ORBITALLY SYMMETRIC SYSTEMS WITH APPLICATIONS TO PLANAR CENTERS

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Abstract. We present a generalization of the most usual symmetries in differential equations known as the time-reversibility and the equivariance ones. We check that the typical properties are also valid for the new definition that unifies both. With it, we are able to present new families of planar polynomial vector fields having equilibrium points of center type. Moreover, we provide the highest lower bound for the local cyclicity of an equilibrium point of polynomial vector fields of degree 6, $M(6) \geq 48$.

1. Introduction

One of the fundamental properties studied in natural science is the existence of symmetries. They appear usually in many physical models describing classical mechanics. The most important studied symmetry is known as the time-reversible one, being Birkhoff one of the first who used it. See, for example, his works on the restricted three-body problem studied in 1915 ([6]) or the billiard ball problem published in 1927 ([7]). There exists an extensive bibliography on symmetries and their properties in all areas of dynamical systems. See for example the nice survey of Lamb & Roberts published in 1998 ([24]). In particular, they describe how this time-symmetry is useful in mathematics and physics for understanding a big list of phenomena: symmetric periodic orbits, local bifurcations, homoclinic and heteroclinic orbits,... They appear also in other research branches as thermodynamics and quantum mechanics.

We recall the well-known definitions of two important symmetries for smooth vector fields: the reversible and the equivariant. Let $U \subset \mathbb{R}^n$ be an open set, $\varphi : U \to U$ be an involution of class $\mathcal{C}^1$, and $\mathcal{X} : U \to \mathbb{R}^n$ a vector field of class $\mathcal{C}^r$. We say that $\mathcal{X}$ is *φ-reversible* or *time-reversible with respect to φ* if

$$D\varphi \cdot \mathcal{X} = -\mathcal{X} \circ \varphi \tag{1}$$

and $\mathcal{X}$ is *φ-equivariant* if

$$D\varphi \cdot \mathcal{X} = \mathcal{X} \circ \varphi. \tag{2}$$

In both cases the phase portrait is symmetric with respect to the fixed points set

$$\operatorname{Fix}\varphi = \{x \in U \subset \mathbb{R}^n : \varphi(x) = x\}.$$

After Birkhoff we can quote the work of Devaney [15] where this definition is also used restricted to manifolds of even dimension $2n$ being n the dimension of the set $\operatorname{Fix}\varphi$. Some years later, Arnol'd and Sevryuk allow that the 'symmetry' φ not to be necessarily an involution, see [3, 4].

The aim of this work is to extend the above definitions not only to treat them in a unified way but also to obtain new symmetric vector fields. We say that $\mathcal{X}$ is *orbitally*

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φ-symmetric or simply *orbitally symmetric* if there exist $F : U \to \mathbb{R}$ a continuous function and an involution $\varphi : U \to U$ such that

$$D\varphi \cdot \mathcal{X} = F\, \mathcal{X} \circ \varphi. \tag{3}$$

It is clear that the classical definitions (1) and (2) are included in (3) when F is constant, i.e. $F \equiv \mp 1$. As we will see in Theorem 1.2, this constant value will be required only at the equilibrium point. But it will be proved that it is satisfied also on $\operatorname{Fix}\varphi$. Also in the classical context, there are other extensions using symmetries and time-reversal symmetries from a compact Lie group. See for example [25, 27, 36]. We remark that in the above definition the set U is almost the full space, because we remove only the set of points where the involution is not well-defined.

The condition (3) says that the factor F can be directly computed from the vector field $\mathcal{X}$ and the involution φ. As we will see in Lemma 2.3, if $\mathcal{X}$ satisfies (3) and $p \in \operatorname{Fix}\varphi$ then $F(p) = 1$ or $F(p) = -1$. This property provides a natural decomposition of the fixed points set of the involution φ with respect to the vector field $\mathcal{X}$ in two disjoint sets. Hence, we split $\operatorname{Fix}\varphi = R_\varphi^{\mathcal{X}} \cup E_\varphi^{\mathcal{X}}$, where $R_\varphi^{\mathcal{X}} = \{p \in \operatorname{Fix}\varphi : F(p) = -1\}$ and $E_\varphi^{\mathcal{X}} = \{p \in \operatorname{Fix}\varphi : F(p) = 1\}$.

