DYNAMICS OF TWO EINSTEIN-FRIEDMANN COSMOLOGICAL MODELS

INNA BASAK AND JAUME LLIBRE

Abstract. We describe completely the dynamics of the two Einstein-Friedmann cosmological models, which can be characterized by the Hamiltonians

$$H = \frac{1}{2}(p_y^2 - p_x^2) + e^{2x}V(y),$$

whith the cosmological potentials $V(y) = e^{\lambda y}$, or $V(y) = (a + by)e^y$ with $\lambda ab \neq 0$.

1. Introduction and statement of the main results

The present work is devoted to the Einstein-Friedmann cosmological models, which can be characterized by the Hamiltonian

$$H = \frac{1}{2}(p_y^2 - p_x^2) + e^{2x}V(y), \tag{1}$$

where $V(y) = e^{\lambda y}$ or $V(y) = (a + by)e^y$ with $\lambda ab \neq 0$ are cosmological potentials. For more details on these two special models see subsections 2.2 and 3.1 [10], and for more details on the general Einstein-Friedmann cosmological models see [4, 7].

The Hamiltonian system with two degrees of freedom associated to the Hamiltonian

$$H = \frac{1}{2}(p_y^2 - p_x^2) + e^{2x+\lambda y} \tag{2}$$

is

$$\begin{aligned}
\dot{x} &= \frac{\partial H}{\partial p_x} = -p_x,\\
\dot{y} &= \frac{\partial H}{\partial p_y} = p_y,\\
\dot{p_x} &= -\frac{\partial H}{\partial x} = -2e^{2x+y\lambda},\\
\dot{p_y} &= -\frac{\partial H}{\partial y} = -\lambda e^{2x+y\lambda}.
\end{aligned} \tag{3}$$

The above Hamiltonian H has the additional first integral

$$F = 2p_y - \lambda p_x, \tag{4}$$

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as it is easy to check. The rank of the 2×4 matrix

$$\begin{pmatrix} 2e^{2x+y\lambda} & \lambda e^{2x+y\lambda} & -p_x & p_y \\ 0 & 0 & -\lambda & 2 \end{pmatrix}$$

is two except at a zero Lebesgue measure set in the phase space $E = \mathbb{R}^4$ with coordinates (x, y, p_x, p_y). Therefore the first integrals H and F are independent.

One can verify that the Poisson bracket $\{H, F\} = 0$, so the first integrals H and F are in *involution*. Therefore, the Hamiltonian system (3) is *completely integrable* in the sense of the Liouville-Arnold Theorem, see for instance [1, 2]. Clearly, the above Hamiltonian system has no equilibrium points. Therefore, all values (h, f) for the map $H \times F$ are regular, in particular the level $(H, F)^{-1}(0, f)$ is regular, so if $(H, F)^{-1}(0, f)$ is not empty, it is a 2-dimensional manifold, for more details see for instance [5].

Since H and F are first integrals, the sets

$$\begin{aligned} I_h &= \{(x, y, p_x, p_y) \in E : H(x, y, p_x, p_y) = h\}, \\ I_f &= \{(x, y, p_x, p_y) \in E : F(x, y, p_x, p_y) = f\}, \\ I_{hf} &= I_h \bigcap I_f. \end{aligned} \tag{5}$$

are invariant by the Hamiltonian flow, i.e. if an orbit solution of the Hamiltonian system has a point in one of the previous three sets, the whole orbit is contained in that set.

From physical reasons we are only interested in the dynamics of the Hamiltonian system (3) on the energy level $H = 0$, see [10]. Following the Liouville-Arnold Theorem, since the values $(0, f)$ are regular for all $f \in \mathbb{R}$, every connected component of the invariant 2-manifold I_{0f} is diffeomorphic either to a torus, to a cylinder or to a plane, see Theorem 3, and the dynamics on them are conjugated to a linear flow when the flow is complete, i.e. when the orbits are defined for all time $t \in \mathbb{R}$.

Theorem 1. *The following statements hold.*

(a) *If $f = 0$ and $|\lambda| > 2$, then I_{0f} is empty.*
(b) *If $f \neq 0$ and $|\lambda| \neq 2$, then I_{0f} is an invariant 2-dimensional manifold diffeomorphic to two copies of $\mathbb{R}^2$.*
(c) *Assume $\lambda = 2$. If $f = 0$, then I_{0f} is empty. If $f \neq 0$, then I_{0f} is an invariant 2-dimensional manifold diffeomorphic to $\mathbb{R}^2$.*
(d) *Assume $\lambda = -2$. If $f = 0$, then I_{0f} is empty. If $f \neq 0$, then I_{0f} is an invariant 2-dimensional manifold diffeomorphic to $\mathbb{R}^2$.*
(d) *All the orbits of the Hamiltonian system (3) restricted to I_{0f} come from the infinity and go to infinity.*

Theorem 1 is proved in section 2.

