$, which has the first integral

$$\begin{aligned} H_3(x, y) = 4\left(x - \frac{y}{10}\right)^2 &+ \frac{-444982 + 18657\sqrt{2} + 35592\sqrt{7} + 2892\sqrt{14}}{9400}x \\ &+ \frac{-72096 + 6009\sqrt{2} + 17224\sqrt{7} + 3072\sqrt{14}}{18800}. \end{aligned}$$

The second linear differential center is

$$\begin{aligned} \dot{x} &= \frac{x}{8} - \frac{5y}{64} + \frac{9475 - 216\sqrt{2} - 3584\sqrt{7}}{64\left(-160 + 9\sqrt{2} + 56\sqrt{7}\right)}, \\ \dot{y} &= x - \frac{y}{8} + \frac{51355 - 3417\sqrt{2} - 18548\sqrt{7} + 480\sqrt{14}}{32\left(-160 + 9\sqrt{2} + 56\sqrt{7}\right)}, \end{aligned} \tag{15}$$

in the region $c_4(x,y)>0$, which has the first integral

$$H_4(x,y)=4\left(x-\frac{y}{8}\right)^2+8\left(\frac{(51355-3417\sqrt{2}-18548\sqrt{7}+480\sqrt{14})x}{32(-160+9\sqrt{2}+56\sqrt{7})}\right.$$
$$\left.-\frac{(9475-216\sqrt{2}-3584\sqrt{7})y}{64(-160+9\sqrt{2}+56\sqrt{7})}\right)+\frac{y^2}{4}.$$

The discontinuous piecewise linear differential system formed by the linear differential centers (14)–(15) has exactly two crossing limit cycles, because the system of equations (9), when $l=3$ and $k=4$ has exactly two real different solutions $(3,3\sqrt{2},10,30)$ and $(5,10,8,8\sqrt{7})$. This completes the proof of statement (c) of Theorem 2. □

Proof of statement (*d*) *of Theorem* 2. For the class C_1 we consider in the region $c_1(x,y)>0$ the linear differential center

(16)

$$\begin{aligned}\dot{x}=&\ \frac{x}{14}+\frac{(14\sqrt{3}-18\sqrt{7}-14\sqrt{14}+28\sqrt{21}-14\sqrt{39}+\sqrt{42}+6\sqrt{91}+\sqrt{546})y}{140\sqrt{3}-98\sqrt{14}+196\sqrt{21}-140\sqrt{39}}+\\&+\frac{A}{7\sqrt{3}(\sqrt{13}-1)(\sqrt{3}+\sqrt{14})(10\sqrt{3}-7\sqrt{14}+14\sqrt{21}-10\sqrt{39})},\\ \dot{y}=&\ x-\frac{y}{14}+\frac{B}{14\sqrt{3}\left(\sqrt{13}-1\right)\left(10\sqrt{3}-7\sqrt{14}+14\sqrt{21}-10\sqrt{39}\right)}\\&+\frac{-420\sqrt{91}+21\sqrt{182}-258\sqrt{273}+28\sqrt{546}}{14\sqrt{3}\left(\sqrt{13}-1\right)\left(10\sqrt{3}-7\sqrt{14}+14\sqrt{21}-10\sqrt{39}\right)},\end{aligned}$$

where $A=63+588\sqrt{2}+502\sqrt{3}-63\sqrt{13}+579\sqrt{14}+84\sqrt{21}-588\sqrt{26}+38\sqrt{39}+21\sqrt{42}-39\sqrt{182}-84\sqrt{273}-21\sqrt{546}$ and $B=63+588\sqrt{2}+502\sqrt{3}-63\sqrt{13}+579\sqrt{14}+84\sqrt{21}-588\sqrt{26}+38\sqrt{39}+21\sqrt{42}-39\sqrt{182}-84\sqrt{273}-21\sqrt{546}$. This differential system has the first integral $H_5(x,y)$ equal to

$$\begin{aligned}&\frac{C}{420+42\sqrt{7}-60\sqrt{13}-7\sqrt{42}-42\sqrt{91}+7\sqrt{546}}\\&+\frac{y((-42-6\sqrt{7}+6\sqrt{13}+\sqrt{42}+6\sqrt{91}-\sqrt{546})y-6\sqrt{3}(\sqrt{13}-1))}{420+42\sqrt{7}-60\sqrt{13}-7\sqrt{42}-42\sqrt{91}+7\sqrt{546}}\\&+\frac{8x((-294\sqrt{2}-322\sqrt{3}-399\sqrt{14}-42\sqrt{21}+294\sqrt{26}-38\sqrt{39}+39\sqrt{182}+42\sqrt{273})y}{14\sqrt{3}(\sqrt{13}-1)(\sqrt{3}+\sqrt{14})(-10\sqrt{3}+7\sqrt{14}-14\sqrt{21}+10\sqrt{39})}\\&+\frac{-3(3286+1722\sqrt{6}+1046\sqrt{7}+439\sqrt{42})+3\sqrt{13}(-38+798\sqrt{6}+650\sqrt{7}+13\sqrt{42}))}{14\sqrt{3}(\sqrt{13}-1)(\sqrt{3}+\sqrt{14})(-10\sqrt{3}+7\sqrt{14}-14\sqrt{21}+10\sqrt{39})}\\&+\frac{+16y(3(42+56\sqrt{6}+24\sqrt{7}+3\sqrt{42}-14\sqrt{78}-6\sqrt{91})y-84\sqrt{273}-39\sqrt{182}+38\sqrt{39}}{14\sqrt{3}(\sqrt{13}-1)(\sqrt{3}+\sqrt{14})(-10\sqrt{3}+7\sqrt{14}-14\sqrt{21}+10\sqrt{39})}\\&+\frac{-588\sqrt{26}+84\sqrt{21}+579\sqrt{14}+502\sqrt{3}+588\sqrt{2})}{14\sqrt{3}(\sqrt{13}-1)(\sqrt{3}+\sqrt{14})(-10\sqrt{3}+7\sqrt{14}-14\sqrt{21}+10\sqrt{39})},\end{aligned}$$

where $C = 4((420+42\sqrt{7}-60\sqrt{13}-7\sqrt{42}-42\sqrt{91}+7\sqrt{546})x^2+4(-204-15\sqrt{7}+33\sqrt{13}+\sqrt{42}+15\sqrt{91}-\sqrt{546})x$. The other linear differential center in the region $c_1(x,y)<0$ is
(17)

$$\begin{aligned}\dot{x} = & \quad \frac{x}{6}+\frac{\left(6\sqrt{3}-18\sqrt{7}-6\sqrt{14}+12\sqrt{21}-6\sqrt{39}+\sqrt{42}+6\sqrt{91}+\sqrt{546}\right)y}{60\sqrt{3}-42\sqrt{14}+84\sqrt{21}-60\sqrt{39}}\\ & +\frac{D}{3(-322-98\sqrt{6}-42\sqrt{7}-38\sqrt{13}-133\sqrt{42}+98\sqrt{78}+42\sqrt{91}+13\sqrt{546})},\\ \dot{y} = & \quad x-\frac{y}{6}+\frac{E}{2\sqrt{3}(\sqrt{13}-1)(10\sqrt{3}-7\sqrt{14}+14\sqrt{21}-10\sqrt{39})}\\ & +\frac{-60\sqrt{91}+7\sqrt{182}-86\sqrt{273}+4\sqrt{546}}{2\sqrt{3}(\sqrt{13}-1)(10\sqrt{3}-7\sqrt{14}+14\sqrt{21}-10\sqrt{39})},\end{aligned}$$

where $D = 502+9\sqrt{3}+196\sqrt{6}+84\sqrt{7}+38\sqrt{13}+9\sqrt{14}-9\sqrt{39}+193\sqrt{42}-196\sqrt{78}-84\sqrt{91}-9\sqrt{182}-13\sqrt{546}$ and $E = 816+196\sqrt{2}+260\sqrt{3}+60\sqrt{7}-132\sqrt{13}+179\sqrt{14}+218\sqrt{21}-196\sqrt{26}-20\sqrt{39}-4\sqrt{42}$. It has the first integral $H_6(x,y)$ equal to

$$\begin{aligned}&\frac{1}{F}(12(-47421-44394\sqrt{6}+3612\sqrt{7}-31207\sqrt{42}+7\sqrt{13}(-531+3066\sqrt{6}\\&+852\sqrt{7}+703\sqrt{42}))x^2)+4x((47421+44394\sqrt{6}-3612\sqrt{7}+3717\sqrt{13}\\&+31207\sqrt{42}-21462\sqrt{78}-5964\sqrt{91}-4921\sqrt{546})y-3(-98442-264782\sqrt{2}\\&-330296\sqrt{3}-89628\sqrt{6}-15774\sqrt{7}+5562\sqrt{13}-170752\sqrt{14}-46296\sqrt{21}\\&+122066\sqrt{26}+45578\sqrt{39}-53461\sqrt{42}+37968\sqrt{78}+13038\sqrt{91}+28726\sqrt{182}\\&+30528\sqrt{273}+9697\sqrt{546})+4y(-3(-5643+15162\sqrt{2}+7791\sqrt{3}-5082\sqrt{6}\end{aligned}$$

$$\begin{aligned}&+1056\sqrt{7}-567\sqrt{13}+5481\sqrt{14}+1088\sqrt{21}-4326\sqrt{26}-1743\sqrt{39}-3751\sqrt{42}\\&+2562\sqrt{78}+636\sqrt{91}-1413\sqrt{182}-464\sqrt{273}+559\sqrt{546})y+15128\sqrt{546}\\&+378\sqrt{273}+1971\sqrt{182}+21672\sqrt{91}+73206\sqrt{78}-103196\sqrt{42}+1278\sqrt{39}\\&+2646\sqrt{26}-2646\sqrt{21}-4077\sqrt{14}+1494\sqrt{13}+7056\sqrt{7}-142002\sqrt{6}\\&+774\sqrt{3}-18522\sqrt{2}-210258),\end{aligned}$$

where $F = 816+196\sqrt{2}+260\sqrt{3}+60\sqrt{7}-132\sqrt{13}+179\sqrt{14}+218\sqrt{21}-196\sqrt{26}-20\sqrt{39}-4\sqrt{42}$. This discontinuous piecewise differential system formed by the linear differential centers (16)–(17) has exactly three crossing limit cycles, because the real solutions of the system (3), when $l=5$ and $k=6$ has exactly three real different solutions $(1,-\sqrt{3},2,\sqrt{14})$, $(2,-\sqrt{14},3,\sqrt{39})$ and $(3,-\sqrt{39},4,\sqrt{84})$. See this configuration in Figure 15.