With the last observation we can update the reversible and equivariant symmetries. We say that a vector field $\mathcal{X} = \mathcal{X}(x)$ is *orbitally φ-reversible* (resp. *orbitally φ-equivariant*) if there exist a diffeomorphism $y = \phi(x)$ and a reparametrization of time $dt/ds = h(x)$ such that the change $(x, t) \mapsto (\phi(x), t/h(x))$ transforms $\mathcal{X}$ to $\mathcal{Y} = \mathcal{Y}(y)$ and $\mathcal{Y}$ is φ-reversible (resp. φ-equivariant).

When we particularize these definitions to planar vector fields, the most usual involutions are $\varphi_1(x, y) = (-x, y)$ and $\varphi_2(x, y) = (-x, -y)$. The first gives a symmetry with respect to the straight line $x = 0$ and the second a symmetry with respect to the origin. In particular, a vector field satisfying (1) for φ_1 writes as

$$(x', y') = (G_1(x^2, y), xG_2(x^2, y)) \tag{4}$$

and it is invariant with respect to the change of variables $(x, y, t) \mapsto (-x, y, -t)$. In this case, we say that (4) is time-reversible with respect to $x = 0$. For simplicity, we have not considered another classical involution $\varphi_3 = \varphi_1 \circ \varphi_2$ for which a vector field will be time-reversible with respect to $y = 0$. Moreover, when a vector field has an equilibrium point of center-focus type a usual sufficient condition to be a center is the above time-reversibility property. Therefore, in this case there exists an affine change of variables such that (4) writes as

$$(x', y') = (-y + g_1(x^2, y), x + xg_2(x^2, y)).$$

The notion of orbital reversibility for centers was presented previously by Giné & Maza in [19] and recently by Algaba, García & Giné in [2], but only for the classical time-reversibility. They use the Montgomery–Bochner theorem, see [29]. This notion can only be extended in the period annulus of the center. Our goal is to consider not only this particular orbital symmetric property for an arbitrary involution but also to give a unified treatment for both symmetries (reversibility and equivariance) and to obtain a global result, obviously, in the full domain where the involution and the vector field are well defined.

Theorem 1.1. *$\mathcal{X}$ is orbitally φ-symmetric if, and only if, $\mathcal{X}$ is orbitally φ-reversible or $\mathcal{X}$ is orbitally φ-equivariant.*

We remark that there are systems exhibiting both classical symmetries simultaneously, but the involutions φ in (1) and (2) are different. Usually up to an affine change of variables they are φ_1 and φ_2. Our approach will provide vector fields being simultaneously orbitally φ-reversible and orbitally φ-equivariant with respect to the same involution φ. Clearly, the symmetries will be with respect to two different sets, the aforementioned $R_\varphi^{\mathcal{X}}$ and $E_\varphi^{\mathcal{X}}$.

Next result generalizes, among others, the existence of a sufficient condition to have a center at an equilibrium point.

Theorem 1.2. *Let $\mathcal{X}$ be an orbitally symmetric planar vector field with respect to an involution φ defined in an open set $U \subset \mathbb{R}^2$. When $\operatorname{Fix}\varphi \cap U$ is a smooth manifold of dimension 1 and $p \in \operatorname{Fix}\varphi \cap U$ is an equilibrium point, the next properties hold:*

(i) If $p \in R_\varphi^{\mathcal{X}}$ and $\det(D\mathcal{X}(p)) > 0\,(< 0)$, then p is a center (saddle) of $\mathcal{X}$.
(ii) If $p \in E_\varphi^{\mathcal{X}}$, then $E_\varphi^{\mathcal{X}} \cap U$ is invariant under the flow of $\mathcal{X}$.

Clearly, from the first conclusion of the last result we can say that the equilibria on the reversible curve $R_\varphi^{\mathcal{X}}$ are φ-reversible.