Now we consider the Hamiltonians of the form

$$H = p_y^2 - p_x^2 + e^x V(y), \tag{6}$$

that can be obtain from Hamiltonian (1) by applying canonical transformation

$$\{x, y, p_x, p_y\} \to \{1/2x, 1/2y, 2p_x, 2p_y\}$$

and substituting

$$\{H, V(y)\} \to \{-2H, -2V(y)\}.$$

Following [10] we consider the potential

$$V(y) = (a + by)e^y \quad \text{with } ab \neq 0. \tag{7}$$

Then the Hamiltonian system with Hamiltonian (6) has the first integral

$$F(x, y, p_x, p_y)p_y^2 - = p_x p_y + e^{x+y}(a + b + by)/2. \tag{8}$$

The condition $\{H, F\} = 0$ proves that F is a first integral of the following Hamiltonian system

$$\begin{aligned}
\dot{x} &= \frac{\partial H}{\partial p_x} = -2p_x, \\
\dot{y} &= \frac{\partial H}{\partial p_y} = 2p_y, \\
\dot{p_x} &= -\frac{\partial H}{\partial x} = -e^{x+y}(a + by), \\
\dot{p_y} &= -\frac{\partial H}{\partial y} = -e^{x+y}(a + b + by).
\end{aligned} \tag{9}$$

Therefore this Hamiltonian system is completely integrable in the sense of Liouville-Arnold Theorem, because the rank of the 2×4 matrix

$$\begin{pmatrix} e^{x+y}(a + by) & e^{x+y}(a + b + by) & -2p_x & 2p_y \\ e^{x+y}(a + b + by)/2 & e^{x+y}(a + 2b + by)/2 & -p_y & 2p_y - p_x \end{pmatrix}$$

is two except in a zero Lebesgue measure set of the phase space $E = \mathbb{R}^4$ again in the coordinates (x, y, p_x, p_y). Hence the first integrals H and F are independent.

As in the previous section the above Hamiltonian system has no equilibrium points and the levels $(H, F)^{-1}(0, f)$ are regular.

Theorem 2. *The following statements hold for the Hamiltonian system* (9)*.*

(a) *The set I_{0f} is empty if $b < 0$ and $f \leq 0$, otherwise I_{0f} is an invariant 2-dimensional manifold diffeomorphic to two copies of $\mathbb{R}^2$.*

(b) *All the orbits of the Hamiltonian system* (3) *restricted to I_{0f} come from the infinity and go to infinity.*

Theorem 2 is proved in section 3.

2. Proof of Theorem 1

In order to describe the dynamics of a complete integrable Hamiltonian system of two degrees of freedom we shall use the Liouville-Arnold Theorem, which can be stated as follows, for more details see [1, 2, 5].

Theorem 3. *Assume that a Hamiltonian system with two degrees of freedom defined on the phase space E has two independent first integrals H and F in involution. Then the following statements hold.*

(a) *Suppose that* (h, f) *is a regular value of the function* (H, F)*, so* I_{0f} *is a 2-dimensional submanifold of* E *invariant under the flow of the system.*
(b) *Under the hypothesis* (*a*) *if the flow on a connected component* I_{hf}^* *of* I_{hf} *is complete, then* I_{hf}^* *is diffeomorphic either to the torus* $\mathbb{S}^1 \times \mathbb{S}^1$*, or to the cylinder* $\mathbb{S}^1 \times \mathbb{R}$*, or to the plane* $\mathbb{R} \times \mathbb{R}$*.*
(c) *Under the hypothesis* (*b*) *the flow on* I_{hf}^* *is conjugated to a linear flow on either* $\mathbb{S}^1 \times \mathbb{S}^1$*, or* $\mathbb{S}^1 \times \mathbb{R}$*, or* $\mathbb{R} \times \mathbb{R}$*.*

Unfortunately we cannot apply to our two Hamiltonian systems (3) and (9) the results of Theorem 3 because we do not know if the flows of these Hamiltonian systems are complete, because we cannot obtain explicitly the solutions of these systems in function of the time. But as we shall see later on we can apply to them the Markus–Neumann–Peixoto theorem.