For the class C_3 we consider the linear differential center

(18)
$$\begin{aligned}\dot{x} = &\ \frac{-2153+2068\sqrt{2}+2283\sqrt{3}+1704\sqrt{6}}{7430}+\frac{x}{5}-\frac{\left(61+4\sqrt{2}+9\sqrt{3}+6\sqrt{6}\right)y}{45\left(-9+4\sqrt{2}+3\sqrt{3}\right)},\\ \dot{y} = &\ x-\frac{y}{5}+\frac{-46751+11159\sqrt{2}+14823\sqrt{3}+10803\sqrt{6}}{66870},,\end{aligned}$$

in the region $c_3(x,y)>0$. This differential system has the first integral $H_5(x,y)$ equal to

$$\begin{aligned}&\frac{4}{45(26-11\sqrt{2}-15\sqrt{3}+9\sqrt{6})}(45(26-11\sqrt{2}-15\sqrt{3}+9\sqrt{6})x^2+(-2079+967\sqrt{2}\\ &+1413\sqrt{3}-633\sqrt{6})x-18(26-11\sqrt{2}-15\sqrt{3}+9\sqrt{6})xy+(18+86\sqrt{2}\\ &+99\sqrt{3}+12\sqrt{6})y^2+9(151-74\sqrt{2}-108\sqrt{3}+42\sqrt{6}-5\sqrt{35+12\sqrt{6}})y).\end{aligned}$$

The other linear differential center is

(19)
$$\begin{aligned}\dot{x} = &\ x-\frac{\left(-7+28\sqrt{2}+21\sqrt{3}+6\sqrt{6}\right)y}{9\left(-9+4\sqrt{2}+3\sqrt{3}\right)}+\frac{-2549+1460\sqrt{2}+1767\sqrt{3}+1272\sqrt{6}}{1486},\\ \dot{y} = &\ x-y+\frac{-26171+7723\sqrt{2}+11379\sqrt{3}+8127\sqrt{6}}{13374},\end{aligned}$$

in the region $c_3(x,y)<0$. It has the first integral $H_6(x,y)$ equal to

$$\begin{aligned}&\frac{4}{9(26-11\sqrt{2}-15\sqrt{3}+9\sqrt{6})}(9(26-11\sqrt{2}-15\sqrt{3}+9\sqrt{6})x^2+(-1243\\ &+579\sqrt{2}+873\sqrt{3}-357\sqrt{6})x-18(26-11\sqrt{2}-15\sqrt{3}+9\sqrt{6})xy+(154\\ &+6\sqrt{2}-9\sqrt{3}+60\sqrt{6})y^2+9(147-74\sqrt{2}-108\sqrt{3}+42\sqrt{6}-\sqrt{35+12\sqrt{6}})y).\end{aligned}$$

This discontinuous piecewise differential system formed by the linear differential centers (18)–(19) has exactly three crossing limit cycles, because the real solutions of the system (6), when $l=5$ and $k=6$, has exactly three real different solutions are $(0,0,1,1)$, $(2,-2\sqrt{2},3,3\sqrt{3})$ and $(3,-3\sqrt{3},4,8)$. See Figure 16.

For the class C_4 we consider the linear differential center

$$\begin{aligned}\dot{x} = &\ \frac{2x}{9}+\frac{1}{963}(33-114\sqrt{2}+32\sqrt{3})y-\frac{4}{9},\\ \dot{y} = &\ x-\frac{2y}{9}+\frac{-5751-2136\sqrt{2}+4384\sqrt{3}}{1926},\end{aligned} \tag{20}$$

in the region $c_4(x,y)>0$. This differential system has the first integral

$$\begin{aligned}H_5(x,y) = &\ \frac{4}{107}(x(107x-539)-7y^2)-\frac{8}{963}(2x(107y-1096\sqrt{3}+534\sqrt{2}+225)\\ &-y((15+57\sqrt{2}-16\sqrt{3})y+428)).\end{aligned}$$

The other linear differential center is

$$\begin{aligned}\dot{x} =&\quad \frac{x}{7}+\frac{1}{749}(34-57\sqrt{2}+16\sqrt{3})y-\frac{2}{7},\\ \dot{y} =&\quad x-\frac{y}{7}+\frac{-4223-1068\sqrt{2}+2192\sqrt{3}}{1498},\end{aligned} \tag{21}$$

in the region $c_4(x,y)<0$. It has the first integral $H_6(x,y)$ equal to

$$\begin{aligned}&\frac{4}{107}(x(107x-539)-7y^2)-\frac{4}{749}(2x(107y-1096\sqrt{3}+534\sqrt{2}+225)\\ &-y((15+57\sqrt{2}-16\sqrt{3})y+428)).\end{aligned}$$

This discontinuous piecewise differential system formed by the linear differential centers (20)–(21) has exactly three crossing limit cycles, because the system of equations 9, when $l=5$ and $k=6$, has exactly three real different solutions $(2,-2,2,-2)$, $(3,-3\sqrt{2},4,4\sqrt{3})$ and $(5,-10,9,18\sqrt{2})$. See Figure 17. □

Proof of statement (e) *of Theorems* 2 *and* 3. We note that for the irreducible cubic curves with $c_2(x,y)=0$ and $c_5(x,y)=0$ which separated the plane into three connected components, the crossing limit cycles having only two points on the cubic curve only can be contained in two of these connected components, so in their definition only appear two linear differential centers. Therefore the proof of statement (e) is the same for all classes of discontinuous piecewise linear differential systems C_i for $i=1,\ldots,5$.

Due to Lemma 1 of [13] any linear differential center in $\mathbb{R}^2$ can be written into the form

$$\dot{x}=-bx-\frac{1}{4}y\left(4b^2+\omega^2\right)+d,\quad \dot{y}=x+by+c, \tag{22}$$

with $\omega>0$. This differential system has the first integral

$$H_7(x,y)=4(by+x)^2+8(cx-dy)+\omega^2y^2.$$

We consider a second arbitrary linear differential center

$$\dot{x}=-Bx-\frac{1}{4}y\left(4B^2+\Omega^2\right)+D,\quad \dot{y}=x+By+C, \tag{23}$$

with $\Omega>0$. This differential system has the first integral

$$H_8(x,y)=4(By+x)^2+8(Cx-Dy)+\Omega^2y^2.$$

In order that the discontinuous piecewise linear differential system formed by the linear differential centers (22)–(23), separated by some irreducible cubic curve $c_k(x,y)=0$ with $k\in\{1,2,3,4,5\}$, has three crossing limit cycles intersecting the curve $c_k(x,y)=0$ in two different points (α_i,β_i) and (γ_i,δ_i), for $i=1,2,3$, these points must satisfy the system of equations

$$\begin{aligned}&H_7(\alpha,\beta)=H_7(\gamma,\delta),\\ &H_8(\alpha,\beta)=H_8(\gamma,\delta),\\ &c_k(\alpha,\beta)=0,\\ &c_k(\gamma,\delta)=0.\end{aligned} \tag{24}$$

From the fact that the points (α_1, β_1) and (γ_1, δ_1) satisfy system (24), we can obtain the following values for the parameters d and D:

$$d = \frac{4\alpha_1^2 + 4b^2\beta_1^2 - 4b^2\delta_1^2 + 8\alpha_1 b\beta_1 - 8b\gamma_1\delta_1 + \beta_1^2\omega^2 + 8\alpha_1 c - 8c\gamma_1 - 4\gamma_1^2 - \delta_1^2\omega^2}{8(\beta_1 - \delta_1)},$$

and we get the same expression for D by changing (b, c, ω) in the expression of d by (B, C, Ω).

Now since the points (α_2, β_2) and (γ_2, δ_2) also satisfy the system (24) we get the following values for the parameters c and C:

$$\begin{aligned} c = \; & \frac{-1}{8(-\alpha_1\beta_2 + \alpha_1\delta_2 + \alpha_2\beta_1 - \alpha_2\delta_1 - \beta_1\gamma_2 + \beta_2\gamma_1 - \gamma_1\delta_2 + \gamma_2\delta_1)} \\ & (-4\alpha_1^2\beta_2 + 4\alpha_1^2\delta_2 + 4\alpha_2^2\beta_1 - 4\alpha_2^2\delta_1 - 4b^2\beta_1^2\beta_2 + 4b^2\beta_1^2\delta_2 + 4b^2\beta_1\beta_2^2 \\ & -4b^2\beta_1\delta_2^2 - 4b^2\beta_2^2\delta_1 + 4b^2\beta_2\delta_1^2 - 4b^2\delta_1^2\delta_2 + 4b^2\delta_1\delta_2^2 - 8\alpha_1 b\beta_1\beta_2 \\ & +8\alpha_1 b\beta_1\delta_2 + 8\alpha_2 b\beta_1\beta_2 - 8\alpha_2 b\beta_2\delta_1 - 8b\beta_1\gamma_2\delta_2 + 8b\beta_2\gamma_1\delta_1 \\ & -8b\gamma_1\delta_1\delta_2 + 8b\gamma_2\delta_1\delta_2 - \beta_1^2\beta_2\omega^2 + \beta_1^2\delta_2\omega^2 + \beta_1\beta_2^2\omega^2 - 4\beta_1\gamma_2^2 \\ & -\beta_1\delta_2^2\omega^2 - \beta_2^2\delta_1\omega^2 + 4\beta_2\gamma_1^2 + \beta_2\delta_1^2\omega^2 - 4\gamma_1^2\delta_2 + 4\gamma_2^2\delta_1 \\ & d - \delta_1^2\delta_2\omega^2 + \delta_1\delta_2^2\omega^2). \end{aligned}$$

If instead of (ω, b) in the expression of c we replace them by (Ω, B), then we get the expression of C.