Although it is important to think about the new symmetry (3) as a global property, we will also study how it varies with a change of coordinates later on. But following this idea, commonly the classical time-reversible property is used locally for classifying equilibria where the involution φ in (1) is taken as the one that changes sign in some of the coordinates. Hence, up to a local change of coordinates, the set $\operatorname{Fix}\varphi$ is a hyperplane. This strategy is used for example in [32, 33]. In the plane, this is equivalent to say that locally all time-reversible vector fields write as (4). In fact our definition works for polynomial vector fields and any change of variables that moves (3) to (1) or (2) goes out from this class and any classification in terms of the degree makes no sense. This is the aim of the initial work of Żołądek [40], later updated in [41], where he uses rational transformations for proving that some cubic polynomial vector fields have center. This rational reversibility is based on taking the pull-back of a vector field $\mathcal{V}$ induced by a noninvertible rational map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ but multiplied by a factor G. That is $\mathcal{X} = G\,\mathrm{Ad}_{\Phi^{-1}*}\mathcal{V}$, where $\mathrm{Ad}_{\Psi*}\mathcal{V} = (D\Psi\,\mathcal{X}) \circ \Psi^{-1}$. The vector field $\mathcal{V}$ is tangent at some point to the line $\Phi(C)$, where $C = \{\det D\Phi = 0\}$ is a line of critical points, and the transformed $\mathcal{X}$ has a center point. This notion of pullback of a differential equation has been recently used in [37, 38] in the context of foliations described by its associated 1-form. But this approach does not take into account how the orbits are traveled. Żołądek's first paper was much more ambitious because he tried to classify all the cubic vector fields having a center but without detailing how his work was exhaustive. This was the main reason for writing, two years later, the second work in which two classes of cubic vector field are considered: The *rationally reversible* class and the *Darboux integrable* one. Although in the first work he consider other integrability classes as the Darboux–Schwartz–Christoffel or Darboux-hyperelliptic classes, in the second one he presents 52 families of cubic polynomial vector fields having a center but all of them belonging only to the first two. In fact the rationally reversibility property was generalized by Christopher & Schlomiuk in [12]. We recall that a polynomial vector field is Darboux integrable if it has a rational first integral. For more details in Darboux or Liouvillian integrability we refer the reader to [16, 28] or the recent book [39]. As we will see also here, some of the families in the rational reversible class are also Darboux integrable vector fields. We have not checked if all the families in [41] or the ones presented here are in the Darboux class. Because, if they are, they could have invariant algebraic curves with a very high degree. Which would make them very difficult to find. Even so, in some of them, the existence of such first integral has been necessary to be found for proving that the equilibrium point is of center type because it is not on the fixed points curve.

The main difference between Żołądek's works and this paper is the fact that here we use the reversibility property directly to the vector field while he uses rational transformations near some special vector fields having a fold type point with respect to a curve, which will be, after the transformation, the symmetry line. Consequently, using the pull-back approach, (3) writes as $\mathrm{Ad}_{\varphi*}\mathcal{X} = F\mathcal{X}$, being φ is the associated involution instead of the noninvertible fold transformation Φ as above. In both works all the properties are considered in a global sense. Recently, Detchenia, Sadovski & Shcheglova have recovered this kind of center studies in [13, 14]. A final remark is the von Bothmer work ([35]) that, with heuristic methods, studied the center components of cubic planar polynomial vector fields giving new evidence for the explained Żołądek's conjecture about the existence of only the aforementioned two type of centers.

This work is not part of a research line dedicated to the classification of centers for a given family, but we think that it provides a new mechanism to check when a vector field has a center. Moreover, it has allowed us to give new families of polynomial vector fields with a center that until now were unknown. Although it was not the initial objective of the work, we have dedicated part of it to studying the number of limit cycles of small amplitude that can bifurcate from an equilibrium of monodromic type. When we are in the class of polynomial vector fields of degree n we will denote by $M(n)$ the maximum number of them. The study of the existence of such $M(n)$ is still an open problem. In fact only for the quadratic family, $M(2) = 3$, it is completely solved (see [5]). In [17] it is conjectured that $M(n) = n^2 + 3n - 7$. Recently, in [18], this conjecture should be updated in one, $M(n) = n^2 + 3n - 6$, because it is false, at least for $n = 3$. In the recent work [21] the local cyclicity of some Darboux cubic centers in [41], the ones with the highest codimension, is studied. The embryo of this idea appears in two relatively old works of Chicone & Jacobs [8, 9] but better developed in [10]. Next main result provides, up to our knowledge, the highest lower bound for the local cyclicity for degree $n = 6$ vector fields that reinforces the new conjecture. In this problem, the natural number of free parameters is $n^2 + 3n - 4$ and to get explicit polynomial systems exhibiting this maximal value of limit cycles of small amplitude is quite hard. Before this paper the conjecture was broken only for cubic family. The difficulties are related with the fact that we are almost using the total number of essential free parameters. For small degrees n, the best lower bound values for $M(n)$ can be found in [18, 21].