From $F(x, y, p_x, p_y) = f$ we obtain that

$$p_y = \frac{1}{2}(f + \lambda p_x). \tag{10}$$

Assume that $|\lambda| \neq 2$. Then, from

$$H\left(x, y, p_x, \frac{1}{2}(f + \lambda p_x)\right) = 0,$$

we get that

$$p_x = \pm\frac{f\lambda - 2\sqrt{f^2 - 2\left(\lambda^2 - 4\right)e^{2x+\lambda y}}}{4 - \lambda^2}. \tag{11}$$

Substituting p_x from (11) in (10) we obtain

$$p_y = \pm\frac{2f - \lambda\sqrt{f^2 - 2\left(\lambda^2 - 4\right)e^{2x+\lambda y}}}{4 - \lambda^2}. \tag{12}$$

In order to prove Theorem 1 we must characterize the topology of the 2-dimensional manifolds I_{0f} given by

$$\left\{\left(x, y, \pm\frac{f\lambda - 2\sqrt{f^2 - 2\left(\lambda^2 - 4\right)e^{2x+\lambda y}}}{4 - \lambda^2}, \pm\frac{2f - \lambda\sqrt{f^2 - 2\left(\lambda^2 - 4\right)e^{2x+\lambda y}}}{4 - \lambda^2}\right) : (x, y) \in \mathbb{R}^2\right\}. \tag{13}$$

Note that when in the expression of p_y there is a plus (respectively minus) in the expression of p_x there is a plus (respectively minus).

In short, when $|\lambda| \neq 2$ from (13) it follows that I_{0f} is empty if $f = 0$ and $|\lambda| > 2$, otherwise I_{0f} is an invariant 2-dimensional manifold diffeomorphic to two copies of $\mathbb{R}^2$. This completes the statements (a) and (b) of Theorem 1.

Assume $\lambda = 2$. Then from $F(x, y, p_x, p_y) = f$ we obtain again that $p_y = (f + 2p_x)/2$, and from $H(x, y, p_x, (f+2p_x)/2) = 0$, we have that $p_x = -2e^{2(x+y))}/f - f/4$. Substituting this expression of p_x into the previous expression of p_y we obtain that $p_y = -2e^{2(x+y))}/f + f/4$. In order to prove statement (c) of Theorem 1 we must characterize the topology of the 2-dimensional manifolds I_{0f} given by

$$\left\{\left(x, y, -\frac{2e^{2(x+y))}}{f} - \frac{1}{4}f, -\frac{2e^{2(x+y))}}{f} + \frac{1}{4}f\right) : (x, y) \in \mathbb{R}^2\right\}. \tag{14}$$

Therefore, when $\lambda = 2$ from (14) it follows that I_{0f} is empty if $f = 0$, otherwise I_{0f} is an invariant 2-dimensional manifold diffeomorphic to $\mathbb{R}^2$. This completes the statement (c) of Theorem 1.

Assume now $\lambda = -2$. Then from $F(x, y, p_x, p_y) = f$ we obtain again that $p_y = (f - 2p_x)/2$, and from $H(x, y, p_x, (f - 2p_x)/2) = 0$, we have that $p_x = 2e^{2(x-y))}/f + f/4$. Substituting this expression of p_x into the previous expression of p_y we obtain that $p_y = -2e^{2(x-y))}/f + f/4$. Again in order to prove statement (d) of Theorem 1 we must characterize the topology of the 2-dimensional manifolds I_{0f} given by

$$\left\{ \left(x, y, \frac{2e^{2(x-y))}}{f} + \frac{1}{4}f, -\frac{2e^{2(x-y))}}{f} + \frac{1}{4}f \right) : (x, y) \in \mathbb{R}^2 \right\}. \tag{15}$$

Therefore, when $\lambda = -2$ from (15) it follows that I_{0f} is empty if $f = 0$, otherwise I_{0f} is an invariant 2-dimensional manifold diffeomorphic to $\mathbb{R}^2$. This completes the statement (d) of Theorem 1.

In order to prove statement (e) we need some preliminary results. A *phase portrait* of the Hamiltonian system (3) restricted to the 2-dimensional manifold I_{0f} is the decomposition of I_{0f} as union of the orbits of this differential system.

Two phase portraits on I_{0f_1} and on I_{0f_2} are *topologically equivalent* if there is a homeomorphism $h : I_{0f_1} \longrightarrow I_{0f_2}$ which send orbits in I_{0f_1} into orbits of I_{0f_2}, preserving or reversing the sense of all the orbits.

A *separatrix* of the Hamiltonian system (3) restricted to the 2-dimensional manifold I_{0f} is one of following orbits: the equilibrium points, the limit cycles, and the two orbits at the boundary of every hyperbolic sector of an equilibrium point, see for more details on the separatrices [6, 8]. Recall that a *limit cycle* of a differential system is a periodic orbit isolated in the set of all periodic orbits of the differential system. For a definition of a hyperbolic sector see page 18 of [3].

The set of all separatrices of the Hamiltonian system (3) restricted to the 2-dimensional manifold I_{0f}, denoted by Σ_{0f}, is a closed set (see [8]). Here

A *canonical region* of I_{0f} is an open connected component of $I_{0f} \setminus \Sigma_{0f}$. The union of the set Σ_{0f} with an orbit of each canonical region form the *separatrix configuration* of the Hamiltonian system (3) restricted to the 2-dimensional manifold I_{0f} and is denoted by Σ'_{0f}.