The last step is to use that (α_3, β_3) and (γ_3, δ_3) satisfy system (24), then we obtain the expressions of $\omega^2 = G/H$ and Ω^2, where

$$\begin{aligned} G = \; & -4(b^2\beta_3\alpha_2\beta_1^2 - b^2\delta_3\alpha_2\beta_1^2 - b^2\beta_2\alpha_3\beta_1^2 + b^2\delta_2\alpha_3\beta_1^2 - b^2\beta_3\gamma_2\beta_1^2 \\ & +b^2\delta_3\gamma_2\beta_1^2 + b^2\beta_2\gamma_3\beta_1^2 - b^2\delta_2\gamma_3\beta_1^2 - a^2\alpha_2\alpha_3^2\beta_1 - 2ab\delta_2\alpha_3\gamma_2\beta_1 \\ & +a^2\alpha_2\gamma_3^2\beta_1 - a^2\gamma_2\gamma_3^2\beta_1 - b^2\beta_3^2\alpha_2\beta_1 + b^2\delta_3^2\alpha_2\beta_1 + 2ab\beta_3\alpha_1\alpha_2\beta_1 \\ & -2ab\delta_3\alpha_1\alpha_2\beta_1 + b^2\beta_2^2\alpha_3\beta_1 - b^2\delta_2^2\alpha_3\beta_1 + a^2\alpha_2^2\alpha_3\beta_1 - 2ab\beta_2\alpha_1\alpha_3\beta_1 \\ & +2ab\delta_2\alpha_1\alpha_3\beta_1 + 2ab\beta_2\alpha_2\alpha_3\beta_1 - 2ab\beta_3\alpha_2\alpha_3\beta_1 + b^2\beta_3^2\gamma_2\beta_1 - b^2\delta_3^2\gamma_2\beta_1 \\ & +a^2\alpha_3^2\gamma_2\beta_1 - 2ab\beta_3\alpha_1\gamma_2\beta_1 + 2ab\delta_3\alpha_1\gamma_2\beta_1 + 2ab\beta_3\alpha_3\gamma_2\beta_1 - a^2\alpha_3\gamma_2^2\beta_1 \\ & -b^2\beta_2^2\gamma_3\beta_1 + b^2\delta_2^2\gamma_3\beta_1 - a^2\alpha_2^2\gamma_3\beta_1 + a^2\gamma_2^2\gamma_3\beta_1 + 2ab\beta_2\alpha_1\gamma_3\beta_1 \\ & -2ab\delta_2\alpha_1\gamma_3\beta_1 - 2ab\beta_2\alpha_2\gamma_3\beta_1 + 2ab\delta_3\alpha_2\gamma_3\beta_1 + 2ab\delta_2\gamma_2\gamma_3\beta_1 \\ & -a^2\beta_3\alpha_1\alpha_2^2 + a^2\delta_3\alpha_1\alpha_2^2 + a^2\beta_2\alpha_1\alpha_3^2 - a^2\delta_2\alpha_1\alpha_3^2 + a^2\delta_1\alpha_2\alpha_3^2 \\ & -a^2\beta_3\alpha_2\gamma_1^2 + a^2\delta_3\alpha_2\gamma_1^2 + a^2\beta_2\alpha_3\gamma_1^2 - a^2\delta_2\alpha_3\gamma_1^2 + 2ab\beta_2\delta_3\alpha_1\alpha_2 \\ & -a^2\delta_3\alpha_1\gamma_2^2 + a^2\delta_1\alpha_3\gamma_2^2 - a^2\beta_3\gamma_1\gamma_2^2 + a^2\delta_3\gamma_1\gamma_2^2 - a^2\beta_2\alpha_1\gamma_3^2 + a^2\beta_3\alpha_1\gamma_2^2 \end{aligned}$$

$$
\begin{aligned}
&+a^2\delta_2\alpha_1\gamma_3^2 - a^2\delta_1\alpha_2\gamma_3^2 + a^2\beta_2\gamma_1\gamma_3^2 - a^2\delta_2\gamma_1\gamma_3^2 + a^2\delta_1\gamma_2\gamma_3^2 + 2ab\beta_2\delta_1\alpha_3\gamma_1\\
&+b^2\beta_2\beta_3^2\alpha_1 + b^2\beta_3\delta_2^2\alpha_1 - b^2\beta_2\delta_3^2\alpha_1 + b^2\delta_2\delta_3^2\alpha_1 - 2ab\delta_3\gamma_2\gamma_3\beta_1\\
&-b^2\beta_3^2\delta_2\alpha_1 + b^2\beta_2^2\delta_3\alpha_1 - b^2\delta_2^2\delta_3\alpha_1 - b^2\beta_3\delta_1^2\alpha_2 - b^2\delta_1\delta_3^2\alpha_2 - b^2\beta_2^2\beta_3\alpha_1\\
&+a^2\beta_3\alpha_1^2\alpha_2 - a^2\delta_3\alpha_1^2\alpha_2 + b^2\beta_3^2\delta_1\alpha_2 + b^2\delta_1^2\delta_3\alpha_2 - 2ab\beta_2\beta_3\alpha_1\alpha_2\\
&+b^2\beta_2\delta_1^2\alpha_3 + b^2\delta_1\delta_2^2\alpha_3 - a^2\beta_2\alpha_1^2\alpha_3 + a^2\delta_2\alpha_1^2\alpha_3 - a^2\delta_1\alpha_2^2\alpha_3 - a^2\delta_3\alpha_2^2\gamma_1\\
&-b^2\beta_2^2\delta_1\alpha_3 - b^2\delta_1^2\delta_2\alpha_3 + 2ab\beta_2\beta_3\alpha_1\alpha_3 - 2ab\beta_3\delta_2\alpha_1\alpha_3 - 2ab\beta_2\delta_1\alpha_2\alpha_3\\
&-b^2\beta_2\beta_3^2\gamma_1 - b^2\beta_3\delta_2^2\gamma_1 + b^2\beta_2\delta_3^2\gamma_1 - b^2\delta_2\delta_3^2\gamma_1 + a^2\beta_3\alpha_2^2\gamma_1 + 2ab\beta_3\delta_1\alpha_2\alpha_3\\
&-a^2\beta_2\alpha_3^2\gamma_1 + a^2\delta_2\alpha_3^2\gamma_1 + b^2\beta_2^2\beta_3\gamma_1 + b^2\beta_3^2\delta_2\gamma_1 - b^2\beta_2^2\delta_3\gamma_1 + b^2\delta_2^2\delta_3\gamma_1\\
&+2ab\beta_2\beta_3\alpha_2\gamma_1 - 2ab\beta_3\delta_1\alpha_2\gamma_1 - 2ab\beta_2\delta_3\alpha_2\gamma_1 + 2ab\delta_1\delta_3\alpha_2\gamma_1 - 2ab\beta_2\beta_3\alpha_3\gamma_1\\
&+2ab\beta_3\delta_2\alpha_3\gamma_1 - 2ab\delta_1\delta_2\alpha_3\gamma_1 + b^2\beta_3\delta_1^2\gamma_2 + b^2\delta_1\delta_3^2\gamma_2 - a^2\beta_3\alpha_1^2\gamma_2 + a^2\delta_3\alpha_1^2\gamma_2\\
&-a^2\delta_1\alpha_3^2\gamma_2 + a^2\beta_3\gamma_1^2\gamma_2 - a^2\delta_3\gamma_1^2\gamma_2 - b^2\beta_3^2\delta_1\gamma_2 - b^2\delta_1^2\delta_3\gamma_2 + 2ab\beta_3\delta_2\alpha_1\gamma_2\\
&-2ab\delta_2\delta_3\alpha_1\gamma_2 - 2ab\beta_3\delta_1\alpha_3\gamma_2 + 2ab\delta_1\delta_2\alpha_3\gamma_2 + 2ab\beta_3\delta_1\gamma_1\gamma_2 - 2ab\beta_3\delta_2\gamma_1\gamma_2\\
&+2ab\delta_2\delta_3\gamma_1\gamma_2 - b^2\beta_2\delta_1^2\gamma_3 - b^2\delta_1\delta_2^2\gamma_3 + a^2\beta_2\alpha_1^2\gamma_3 - a^2\delta_2\alpha_1^2\gamma_3 + a^2\delta_1\alpha_2^2\gamma_3\\
&-a^2\beta_2\gamma_1^2\gamma_3 + a^2\delta_2\gamma_1^2\gamma_3 - a^2\delta_1\gamma_2^2\gamma_3 + b^2\beta_2^2\delta_1\gamma_3 + b^2\delta_1^2\delta_2\gamma_3 - 2ab\beta_2\delta_3\alpha_1\gamma_3\\
&+2ab\delta_2\delta_3\alpha_1\gamma_3 + 2ab\beta_2\delta_1\alpha_2\gamma_3 - 2ab\delta_1\delta_3\alpha_2\gamma_3 - 2ab\beta_2\delta_1\gamma_1\gamma_3 + 2ab\delta_1\delta_2\gamma_1\gamma_3\\
&+2ab\beta_2\delta_3\gamma_1\gamma_3 - 2ab\delta_2\delta_3\gamma_1\gamma_3 - 2ab\delta_1\delta_2\gamma_2\gamma_3 + 2ab\delta_1\delta_3\gamma_2\gamma_3 - 2ab\delta_1\delta_3\gamma_1\gamma_2)
\end{aligned}
$$

and

$$
\begin{aligned}
H = \ & -\alpha_3\beta_1^2\beta_2 + \gamma_3\beta_1^2\beta_2 + \alpha_2\beta_1^2\beta_3 - \gamma_2\beta_1^2\beta_3 + \alpha_3\beta_1^2\delta_2 - \gamma_3\beta_1^2\delta_2 - \alpha_2\beta_1^2\delta_3\\
&+\gamma_3\beta_1\delta_2^2 + \gamma_2\beta_1^2\delta_3 + \alpha_3\beta_1\beta_2^2 - \gamma_3\beta_1\beta_2^2 - \alpha_2\beta_1\beta_3^2 + \gamma_2\beta_1\beta_3^2 - \alpha_3\beta_1\delta_2^2\\
&+\alpha_2\beta_1\delta_3^2 - \gamma_2\beta_1\delta_3^2 - \alpha_1\beta_2^2\beta_3 + \gamma_1\beta_2^2\beta_3 - \alpha_3\beta_2^2\delta_1 + \gamma_3\beta_2^2\delta_1 + \alpha_1\beta_2^2\delta_3\\
&-\gamma_1\beta_2^2\delta_3 + \alpha_1\beta_2\beta_3^2 - \gamma_1\beta_2\beta_3^2 + \alpha_3\beta_2\delta_1^2 - \gamma_3\beta_2\delta_1^2 - \alpha_1\beta_2\delta_3^2 + \gamma_1\beta_2\delta_3^2\\
&-\gamma_2\beta_3^2\delta_1 - \alpha_1\beta_3^2\delta_2 + \gamma_1\beta_3^2\delta_2 - \alpha_2\beta_3\delta_1^2 + \gamma_2\beta_3\delta_1^2 + \alpha_1\beta_3\delta_2^2 + \alpha_2\beta_3^2\delta_1\\
&-\gamma_1\beta_3\delta_2^2 - \alpha_3\delta_1^2\delta_2 + \gamma_3\delta_1^2\delta_2 + \alpha_2\delta_1^2\delta_3 - \gamma_2\delta_1^2\delta_3 + \alpha_3\delta_1\delta_2^2 - \gamma_3\delta_1\delta_2^2\\
&-\alpha_2\delta_1\delta_3^2 + \gamma_2\delta_1\delta_3^2 - \alpha_1\delta_2^2\delta_3 + \gamma_1\delta_2^2\delta_3 + \alpha_1\delta_2\delta_3^2 - \gamma_1\delta_2\delta_3^2.
\end{aligned}
$$

Again we get Ω^2 to changing b by B in the expression of ω^2.