Theorem 1.3. *The local cyclicity of a monodromic equilibrium point for polynomial systems of degree six is at least* 48. *That is,* $M(6) \geq 48$.

For more details about centers, local cyclicity, and other related problems on bifurcation of limit cycles in planar polynomial vector fields we refer the reader to the books of Roussarie ([31]), Christopher & Li ([11]), and Romanovskii & Shafer ([30]). Or the more recent monographies of Han ([22]) and Han & Yu ([23]).

This paper is structured as follows. In Section 2 we introduce some general properties about how the orbital φ-symmetry property acts over the solution curves of a differential equation, proving the first two main results, Theorems 1.1 and 1.2. In particular about the sufficient conditions for a vector field to have a center or a saddle on the fixed points set of its corresponding involution φ. Among others, the involutions associated to the folded rational transformations introduced by Żołądek in [40], to classify cubic centers, are detailed in Section 3. In Section 4, together with other details, we show polynomial vector fields that are orbitally φ-symmetric having centers out of the fixed points set of the involution φ. More concretely, they are out of the domain of definition of φ. In Sections 5 and 6 we check that vector fields having the rational reversibility property introduced in

[40] also satisfy our definition, providing which are the respective involutions. We show the existence of equilibrium points of center and saddle type for such systems. In some cases the involutions are explicit and in some others are implicit. New families of vector fields exhibiting centers satisfying the orbital φ-reversibility property are also given. Finally, in Section 7 we study lower bounds for the local cyclicity for some of the presented systems when we perturb them inside the class of polynomial vector fields maintaining the degree. We finish proving our third main result, Theorem 1.3, providing a polynomial system of degree six that unfolds, in the class of polynomial vector fields of degree six, 48 limit cycles of small amplitude.

2. General properties

This section is devoted to present interesting properties that generalize some of the usual ones for time-reversible vector fields as appeared in [15, 24]. They will be very useful for the proofs of our results. Proposition 2.1 is just a simple version of Montgomery–Bochner Theorem given in [29, page 206]. Lemma 2.4 provides the sufficient condition for a solution to be periodic. The situation of symmetric equilibria with respect to $\operatorname{Fix}\varphi$ is given in Lemma 2.5 and their stabilities in Lemma 2.7. Lemma 2.8 shows how the definition of φ-symmetry and the factor F behaves with a change of variables. We finish proving Theorems 1.1 and 1.2.

Consider $\mathcal{C}^r$ differential systems of the form

$$x' = \mathcal{X}(x), \qquad x \in U \subset \mathbb{R}^n,$$

where $r \in \mathbb{N}\cup\{\infty, \omega\}$, the prime symbol denotes derivative with respect to the independent variable t and U is an open set. For our purposes, the natural number r will be taken big enough. Throughout this paper we are working with $\mathcal{C}^r$-*involutions* $\varphi : U \to U$ and all the properties are satisfied on the domain U. In particular, $\varphi^2 = \varphi \circ \varphi = I$ in U.

Proposition 2.1. *Each involution φ can be linearized in a neighborhood of a fixed point p, $\varphi(p) = p$. In other words there exist neighborhoods U_p of p and W_0 of 0, and a diffeomorphism $g_p : U_p \to W_0$ such that the transformed involution $\widetilde{\varphi} : W_0 \to W_0$ is linear, being $\widetilde{\varphi} = g_p \circ \varphi \circ g_p^{-1}$.*

Proposition 2.2. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a linear involution. There exists a basis of $\mathbb{R}^n$ such that φ is expressed as $\varphi(x) = (x_1, \ldots, x_s, -x_{s+1}, \ldots, -x_n)$, where $s = \dim \operatorname{Fix}\varphi$.*