We say that the flow of the Hamiltonian system (3) restricted to a 2-dimensional manifold I_{0f} is *parallel* if it is topologically equivalent to one of the following flows:

(i) The flow defined on $\mathbb{R}^2$ by the differential system $\dot{x} = 1$, $\dot{y} = 0$, which it is called the *strip flow*.

(ii) The flow defined on $\mathbb{R}^2 \setminus \{0\}$ by the differential system given in polar coordinates by $r' = 0$, $\theta' = 1$, which it is called the *annulus flow*.

(iii) The flow defined on $\mathbb{R}^2 \setminus \{0\}$ by the differential system given in polar coordinates by $r' = r$, $\theta' = 0$, which it is called the *spiral* or *nodal flow*.

A main result is the following: The flow at every canonical region of a flow on a 2-dimensional manifold is parallel, given by either a strip, an annular or a spiral flow, see [8].

Two separatrix configurations Σ'_{0f_1} and Σ'_{0f_2} are *topologically equivalent* if there is a homeomorphism $\Sigma'_{0f_1} \longrightarrow \Sigma'_{0f_2}$ such that $h(\Sigma'_{0f_1}) = \Sigma'_{0f_2}$.

According to the following theorem, which was proved by Markus [6], Neumann [8] and Peixoto [9], it is sufficient to investigate the separatrix configuration of a differential system on a 2-dimensional manifold, for determining its phase portrait.

Theorem 4. *Two phase portraits on I_{0f_1} and on I_{0f_2} with finitely many separatrices are topologically equivalent if and only if the two separatrix configurations Σ'_{0f_1} and Σ'_{0f_2} are topologically equivalent.*

We note that the Hamiltonian system (3) restricted to every 2-dimensional manifold I_{0f} has no separatrices because: First, these systems have no equilibrium points. Second, these systems have no periodic orbits because in the region limited by a periodic orbit a differential system in dimension two must have at least one equilibrium point (see for instance Theorem 1.31 of [3]), consequently these system have no limit cycles. Finally these systems have no hyperbolic sectors again because they do not have equilibrium points.

In summary Σ_{0f} is empty for all $f \in \mathbb{R}$. Therefore, by Theorem 4 the phase portrait of the Hamiltonian system (3) restricted to an arbitrary connected component of a 2-dimensional manifold I_{0f} is topologically the same for all $f \in \mathbb{R}$, in other words a connected component of I_{0f_1} and a connected component of I_{0f_2} are topologically equivalent for all $f_1, f_2 \in \mathbb{R}$. Moreover, every connected component I_{0f} is a canonical region. Since every one of these connected components are diffeomorphic to $\mathbb{R}^2$ it follows that the flow on each of these connected components is strip. So their orbits come and go to the infinity of the phase space E. Therefore statement (e) of Theorem 1 is proved.

This completes the proof of Theorem 1.

3. Proof of Theorem 2

From $F(x, y, p_x, p_y) = f$ we obtain that

$$p_x = \frac{e^{x+y}(a+b+by) - 2f + 2p_y^2}{2p_y}, \tag{16}$$

and from

$$H\left(x, y, \frac{e^{x+y}(a+b+by) - 2f + 2p_y^2}{2p_y}, p_y\right) = 0,$$

we get that

$$p_y = \pm\frac{2f - e^{x+y}(a+b+by)}{2\sqrt{2f - be^{x+y}}}. \tag{17}$$

Substituting p_y from (17) in (16) we obtain

$$p_x = \mp \frac{e^{x+y}(a-b+by)+2f}{2\sqrt{2f-be^{x+y}}}. \tag{18}$$

In order to prove Theorem 2 we must characterize the topology of the 2-dimensional manifolds I_{0f} given by

$$\left\{ \left(x, y, \mp \frac{e^{x+y}(a-b+by)+2f}{2\sqrt{2f-be^{x+y}}}, \pm \frac{2f-e^{x+y}(a+b+by)}{2\sqrt{2f-be^{x+y}}} \right) : (x,y) \in \mathbb{R}^2 \right\}. \tag{19}$$

Note that when in the expression of p_y there is a plus (respectively minus) in the expression of p_x there is a minus (respectively plus).

From (19) it follows that I_{0f} is empty if $b < 0$ and $f \leq 0$, otherwise I_{0f} is an invariant 2-dimensional manifold diffeomorphic to two copies of $\mathbb{R}^2$. So statement (a) of Theorem 2 is proved.

The proof of statement (b) of Theorem 2 is exactely the same than the proof of statement (e) of Theorem 1. This completes the proof of Theorem 2.

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Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

Email address: innabasak@gmail.com

Email address: jllibre@mat.uab.cat