Substituting the obtained values of d, c and ω into the first integral $H_7(x,y)$ it takes the form

$$
\begin{aligned}
H_7(x,y) = \ & f(x,y,\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma_1,\gamma_2,\gamma_3,\delta_1,\delta_2,\delta_3)\\
&+bg(x,y,\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma_1,\gamma_2,\gamma_3,\delta_1,\delta_2,\delta_3).
\end{aligned}
$$

The expression of the first integral $H_8(x,y)$ is the same than the expression of $H_7(x,y)$ changing b by B.

Now assume that we have a fourth limit cycle which intersect the cubic in the points (α_4,β_4) and (γ_4,δ_4). Then we must have

$$
H_7(\alpha_4,\beta_4) = H_7(\gamma_4,\delta_4), \quad \text{and} \quad H_8(\alpha_4,\beta_4) = H_8(\gamma_4,\delta_4). \tag{25}
$$

From the expressions of H_7, H_8 and the equations (25) it follows that $b = B$, and consequently the first integrals H_7 and H_8 are equal. Then the two linear centers defining the piecewise linear differential system are the same, and consequently the system is linear and has no limit cycles in contradiction, that we are assuming that it has four limit cycles. So the piecewise linear differential systems here studied have at most three limit cycles. □

3. Proof of Theorem 3

This section is dedicated to prove statements (b), (c) and (d) of Theorem 3.

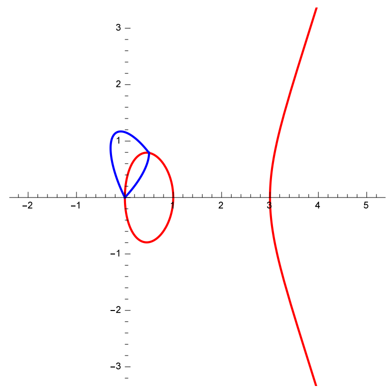

FIGURE 18. The unique limit cycle of the discontinuous piecewise linear differential system (26)–(27).

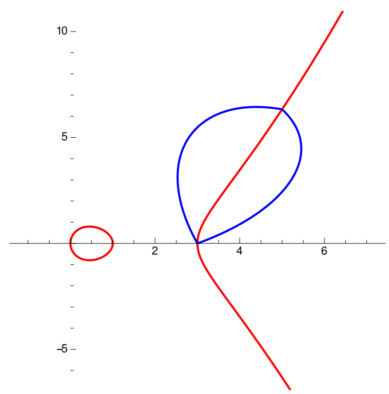

FIGURE 19. The unique limit cycle of the discontinuous piecewise linear differential system (29)–(30).

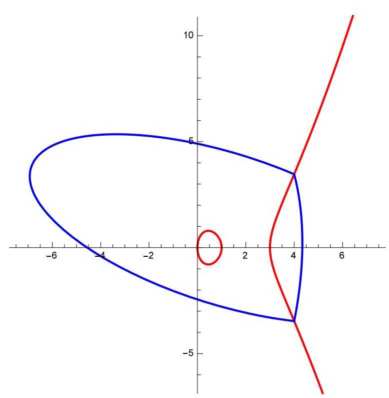

FIGURE 20. The unique limit cycle of the discontinuous piecewise linear differential system (31)–(32).

Proof of statement (b) *of Theorem* 3. First we prove the existence of three different configurations of one crossing limit cycle for the class C_2.

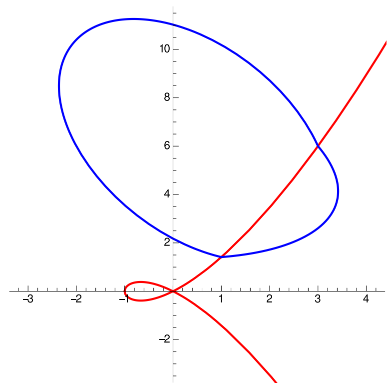

FIGURE 21. The unique limit cycle of the discontinuous piecewise linear differential system (33)–(34).

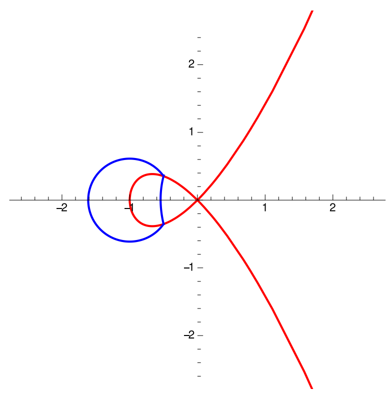

FIGURE 22. The unique limit cycle of the discontinuous piecewise linear differential system (36)–(37).

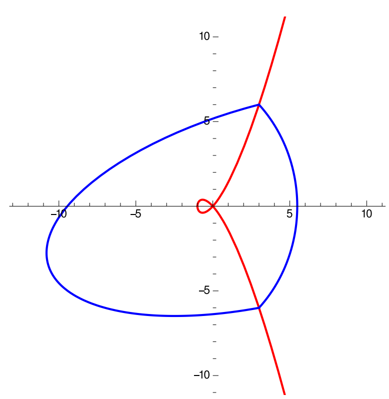

FIGURE 23. The unique limit cycle of the discontinuous piecewise linear differential system (38)–(39).

For the first possible configuration we consider the linear differential center

$$\dot{x} = -\frac{x}{2} - \frac{y}{2} + \frac{1}{80}(20 + \sqrt{10}), \quad \dot{y} = x + \frac{y}{2} - \frac{1}{2}, \tag{26}$$

in the region $c_2(x, y) > 0$, with its first integral

$$H_1(x, y) = 4x^2 + 4x(y-1) + y(2y - \frac{1}{\sqrt{10}} - 2).$$

In the region $c_2(x, y) < 0$ we consider the linear center

$$\dot{x} = x - 2y + \sqrt{\frac{5}{2}} - \frac{1}{2}, \quad \dot{y} = 1 + x - y, \tag{27}$$

with its first integral

$$H_2(x, y) = -4(2x + \sqrt{10} - 1)y + 4x(x+2) + 8y^2.$$

For the discontinuous piecewise linear differential system (26)–(27) the unique solution of the system of equations

$$\begin{aligned} &H_l(\alpha, \beta) - H_l(\gamma, \delta) = 0, \\ &H_k(\alpha, \beta) - H_k(\gamma, \delta) = 0, \\ &\beta^2 - \alpha(\alpha-1)(\alpha-3) = 0, \\ &\delta^2 - \gamma(\gamma-1)(\gamma-3) = 0, \end{aligned} \tag{28}$$

when $l = 1$ and $k = 2$, is $(\alpha, \beta, \gamma, \delta) = (0, 0, 1/2, \sqrt{5}/(2\sqrt{2}))$. This proves the uniqueness of the crossing limit cycle. See this configuration in Figure 18.

For the second possible configuration for the class C_2 we consider in the region $c_2(x, y) > 0$ the linear differential center

$$\dot{x} = \frac{1}{80}\left(-20x - 25y + 9\sqrt{10} + 100\right), \quad \dot{y} = x + \frac{y}{4} - 6, \tag{29}$$

with its first integral

$$H_1(x, y) = 4\left(x + \frac{y}{4}\right)^2 + 8\left(\frac{\left(-18 - 20\sqrt{10}\right)y}{16\sqrt{10}} - 6x\right) + y^2.$$

In the region $c_2(x, y) < 0$ we consider the linear differential center

$$\dot{x} = \frac{1}{25}\left(5x - 26y + 36\sqrt{10} - 25\right), \quad \dot{y} = x - \frac{y}{5}, \tag{30}$$

with its first integral

$$H_2(x, y) = 4\left(x - \frac{y}{5}\right)^2 + 4y^2 + \frac{\left(16\sqrt{10} - \frac{1152}{5}\right)}{2\sqrt{10}}y.$$

For this case the unique real solution of system (28), when $l = 1$ and $k = 2$, is $(\alpha, \beta, \gamma, \delta) = (3, 0, 5, 2\sqrt{10})$. Hence the discontinuous piecewise linear differential system (29)–(30) has a unique crossing limit cycle, see Figure 19.

Finally we prove the statement for the last configuration of the class C_2. We consider the linear differential center

$$\dot{x} = \frac{x}{5} - \frac{29y}{100} - \frac{4}{5}, \quad \dot{y} = x - \frac{y}{5} + 1, \tag{31}$$

in the region $c_2(x, y) < 0$, with its first integral

$$H_1(x, y) = 4(x - \frac{y}{5})^2 + 8(x + \frac{4y}{5}) + y^2.$$

The second linear differential center is

$$\dot{x} = -x - \frac{13y}{4} + 4, \quad \dot{y} = -2 + x + y, \tag{32}$$

in the region $c_2(x, y) > 0$, with its first integral

$$H_2(x, y) = 8(-2x - 4y) + 9y^2 + 4(x + y)^2.$$

Since system (28), when $l = 1$ and $k = 2$, has the unique real solution $(\alpha, \beta, \gamma, \delta) = (4, -2\sqrt{3}, 4, 2\sqrt{3})$, the discontinuous piecewise linear differential system (31)–(32) has a unique crossing limit cycle, see it in Figure 20.