Proof. It is easy to see that all $x \in \mathbb{R}^n$ can be written as $x = \left(\frac{x+\varphi(x)}{2}\right) + \left(\frac{x-\varphi(x)}{2}\right)$. So $\mathbb{R}^n$ can be decomposed as the direct sum $\operatorname{Fix}\varphi \oplus \operatorname{Fix}(-\varphi)$. It is enough to choose a linear basis on each subspace. □

Lemma 2.3. *Let $\mathcal{X}$ be an orbitally symmetric vector field. If p satisfies $\varphi(p) = p$ and $\mathcal{X}(p) \neq 0$, then $F(p) = 1$ or $F(p) = -1$. In particular, if $\operatorname{Fix}\varphi$ is a connected set, then the restriction of F to $\operatorname{Fix}\varphi$ is identically 1 or identically -1.*

Proof. First of all we observe that if φ is an involution and $p \in \operatorname{Fix}\varphi$, then the linear transformation $D\varphi_p : \mathbb{R}^n \to \mathbb{R}^n$ is also an involution. We have only to do the derivative of the expression $\varphi \circ \varphi = I$ being evaluated at the fixed point p. Applying the definition of φ-symmetric systems at the point p and taking into account that $\varphi(p) = p$ we obtain

$$D\varphi_p \cdot \mathcal{X}(p) = F(p)\, \mathcal{X}(p).$$

The hypothesis that $\mathcal{X}(p) \neq 0$ implies that $\mathcal{X}(p)$ is an eigenvector of the linear involution $D\varphi_p$ associated to the eigenvalue $F(p)$. From Proposition 2.2 we have that $F(p) = 1$ or $F(p) = -1$. The fact that F is of class $\mathcal{C}^r$ implies that on each connected component of the set $\operatorname{Fix}\varphi$ we have that F is identically 1 or -1. □

In the literature it is easy to find planar vector fields that are equivariant with respect to the involution $\varphi_1(x, y) = (-x, -y)$ and reversible with respect to another involution $\varphi_2(x, y) = (-x, y)$. We remark that we show (see for example Propositions 5.1, 5.2, and 6.1) the existence of planar vector fields which are simultaneously orbitally φ-reversible and orbitally φ-equivariant with the same involution φ. In these cases, both components, $R_\varphi^{\mathcal{X}}$ and $E_\varphi^{\mathcal{X}}$, of $\operatorname{Fix}\varphi$ are non empty. For example, the vector field $(x', y') = ((6-5x)(2x^2+y), 2y(4x^2+2y^2+6x+9y))$ is orbitally φ-symmetric taking $\varphi(x, y) = (x, -x^2y/(x^2+y))$ as its corresponding involution and being the factor $F(x, y) = (x^2+y)/x^2$. In this case, $E_\varphi^{\mathcal{X}}$ is the straight line $y = 0$ and $R_\varphi^{\mathcal{X}}$ is the parabola $2x^2+y = 0$. The straight line $x = 0$ and the parabola $x^2+y = 0$ should be removed because they are outside of the domains of definition of the involution and the factor. Clearly, by Theorem 1.2, $y = 0$ is an invariant straight line and the three equilibria $\{(1, -2), (1/2, -1/2), (-3/2, -9/2)\}$, that are on the parabola $2x^2+y = 0$, are φ-reversible centers (the first) or saddles (the other two).

Lemma 2.4. *Let $\mathcal{X}$ be an orbitally φ-symmetric vector field such* $\operatorname{Fix}\varphi = R_\varphi^{\mathcal{X}}$. *If $\alpha(t)$ is a solution, defined in U, of the differential system $x' = \mathcal{X}(x)$ that connects two different fixed points of φ and $F(\alpha(t)) \neq 0$ for all t, then $\alpha(t)$ is periodic.*

Proof. The first step of the proof is to find a function $h(t)$ of class $\mathcal{C}^r$ that satisfies $h'(t)F(\alpha(h(t))) = 1$. Taking into account that F and α are of class $\mathcal{C}^r$, we consider $G(y) = \int_0^y F(\alpha(t))dt$ a primitive of $F(\alpha(y))$. Using the hypothesis $F(\alpha(t)) \neq 0$ for all $t \in \mathbb{R}$ we obtain $G'(y) \neq 0$ for all $y \in \mathbb{R}$. By the Inverse Function Theorem, G is a $\mathcal{C}^r$ diffeomorphism from $\mathbb{R}$ to an open interval $J \subset \mathbb{R}$. Let h be the inverse of G. It is easy to see that differentiating the identity $G(h(t)) = t$ with respect to t we obtain the expression $h'(t)F(\alpha(h(t))) = 1$ for all $t \in J$.