The piecewise linear differential systems of the class C_2 having one crossing limit cycle can exhibit this limit cycle in three different configurations, see Figures 18, 19 and 20.

In short statement (b) is proved for the class C_2.

Now we prove the statement for the first configuration of the class C_5. In the region $c_5(x, y) < 0$ we consider the linear center

$$\dot{x} = -\frac{x}{2} - \frac{5y}{4} + \frac{1}{136}(770 + 117\sqrt{2}), \quad \dot{y} = x + \frac{y}{2} - 1, \tag{33}$$

which has the first integral

$$H_1(x, y) = 4(x + \frac{y}{2})^2 + 8(-x - \frac{1}{136}(770 + 117\sqrt{2})y) + 4y^2.$$

In the region $c_5(x, y) > 0$ we consider the linear center

$$\dot{x} = -\frac{x}{4} - \frac{5y}{16} + \frac{1}{544}(934 + 133\sqrt{2}), \quad \dot{y} = x + \frac{y}{4} - 2, \tag{34}$$

with its first integral

$$H_2(x, y) = 4\left(x + \frac{y}{4}\right)^2 + 8\left(-2x - \frac{1}{544}\left(934 + 133\sqrt{2}\right)y\right) + y^2.$$

The discontinuous piecewise linear differential system (33)–(34) has exactly one crossing limit cycle, because the system

$$\begin{aligned} &H_l(\alpha, \beta) - H_l(\gamma, \delta) = 0, \\ &H_k(\alpha, \beta) - H_k(\gamma, \delta) = 0, \\ &\beta^2 - \alpha^2(\alpha + 1) = 0, \\ &\delta^2 - \gamma^2(\gamma + 1) = 0, \end{aligned} \tag{35}$$

when $l = 1$ and $k = 2$, has the unique real solution $(\alpha, \beta, \gamma, \delta) = (1, \sqrt{2}, 3, 6)$. This proves the statement for the first configuration, see Figure 21.

Now we prove the statement for the second configuration of the class C_5.

In the region $c_5(x, y) > 0$ we consider the linear differential center

$$\dot{x} = -y, \quad \dot{y} = 1 + x, \tag{36}$$

with its first integral

$$H_1(x, y) = 4x^2 + 8x + 4y^2.$$

In the region $c_5(x, y) < 0$ we consider the linear differential center

$$\dot{x} = -\frac{x}{4} - \frac{17y}{16} - \frac{1}{8}, \quad \dot{y} = x + \frac{y}{4} - 1, \tag{37}$$

with its first integral

$$H_2(x,y) = 4\left(x + \frac{y}{4}\right)^2 + 8\left(\frac{y}{8} - x\right) + 4y^2.$$

The discontinuous piecewise linear differential system (36)–(37) has exactly one crossing limit cycle, because the system of equations (35), when $l = 1$ and $k = 2$, has exactly one real solution $(-1/2, 1/(2\sqrt{2}), -1/2, -1/(2\sqrt{2}))$. See this configuration in Figure 22.

Finally we prove the statement for the last possible configuration for the class C_5. We consider the differential center

$$\dot{x} = -3 + x - 5y, \quad \dot{y} = -4 + x - y, \tag{38}$$

in the region $c_5(x,y) > 0$, with the first integral

$$H_1(x,y) = 4(x-y)^2 + 16y^2 + 8(-4x + 3y).$$

For the second differential linear center we consider

$$\dot{x} = -y, \quad \dot{y} = x + 3, \tag{39}$$

in the region $c_5(x,y) < 0$, with its first integral

$$H_2(x,y) = 24x + 4x^2 + 4y^2.$$

Since the unique intersection point between the three curves $H_1(x,y) = 0$, $H_2(x,y) = 0$ and $c_5(x,y) = 0$ are the two points $(3,-6)$ and $(3,6)$, the discontinuous piecewise linear differential system (38)–(39) has exactly one crossing limit cycle, the one of Figure 23.

This completes the proof of statement (b) of Theorem 3. $\square$

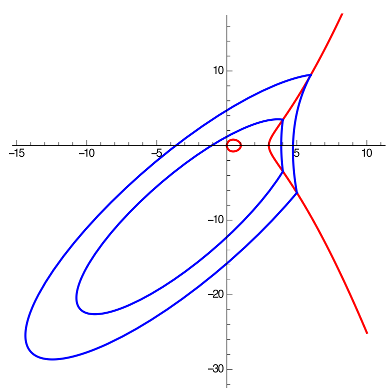

FIGURE 24. Two crossing limit cycles of the discontinuous piecewise linear differential system (40)–(41).

Proof of statement (*c*) *of Theorem* 3. First we prove this statement for the six possible configurations of two crossing limit cycles of the class C_2. For the first configuration for the class C_2, we consider the differential linear center

$$\dot{x} = \frac{x}{2} - \frac{13y}{36} - 2, \quad \dot{y} = x - \frac{y}{2} + 4\sqrt{10} - \frac{523}{36}, \tag{40}$$

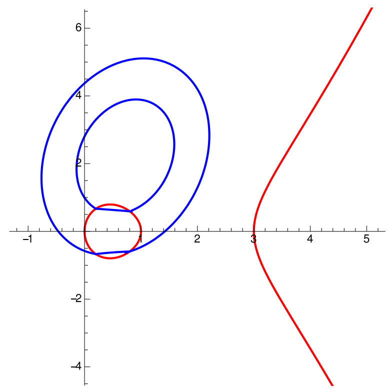

FIGURE 25. Two crossing limit cycles of the discontinuous piecewise linear differential system (42)–(43).

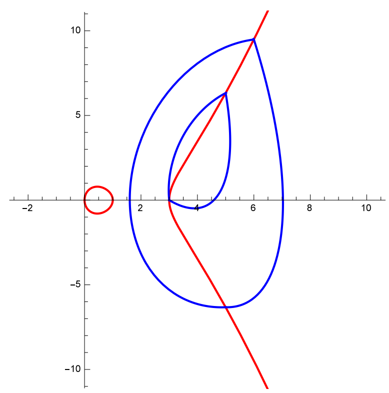

FIGURE 26. Two crossing limit cycles of the discontinuous piecewise linear differential system (44)–(45).

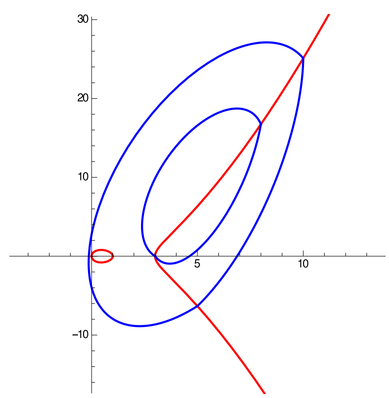

FIGURE 27. Two crossing limit cycles of the discontinuous piecewise linear differential system (46)–(47).

in the region $c_2(x, y) > 0$. A first integral of this system is

$$H_3(x, y) = 4x^2 + x\left(-4y + 32\sqrt{10} - \frac{1046}{9}\right) + \frac{1}{9}y(13y + 144).$$

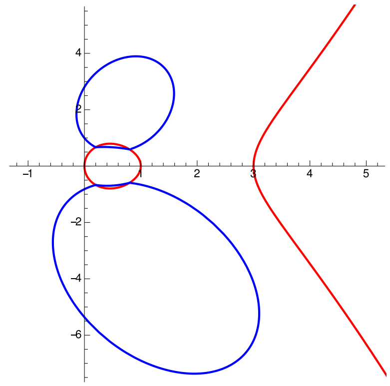

FIGURE 28. Two crossing limit cycles of the discontinuous piecewise linear differential system (48)–(49).

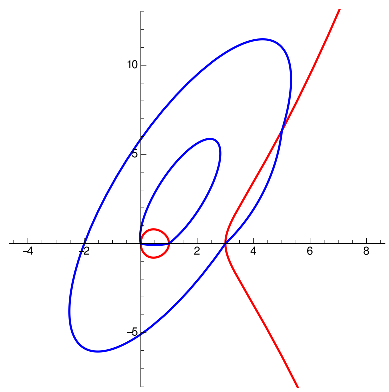

FIGURE 29. Two crossing limit cycles of the discontinuous piecewise linear differential system (50)–(51).

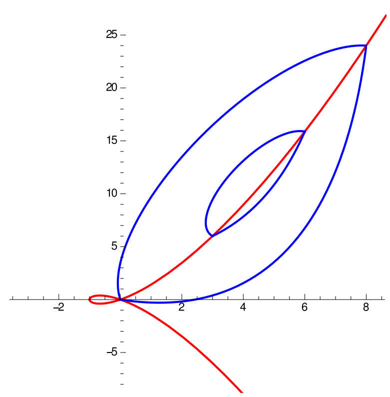

FIGURE 30. Two crossing limit cycles of the discontinuous piecewise linear differential system (52)–(53).

The second linear center is

$$\dot{x} = -x - \frac{37y}{36} + 4, \quad \dot{y} = x + y - 8\sqrt{10} - \frac{1123}{36}, \tag{41}$$

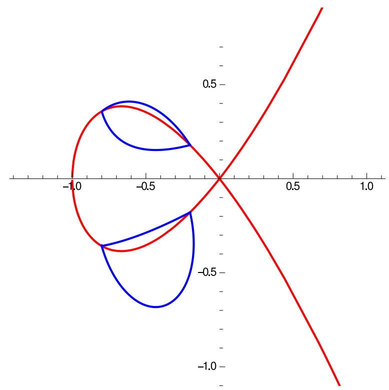

FIGURE 31. Two crossing limit cycles of the discontinuous piecewise linear differential system (54)–(55).

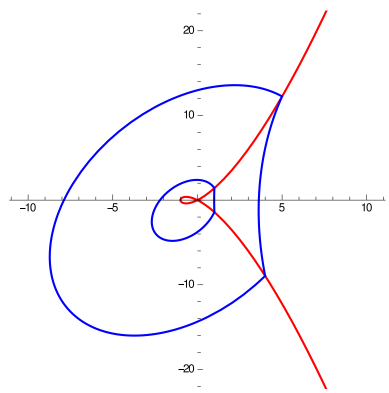

FIGURE 32. Two crossing limit cycles of the discontinuous piecewise linear differential system (56)–(57).