Next step is consider $\beta(t) = \varphi(\alpha(h(t)))$ for $t \in J$. Differentiating with respect to t we get

$$\begin{aligned} \beta'(t) =& D\varphi(h'(t)\,\alpha'(h(t))) = h'(t)\,D\varphi \cdot \mathcal{X}(\alpha(h(t))) \\ =& h'(t)\,F(\alpha(h(t)))\,\mathcal{X}(\varphi(\alpha(h(t)))) = \mathcal{X}(\beta(t)). \end{aligned}$$

So, β is a solution of $x' = \mathcal{X}(x)$. Denoting by $p = \alpha(0)$ and $q = \alpha(T)$ the two different fixed points of φ, we observe that $\beta(0) = \varphi(\alpha(h(0))) = \varphi(\alpha(0)) = \varphi(p) = p = \alpha(0)$. By the Existence and Uniqueness Theorem for solutions of ordinary differential equations we get that $\alpha(t) = \beta(t)$, for all t in the maximal interval of existence. Evaluating at $t = T$ we obtain

$$q = \alpha(T) = \beta(T) = \varphi(\alpha(h(T))). \tag{5}$$

Using that $q \in \operatorname{Fix}\varphi$ and (5) we conclude that $\alpha(T) = \alpha(h(T))$. So, the solution α is periodic with period $h(T) - T$. It is easy to see that $h(T) \neq T$, because the hypotheses $F(p) = F(q) = -1$ and the fact $F(\alpha(t)) \neq 0$ for all t imply that $h'(t) < 0$ for all t. As a consequence, 0 is the unique fixed point of h. □

Lemma 2.5. *Let $\mathcal{X}$ be an orbitally φ-symmetric vector field. If $p \in U$ is an equilibrium of $\mathcal{X}$ that is neither fixed by φ nor vanishes F, then $\varphi(p)$ is also an equilibrium.*

Proof. From the hypotheses, $\mathcal{X}(p) = 0$ and $\varphi(p) = p$, we have that (3) allow us to write

$$F(p)\,\mathcal{X}(\varphi(p)) = D\varphi \cdot \mathcal{X}(p) = D\varphi \cdot 0 = 0.$$

As $F(p) \neq 0$ we can conclude that $\mathcal{X}(\varphi(p)) = 0$ and the statement follows. □

Lemma 2.6. *Let $\mathcal{X}$ be an orbitally φ-symmetric vector field. Assume that $\gamma \subset U$ is an orbit of $\mathcal{X}$ that F is negative on it and $p \in U$ is an equilibrium point of $\mathcal{X}(p)$. If p is in the ω-limit set of γ, then $\varphi(p)$ is in the α-limit set of γ.*

Proof. Let $\alpha(t)$, with $t \in \mathbb{R}$, be a time-parametrization of the solution γ. Assume that $\alpha(0) \in \operatorname{Fix}\varphi$. According to the proof of Lemma 2.4 there exists a function $h : J \subset \mathbb{R} \to \mathbb{R}$, with $h'(t) < 0$ for all $t \in J$, such that $\alpha(t) = \varphi(\alpha(h(t)))$ for all $t \in J$. By hypothesis there exists $\delta > 0$ such that the restriction $F|_\gamma < -\delta < 0$ implies that $J = \mathbb{R}$, because the inverse function of h is given by $h^{-1}(y) = \int_0^y F(\alpha(t))dt$ and it tends to $\mp\infty$ as y tends to $\pm\infty$. The hypothesis that p is in the ω-limit set of γ implies the existence of a sequence $\{t_n\}$ such that t_n tends to $+\infty$ and $\alpha(t_n)$ tends to p as n tends to infinity. For each t_n there exists $s_n \in \mathbb{R}$ such that $h(s_n) = t_n$. The sequence $\{s_n\}$ tends to $-\infty$. Thus $\alpha(s_n) = \varphi(\alpha(h(s_n))) = \varphi(\alpha(t_n))$ tends to $\varphi(p)$ as n tends to infinity. It implies that $\varphi(p)$ is in the α-limit set of γ. □

We remark that the converse is also true. Moreover, the equilibrium point p can be substituted by a limit cycle and the result also holds.

Lemma 2.7. *Let $\mathcal{X}$ be an orbitally φ-symmetric vector field having an equilibrium point at $p \in U$ with $F(p) \neq 0$. If λ is an eigenvalue of the Jacobian matrix of $\mathcal{X}$ at p, $D\mathcal{X}(p)$, then $\lambda/F(p)$ is an eigenvalue of the Jacobian matrix of $\mathcal{X}$ at $\varphi(p)$. In particular, when $p \notin \operatorname{Fix}\varphi$ and $F(p) < 0$, we have that if p is an attracting (repelling) hyperbolic equilibrium point, then $\varphi(p)$ is a repelling (attracting) hyperbolic equilibrium point. Moreover, when $p \in \operatorname{Fix}\varphi$ and $F(p) = -1$, if λ is an eigenvalue of $D\mathcal{X}(p)$, then $-\lambda$ is also an eigenvalue.*

Proof. Let us call $L = D\varphi(p)$, $A = D\mathcal{X}(p)$, and $B = D\mathcal{X}(\varphi(p))$. Differentiating the expression (3) at the point p we obtain $L \cdot A = F(p)B \cdot L$. Consider an eigenvector $v \neq 0$ of eigenvalue λ of the matrix A, $Av = \lambda v$. Thus we have

$$B \cdot L \cdot v = \frac{1}{F(p)} L \cdot A \cdot v = \frac{1}{F(p)} L \cdot \lambda v = \frac{\lambda}{F(p)} L \cdot v$$

and the proof follows, being $L \cdot v$ an eigenvector of eigenvalue $\lambda/F(p)$ of the matrix B. We notice that when $p \in \operatorname{Fix}\varphi$ we have $A = B$. □

We observe that if $\mathcal{X}$ is a φ-symmetric vector field and $p \in \operatorname{Fix}\varphi$ is such that $\mathcal{X}(p) = 0$ and $F(p) = 1$ it does not imply that the trace of $D\mathcal{X}(p)$ is zero. Consider the simple linear vector field $\mathcal{X}(x, y) = (x, y)$, $\varphi(x, y) = (x, -y)$, and $F(x, y) = 1$. We have that (3) is satisfied, $\mathcal{X}(0, 0) = (0, 0)$, and the trace of the differential matrix $D\mathcal{X}(0, 0)$ is equal to two.

Lemma 2.8. *Let $\mathcal{X}$ be an orbitally φ-symmetric vector field. If ψ is a change of coordinates, then the transformed vector field $\widetilde{\mathcal{X}} = D\psi \cdot \mathcal{X} \circ \psi^{-1}$ is an orbitally $\widetilde{\varphi}$-symmetric vector field, being $\widetilde{\varphi} = \psi \circ \varphi \circ \psi^{-1}$ and $\widetilde{F} = F \circ \psi^{-1}$ the respective transformed involution and factor.*

Proof. The proof follows just doing some simple computations. From $\widetilde{\varphi} = \psi \circ \varphi \circ \psi^{-1}$ we obtain $D\widetilde{\varphi} = D\psi \cdot D\varphi \cdot D\psi^{-1}$. Thus, at every point p,

$$\begin{gathered} D\widetilde{\varphi} \cdot \widetilde{\mathcal{X}}(p) = D\psi \cdot D\varphi \cdot D\psi^{-1} \cdot D\psi \cdot \mathcal{X}(\psi^{-1}(p)) = \\ D\psi \cdot D\varphi \cdot \mathcal{X}(\psi^{-1}(p)) = D\psi\big(F(\psi^{-1}(p)) \cdot \mathcal{X}(\varphi(\psi^{-1}(p)))\big) = \\ F(\psi^{-1}(p))\, D\psi \cdot (\mathcal{X} \circ \psi^{-1} \circ \psi \circ \varphi \circ \psi^{-1})(p) = \widetilde{F}(p)\, (\widetilde{\mathcal{X}} \circ \widetilde{\varphi})(p). \end{gathered}$$