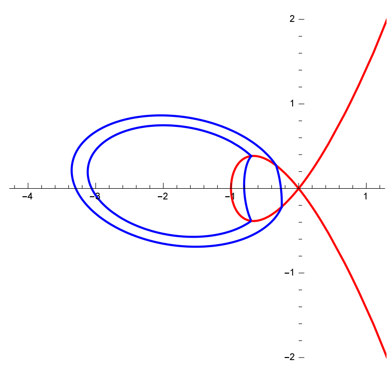

FIGURE 33. Two crossing limit cycles of the discontinuous piecewise linear differential system (58)–(59).

in the region $c_2(x,y) < 0$. Its first integral is

$$H_4(x,y) = 4x^2 + x\left(8y - 64\sqrt{10} - \frac{2246}{9}\right) + \frac{1}{9}y(37y - 288).$$

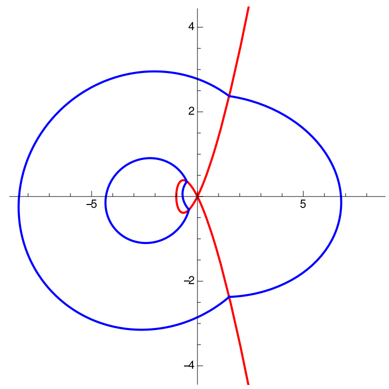

FIGURE 34. Two crossing limit cycles of the discontinuous piecewise linear differential system (60)–(61).

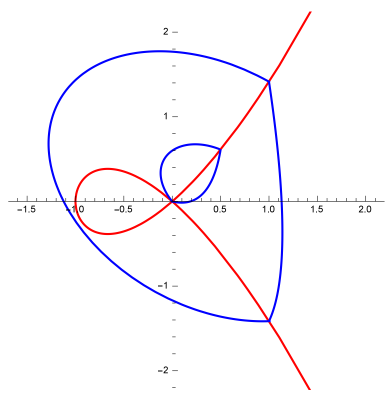

FIGURE 35. Two crossing limit cycles of the discontinuous piecewise linear differential system (62)–(63).

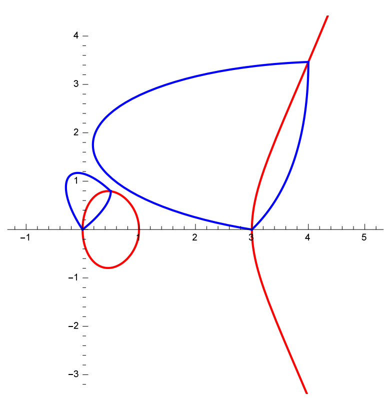

FIGURE 36

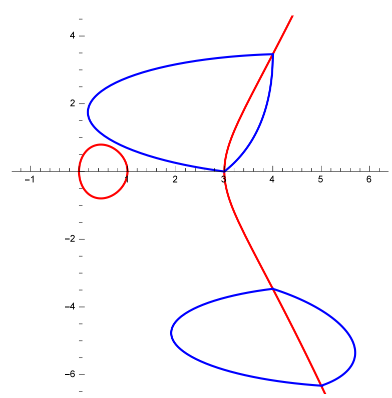

FIGURE 37

The real solutions of the system of equations (28) when $l = 3$ and $k = 4$, for the discontinuous piecewise linear differential system (40)–(41) are $(4, -2\sqrt{3}, 4, 2\sqrt{3})$ and $(5, -2\sqrt{10}, 6, 3\sqrt{10})$, producing the two crossing limit cycles of Figure 24.

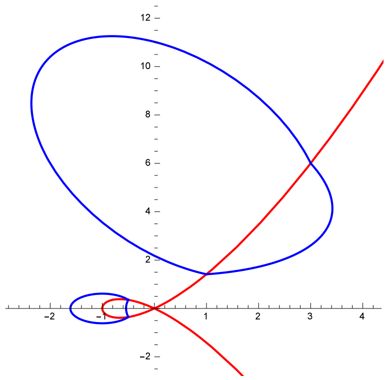

FIGURE 38

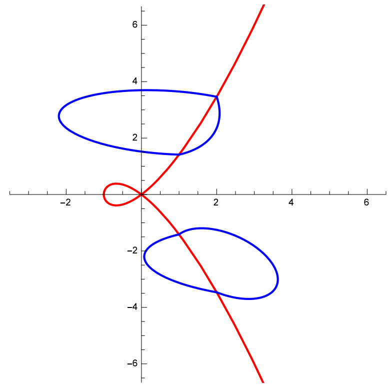

FIGURE 39

For the second configuration in the class C_2 we consider the differential linear center

$$\dot{x} = \frac{x}{9} - \frac{85y}{324} + \frac{1}{45}\left(10 + \sqrt{154}\right), \quad \dot{y} = x - \frac{y}{9} - \frac{194}{405}, \tag{42}$$

in the region $c_2(x, y) > 0$. Its first integral is

$$H_3(x, y) = \frac{1}{405}\left(-72\left(5x + \sqrt{154} + 10\right)y + 4x(405x - 388) + 425y^2\right).$$

The second linear center is

$$\dot{x} = -2x - \frac{17y}{4} - \frac{2\sqrt{154}}{5} - 4, \quad \dot{y} = x + 2y - \frac{4}{25}, \tag{43}$$

in the region $c_2(x, y) < 0$. Its first integral is

$$H_4(x, y) = 4x^2 + \frac{16}{5}\left(5x + \sqrt{154} + 10\right)y - \frac{32x}{25} + 17y^2.$$

The real solutions of the system of equations (28) when $l = 3$ and $k = 4$, for the discontinuous piecewise linear differential system (42)–(43) are $(1/5, 2\sqrt{14/5}/5, 4/5, 2\sqrt{11/5}/5)$ and $(1/5, -2\sqrt{14/5}/5, 4/5, -2\sqrt{11/5}/5)$, given place to the two crossing limit cycles of Figure 25.

Now we prove the statement for the third configuration. In the region $c_2(x, y) > 0$ we consider the linear differential center

$$\begin{aligned} \dot{x} &= \frac{1}{200}\left(20x - 52y + 27\sqrt{10} - 115\right), \\ \dot{y} &= x + \frac{1}{40}\left(-4y - 3\left(70 + \sqrt{10}\right)\right), \end{aligned} \tag{44}$$

which has the first integral

$$H_3(x, y) = 4x^2 - \frac{1}{5}x\left(4y + 3\left(70 + \sqrt{10}\right)\right) + \frac{1}{25}y\left(26y - 27\sqrt{10} + 115\right).$$

In the region $c_2(x, y) < 0$ we consider the linear differential center

$$\begin{aligned} \dot{x} &= \frac{-640x - 328y + 243\sqrt{10} + 3680}{3200}, \\ \dot{y} &= x + \frac{y}{5} + \frac{3}{2\sqrt{10}} - \frac{273}{64}, \end{aligned} \tag{45}$$

with its first integral

$$H_4(x,y)=4x^2+x\left(\frac{8y}{5}+6\sqrt{\frac{2}{5}}-\frac{273}{8}\right)+\frac{1}{400}y\left(164y-243\sqrt{10}-3680\right).$$

The real solutions of the system of equations (28) when $l=3$ and $k=4$, for the discontinuous piecewise linear differential system (44)–(45) are $(5,-2\sqrt{10},6,3\sqrt{10})$ and $(3,0,5,2\sqrt{10})$, producing the two crossing limit cycles of Figure 26.

We prove the statement for the forth configuration. In the region $c_2(x,y)>0$ we consider the linear differential center

$$\begin{aligned}\dot{x}=&\ \frac{x}{6}-\frac{13y}{144}+\frac{1}{864}\left(691\sqrt{10}\left(\sqrt{7}-2\right)+288\left(3\sqrt{7}-13\right)\right),\\ \dot{y}=&\ x-\frac{y}{6}-4\sqrt{\frac{7}{5}\left(8-3\sqrt{7}\right)}+\frac{1}{216}\left(3103-1382\sqrt{7}\right),\end{aligned}\tag{46}$$

which has the first integral

$$\begin{aligned}H_3(x,y)=&\ 4\left(x-\frac{y}{6}\right)^2+\frac{1}{27}\left(3103-1382\sqrt{7}-864\sqrt{\frac{7}{5}\left(8-3\sqrt{7}\right)}\right)x+\frac{y^2}{4}\\ &+\frac{1}{108}\left(3744-864\sqrt{7}-691\sqrt{10}\left(\sqrt{7}-2\right)\right)y.\end{aligned}$$

In the region $c_2(x,y)<0$ we consider the linear differential center

$$\begin{aligned}\dot{x}=&\ \frac{9x}{70}-\frac{977y}{22050}+\frac{74387\sqrt{10}\left(2+3\sqrt{7}\right)-11340\left(157+29\sqrt{7}\right)}{44100\left(25+8\sqrt{7}\right)},\\ \dot{y}=&\ x-\frac{9y}{70}+\frac{85075-\dfrac{297548}{\sqrt{7}}-5832\sqrt{35\left(8-3\sqrt{7}\right)}}{9450},\end{aligned}\tag{47}$$

with its first integral

$$\begin{aligned}H_4(x,y)=&\ 4\left(x-\frac{9y}{70}\right)^2+\frac{4\left(85075-\dfrac{297548}{\sqrt{7}}-5832\sqrt{35\left(8-3\sqrt{7}\right)}\right)x}{4725}+\frac{y^2}{9}\\ &+\frac{2\left(11340\left(157+29\sqrt{7}\right)-74387\sqrt{10}\left(2+3\sqrt{7}\right)\right)y}{11025\left(25+8\sqrt{7}\right)}.\end{aligned}$$

The real solutions of the system of equations (28) when $l=3$ and $k=4$, are $(5,-2\sqrt{10},10,3\sqrt{70})$ and $(3,0,8,2\sqrt{70})$, therefore the discontinuous piecewise linear differential system (46)–(47) has the two crossing limit cycles of Figure 27.

For the fifth configuration. In the region $c_2(x,y)>0$ we consider the linear differential center

$$\begin{aligned}\dot{x}=&\ \frac{x}{9}-\frac{85y}{324}+\frac{1}{45}\left(10+\sqrt{154}\right),\\ \dot{y}=&\ x-\frac{y}{9}-\frac{194}{405},\end{aligned}\tag{48}$$

which has the first integral

$$H_3(x,y)=4\left(x-\frac{y}{9}\right)^2-\frac{8}{405}\left(194x+9\left(10+\sqrt{154}\right)y\right)+y^2.$$

In the region $c_2(x,y)<0$ we consider the linear differential center

$$\tag{49} \begin{aligned} \dot{x} &= \frac{1}{100}\left(-20x-29y-4\left(10+\sqrt{154}\right)\right),\\ \dot{y} &= x+\frac{y}{5}-\frac{298}{625}, \end{aligned}$$

with its first integral

$$H_4(x,y)=4\left(x+\frac{y}{5}\right)^2+\frac{8}{625}\left(25\left(10+\sqrt{154}\right)y-298x\right)+y^2.$$

The real solutions of the system of equations (28) when $l=3$ and $k=4$, are $(1/5,2\sqrt{14/5}/5,4/5,2\sqrt{11/5}/5)$ and $(1/5,-2\sqrt{14/5}/5,4/5,-2\sqrt{11/5}/5)$, then the discontinuous piecewise linear differential system (48)–(49) has the two crossing limit cycles of Figure 28.

For the sixth configuration we consider the linear differential center

$$\tag{50} \begin{aligned} \dot{x} &= \frac{1}{900}\left(300x-181y+496\sqrt{10}-1500\right),\\ \dot{y} &= x-\frac{y}{3}-\frac{1}{2}, \end{aligned}$$

in the region $c_2(x,y)>0$, this system has the first integral

$$H_3(x,y)=4x^2-\frac{4}{3}x(2y+3)+\frac{1}{225}y\left(181y-992\sqrt{10}+3000\right).$$

The second linear differential center is

$$\tag{51} \begin{aligned} \dot{x} &= \frac{1}{980}\left(140x-265y+608\sqrt{10}-700\right),\\ \dot{y} &= x-\frac{y}{7}-\frac{1}{2}, \end{aligned}$$

in the region $c_2(x,y)<0$, this system has the following first integral

$$H_4(x,y)=-\frac{8}{245}\left(35x+152\sqrt{10}-175\right)y+4(x-1)x+\frac{53y^2}{49}.$$

The real solutions of the system of equations (28) when $l=3$ and $k=4$, are $(3,0,5,2\sqrt{10})$ and $(0,0,1,0)$, so the discontinuous piecewise linear differential system (50)–(51) has the two crossing limit cycles of Figure 29.

The second part of this proof analyze all the possible configurations of two crossing limit cycles of the class C_5. For the first configuration we consider the linear differential center

$$\tag{52} \dot{x}=\frac{x}{4}-\frac{y}{8}+\frac{1}{3}\left(\sqrt{7}-2\right),\quad \dot{y}=x-\frac{y}{4}+\sqrt{7}-\frac{9}{2},$$

in the region $c_5(x,y)>0$, with its first integral

$$H_5(x,y)=4x^2+x\left(-2y+8\sqrt{7}-36\right)+\frac{1}{6}y\left(3y-16\sqrt{7}+32\right).$$

The second linear differential center is

$$\tag{53} \dot{x}=\frac{x}{4}-\frac{y}{8}+\frac{1}{3}\left(\sqrt{7}-2\right),\quad \dot{y}=x-\frac{y}{4}+\sqrt{7}-\frac{9}{2},$$

in the region $c_5(x,y)<0$, with its first integral

$$H_6(x,y) = -\frac{x}{6} - \frac{17y}{450} + \frac{1}{6}\left(11+2\sqrt{7}\right).$$

For this case the real solutions of the system of equations (35) when $l=5$ and $k=6$, for the discontinuous piecewise linear differential system (52)–(53) are $(0,0,8,24)$ and $(3,6,6,6\sqrt{7})$. So the discontinuous piecewise linear differential system (52)–(53) has two crossing limit cycles, see them in Figure 30.

For the second possible configuration we consider the first linear differential center

$$\dot{x} = -\frac{x}{6} - \frac{85y}{144} - \frac{7}{30}, \quad \dot{y} = x + \frac{y}{6} + \frac{197}{360}, \tag{54}$$

in the region $c_5(x,y)>0$, with its first integral

$$H_5(x,y) = 4x^2 + \frac{1}{45}x(60y+197) + \frac{1}{180}y(425y+336).$$

In the region $c_5(x,y)<0$ we consider the linear differential center

$$\dot{x} = x - \frac{17y}{16} + \frac{7}{5}, \quad \dot{y} = x - y + \frac{117}{200}, \tag{55}$$

with its first integral

$$H_6(x,y) = 4x^2 + x\left(\frac{117}{25} - 8y\right) + \frac{1}{20}y(85y-224).$$

The real solutions of the system of equations (35) when $l=5$ and $k=6$, for the discontinuous piecewise linear differential system (54)–(55) are $(-1/5, 2/(5\sqrt{5}), -4/5, 4/(5\sqrt{5}))$ and $(-1/5, -2/(5\sqrt{5}), -4/5, -4/(5\sqrt{5}))$, producing the two crossing limit cycles of Figure 31.

For the third possible configuration for the class C_5 we consider in the region $c_5(x,y)>0$ the linear differential center

$$\dot{x} = \frac{x}{5} - \frac{29y}{100} - \frac{1}{5}, \quad \dot{y} = x - \frac{y}{5} + 4\sqrt{6} + \frac{12}{\sqrt{5}} - \frac{293}{20}, \tag{56}$$

with its first integral

$$H_5(x,y) = 4x^2 + \frac{2}{5}x\left(-4y + 80\sqrt{6} + 48\sqrt{5} - 293\right) + \frac{1}{25}y(29y+40).$$

In the region $c_5(x,y)<0$ we consider the linear differential center

$$\dot{x} = -\frac{x}{8} - \frac{17y}{64} + \frac{1}{8}, \quad \dot{y} = x + \frac{1}{64}\left(8y - 160\sqrt{6} - 96\sqrt{5} - 883\right), \tag{57}$$

with its first integral

$$H_6(x,y) = 4x^2 + x\left(y - 20\sqrt{6} - 12\sqrt{5} - \frac{883}{8}\right) + \frac{17y^2}{16} - y.$$

In this case the real solutions of the system of equations (35) when $l=5$ and $k=6$, are $(1,-\sqrt{2},1,\sqrt{2})$ and $(4,-4\sqrt{5},5,5\sqrt{6})$. So the discontinuous piecewise linear differential system (56)–(57) has the two crossing limit cycles of Figure 32.

For the fourth possible configuration of the class C_5 we consider in the region $c_5(x,y)<0$ the linear differential center

$$\dot{x} = \frac{1}{100}(-10x-26y-7), \quad \dot{y} = x + \frac{1440y + 704\sqrt{6} + 972\sqrt{3} + 4811}{14400}, \tag{58}$$

with its first integral

$$H_5(x,y) = \frac{7200x^2 + x(1440y + 704\sqrt{6} + 972\sqrt{3} + 4811) + 144y(13y+7)}{1800}.$$

In the region $c_5(x,y)>0$ we consider the linear differential center

$$\dot{x} = -\frac{x}{3} - \frac{37y}{9} - \frac{7}{30}, \quad \dot{y} = x + \frac{y}{3} + \frac{9\sqrt{3}}{40} + \frac{22\sqrt{2/3}}{45} + \frac{2495}{2592}, \tag{59}$$

with its first integral

$$H_6(x,y) = 4x^2 + x(\frac{8y}{3} + \frac{9\sqrt{3}}{5} + \frac{176\sqrt{2/3}}{45} + \frac{2495}{324}) + \frac{4}{45}y(185y+21).$$

In this case the real solutions of the system of equations (35) when $l=5$ and $k=6$, are $(-7/10,(7/10)\sqrt{3/10},7/10,(7/10)\sqrt{3/10})$ and $(-1/3,(1/3)\sqrt{2/3},-1/4,-\sqrt{3}/8)$. So the discontinuous piecewise linear differential system (58)–(59) has the two crossing limit cycles of Figure 33.

For the fifth possible configuration of the class C_5 we consider in the region $c_5(x,y)<0$ the linear differential center

$$\dot{x} = -\frac{x}{6} - \frac{113y}{18} + \frac{1}{4}, \quad \dot{y} = x + \frac{y}{6} - \frac{19}{5\sqrt{15}} - \frac{5}{3\sqrt{2}} + \frac{4897}{3600}, \tag{60}$$

with its first integral

$$H_5(x,y) = 4x^2 - \frac{1}{450}x\left(-600y + 912\sqrt{15} + 3000\sqrt{2} - 4897\right) + \frac{2}{9}y(113y-9).$$

In the region $c_5(x,y)>0$ we consider the linear differential center

$$\dot{x} = \frac{x}{10} - \frac{401y}{100} - \frac{3}{20}, \quad \dot{y} = x - \frac{y}{10} + \frac{1}{\sqrt{2}} + \frac{19\sqrt{\frac{3}{5}}}{25} + \frac{20629}{20000}, \tag{61}$$

with its first integral

$$H_6(x,y) = 4x^2 + x(-\frac{4y}{5} + 4\sqrt{2} + \frac{152\sqrt{\frac{3}{5}}}{25} + \frac{20629}{2500}) + \frac{1}{25}y(401y+30).$$

The real solutions of the system of equations (35) when $l=5$ and $k=6$, are $(-1/2,1/(2\sqrt{2}),-2/5,(3/2)\sqrt{5/2})$ and $(3/2,(3/2)\sqrt{5/2},3/2,(-3/2)\sqrt{5/2})$. So the discontinuous piecewise linear differential system (60)–(61) has the two crossing limit cycles of Figure 34.

For the sixth and last possible configuration of the class C_5 we consider in the region $c_5(x,y)>0$ the linear differential center

$$\dot{x} = -\frac{x}{3} - \frac{10y}{9} + \frac{1}{3}, \quad \dot{y} = x + \frac{y}{3} + \frac{1}{12}(\sqrt{6}-8), \tag{62}$$

with its first integral

$$H_5(x,y) = 4x^2 + \frac{2}{3}x(4y + \sqrt{6} - 8) + \frac{8}{9}y(5y - 3).$$

In the region $c_5(x,y) < 0$ we consider the linear differential center

$$\dot{x} = -\frac{x}{6} - \frac{5y}{18} + \frac{1}{6}, \quad \dot{y} = x + \frac{y}{3} + \frac{1}{2\sqrt{6}} - \frac{457}{1536}, \tag{63}$$

with its first integral

$$H_6(x,y) = 4x^2 + x(\frac{8y}{3} + 2\sqrt{\frac{2}{3}} - \frac{457}{192}) + \frac{1}{144}y(73y - 384).$$

The real solutions of the system of equations (35) when $l = 5$ and $k = 6$, are $(3, 6, 3, -6)$ and $(0, 0, 1, \sqrt{2})$. So the discontinuous piecewise linear differential system (62)–(63) has the two crossing limit cycles of Figure 35.

This completes the proof of statement (c) of Theorem 3. □

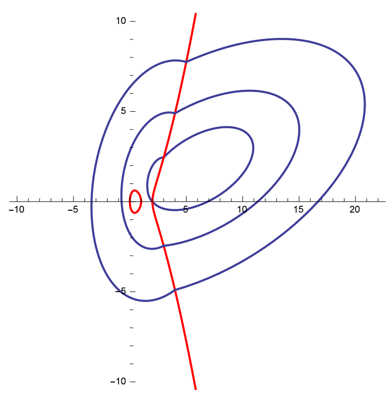

Figure 40. Three limit cycles of the discontinuous piecewise linear differential system (64)–(65).

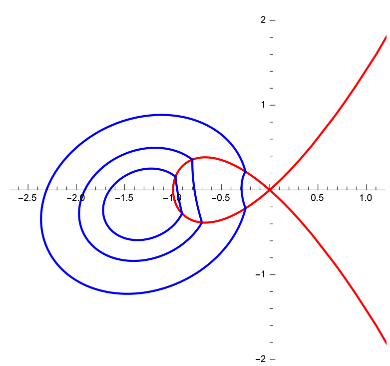

Figure 41. Three limit cycles of the discontinuous piecewise linear differential system (66)–(67).

Proof of statement (d) of Theorem 3. First we prove the statement for the class C_2. For this class we consider the linear differential center

(64)
$$\begin{aligned}\dot{x} =&\ \frac{x}{5}+\frac{1}{180}(10-28\sqrt{6}-5\sqrt{10}-20\sqrt{15})y+\frac{1}{36}(-12+2\sqrt{6}+6\sqrt{10}+\sqrt{15}),\\ \dot{y} =&\ x-\frac{y}{5}-\frac{\sqrt{6}}{5}-2,\end{aligned}$$

in the region $c_2(x,y)>0$. This differential system has the first integral

$$\begin{aligned}H_7(x,y) =&\ -\frac{-8x((\sqrt{15}-2\sqrt{6})y+3(\sqrt{10}-4))-4y(\sqrt{10}y+y-10\sqrt{15}+15\sqrt{6})}{5\sqrt{39-12\sqrt{10}}}\\ &\ \frac{12\sqrt{39-12\sqrt{10}}(x-4)x+y(2\sqrt{15}y-3\sqrt{6}y-6)}{6\sqrt{6}-3\sqrt{15}}.\end{aligned}$$

The other linear differential center is

(65)
$$\begin{aligned}\dot{x} =&\ x+\frac{1}{36}(2-28\sqrt{6}-\sqrt{10}-20\sqrt{15})y+\frac{1}{36}(-60+2\sqrt{6}+30\sqrt{10}+\sqrt{15}),\\ \dot{y} =&\ x-y-\sqrt{6}-2,\end{aligned}$$

in the region $c_2(x,y)<0$. It has the first integral

$$\begin{aligned}H_8(x,y) =&\ -\frac{-8x((\sqrt{15}-2\sqrt{6})y+3(\sqrt{10}-4))-4y(\sqrt{10}y+y-10\sqrt{15}+15\sqrt{6})}{\sqrt{39-12\sqrt{10}}}\\ &\ +\frac{12\sqrt{39-12\sqrt{10}}(x-4)x+y(2\sqrt{15}y-3\sqrt{6}y-6)}{6\sqrt{6}-3\sqrt{15}}.\end{aligned}$$

This discontinuous piecewise differential system formed by the linear differential centers (64)–(65) has exactly three crossing limit cycles, because the real solutions of the system (28) when $l=7$ and $k=8$, are $(2,0,3,\sqrt{6})$, $(3,-\sqrt{6},4,2\sqrt{6})$ and $(4,-2\sqrt{6},5,2\sqrt{15})$. See Figure 40.

Second we prove the statement for the class C_5. For this class we consider the linear differential center

$$\begin{aligned}\dot{x} =&\ \frac{x}{6}-\frac{37y}{36}+\frac{1}{24},\\ \dot{y} =&\ x-\frac{y}{6}+\frac{4697+1056\sqrt{5}+378\sqrt{30}}{7200},\end{aligned} \tag{66}$$

in the region $c_5(x,y)>0$. This differential system has the first integral

$$H_7(x,y)=4\left(x-\frac{y}{6}\right)^2+8\left(\frac{\left(4697+1056\sqrt{5}+378\sqrt{30}\right)x}{7200}-\frac{y}{24}\right)+4y^2.$$

The other linear differential center is

$$\begin{aligned}\dot{x} =&\ -\frac{x}{6}-\frac{5y}{18}-\frac{1}{24},\\ \dot{y} =&\ x+\frac{y}{6}+\frac{2605-528\sqrt{5}-189\sqrt{30}}{3600},\end{aligned} \tag{67}$$

in the region $c_5(x,y)<0$. It has the first integral

$$H_8(x,y)=\frac{1}{450}\left(1800x^2+x\left(600y-189\sqrt{30}-528\sqrt{5}+2605\right)+50y(10y+3)\right).$$

This discontinuous piecewise differential system formed by the linear differential centers (66)–(67) has exactly three crossing limit cycles, because the real solutions of the system (35) when $l=7$ and $k=8$, are $(-1/4,\sqrt{3}/8,-1/4,-\sqrt{3}/8)$, $(-4/5,4/(5\sqrt{5}),-7/10,(-7/10)\ \sqrt{3/10})$ and

$$(-0.9063569280622145544995606620..,-0.2773556931549898416488535525..,\\ -0.9742505235096835971802596877..,0.1563345074809235486186539264..).$$

See Figure 41. □

References

[1] A. Andronov, A. Vitt and S. Khaikin, *Theory of Oscillations*, Pergamon Press, Oxford, 1966.

[2] J.C. Artes, J. Llibre, J.C. Medrado and M.A. Teixeira, *Piecewise linear with two real saddles*, Math. Comput. Simulation **95** (2014), 13–22.

[3] M. Di Bernardo, C.J. Budd, A.R. Champneys and P. Kowalczyk, *Piecewise-Smooth Dynamical Systems: Theory and Applications*, Appl. Math. Sci. Series **163**, Springer-Verlag, London, 2008.

[4] R. Bix, *Conics and cubics*, Undergraduat Texts in Mathematics, Second Edition, Springer, 2006.

[5] R.D. Euzébio and J. Llibre, *On the number of limit cycles in discontinuous piecewise linear differential systems with two pieces separated by a straight line*, J. Math. Anal. Appl. **424**(1) (2015), 475–486.

[6] E. Freire, E. Ponce, F. Rodrigo and F. Torres, *Bifurcation sets of continuous piecewise linear systems with two zones*, Int. J. Bifurcation and Chaos **8** (1998), 2073–2097.

[7] E. Freire, E. Ponce and F. Torres, *Canonical discontinuous planar piecewise linear systems*, SIAM J. Appl. Dyn. Syst. **11**(1)(2012), 181–211.

[8] M. Han and W. Zhang, *On hopf bifurcation in non-smooth planar systems*, J. Differential Equations **248**(9) (2010), 2399–2416.

[9] S.M. Huan and X.S. Yang, *On the number of limit cycles in general planar piecewise linear systems*, Disc. Cont. Dyn. Syst. **32**(6) (2012), 2147–2164.

[10] J. Llibre, D.D. Novaes and M.A. Teixeira, *Maximum number of limit cycles for certain piecewise linear dynamical systems*, Nonlinear Dyn. **82** (2015), 1159–1175.

[11] J. Llibre, M. Ordóñez and E. Ponce, *On the existence and uniqueness of limit cycles in a planar piecewise linear systems without symmetry*, Nonlinear Analysis Series B: Real World Applications **14** (2013), 2002–2012.

[12] J. Llibre and E. Ponce, *Three nested limit cycles in discontinuous piecewise linear differential systems with two zones*, Dyn. Contin. Discr. Impul. Syst., Ser. B **19** (2012), 325–335.

[13] J. Llibre and M.A. Teixeira, *Piecewise linear differential systems with only centers can create limit cycles*? Nonlinear Dyn. **91** (2018), 249–255.

[14] J. Llibre and M.A. Teixeira, *Limit cycles in Filippov systems having a circle as switching manifold*, preprint, (2018).

[15] J. Llibre and X. Zhang, *Limit cycles for discontinuous planar piecewise linear differential systems separated by an algebraic curve*, to appear Int. J. Bifurcation and Chaos (2019).

[16] R. Lum and L.O. Chua, *Global propierties of continuous piecewise-linear vector fields. Part I: Simplest case in $\mathbb{R}^2$*, Int. J. of Circuit Theory and Appl. **19**(3) (1991), 251–307.

[17] R. Lum and L.O. Chua, *Global properties of continuous piecewise linear vector fields. II. Simplest symmetric case in $\mathbb{R}^2$*, Int. J. of Circuit Theory and Appl.**20**(1) (1992), 9–46.

[18] S. Shui, X. Zhang and J. Li, *The qualitative analysis of a class of plana Filippov systems*, Nonlinear Anal. **73**(5) (2010), 1277–1288.

[19] D.J.W. Simpson, *Bifurcations in piecewise-Smooth Continuous Systems*, World Scientific series on Nonlinear Science A, vol. **69**, World scientific, singapure, 2010.

[1] Département de Mathématiques, Université Bachir El Ibrahimi, Bordj Bou Arréridj, Bordj Bou Arréridj 34265, El Anasser, Algeria

Email address: r.benterki@univ-bba.dz

[2] Departament de Matematiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

Email address: jllibre@mat.uab.cat