Observe that we have used the hypothesis that $\mathcal{X}$ is an orbitally φ-symmetric vector field on the above third equality. □

Lemma 2.9. *Let $\mathcal{X}$ be an orbitally φ-symmetric vector field. Assume that $p \in \operatorname{Fix}\varphi = R_\varphi^{\mathcal{X}} \cup E_\varphi^{\mathcal{X}}$ and $\mathcal{X}(p) \neq 0$. If $p \in R_\varphi^{\mathcal{X}}$, then $\mathcal{X}(p) \notin T_p \operatorname{Fix}\varphi$ and if $p \in E_\varphi^{\mathcal{X}}$, then $\mathcal{X}(p) \in T_p \operatorname{Fix}\varphi$, where T_pM denotes the tangent space of a manifold M at p.*

Proof. First of all we observe that if φ is an involution and $\varphi(p) = p$, then the differential $L = D\varphi(p)$ is a linear involution. In fact, differentiating the expression $(\varphi \circ \varphi)(x) = x$ at the point $x = p$ we obtain $D\varphi(\varphi(p)) \cdot D\varphi(p) = I$. Thus, we have $L \cdot L = I$. Accordingly to Proposition 2.2 we can decompose the total space $\mathbb{R}^n = \operatorname{Fix} L \oplus \operatorname{Fix}(-L)$. The sets $\operatorname{Fix} L$ and $\operatorname{Fix}(-L)$ are the eigenspaces associated to the eigenvalues 1 and -1 of L, respectively. Observe that $\operatorname{Fix} L = T_p \operatorname{Fix} \varphi$ and so, $\operatorname{Fix}(-L) \pitchfork T_p \operatorname{Fix} \varphi$. As usual $\pitchfork$ denotes the transversal intersection of two manifolds. As the vector field $\mathcal{X}$ is orbitally φ-symmetric (i.e. $D\varphi(p) \cdot \mathcal{X}(p) = F(p)\,\mathcal{X}(\varphi(p)))$, when $p \in R_\varphi^{\mathcal{X}}$, that is $F(p) = -1$, we have $L \cdot \mathcal{X}(p) = -\mathcal{X}(p)$. It implies that $\mathcal{X}(p) \in \operatorname{Fix}(-L)$ and so $\mathcal{X}(p) \notin T_p \operatorname{Fix} \varphi$. Additionally, when $p \in E_\varphi^{\mathcal{X}}$, that is $F(p) = 1$, we have $L \cdot \mathcal{X}(p) = \mathcal{X}(p)$ and $\mathcal{X}(p) \in \operatorname{Fix}(L) = T_p \operatorname{Fix} \varphi$. □

We end this section with the proofs of our first two main results.

Proof of Theorem 1.1. Assume that $\mathcal{X}$ is orbitally symmetric. So, there exist a continuous function F and an involution φ that satisfy (3). From the property $D\varphi_{\varphi(x)} \cdot D\varphi_x = I$ we have that $X(x) = D\varphi_{\varphi(x)} \cdot D\varphi_x \cdot X(x) = D\varphi_{\varphi(x)} \cdot F(x) \cdot X \circ \varphi(x) = F(x) \cdot D\varphi_{\varphi(x)} \cdot X \circ \varphi(x) = F(x) \cdot F \circ \varphi(x) \cdot X(x)$, consequently $F(x) \cdot F(\varphi(x)) = 1$ for all x and F never vanishes. In a connected component where $F > 0$ we take the reparametrization of time $dt/ds = h(x)$ where $h(x) = 2/(F(x) + 1)$. We have that $\mathcal{Y}(x) = h(x) \cdot \mathcal{X}(x)$ is φ-equivariant. In fact,

$$D\varphi_x \cdot \mathcal{Y}(x) = D\varphi_x \cdot h(x) \cdot \mathcal{X}(x) = \frac{2}{F(x)+1} \cdot D\varphi_x \cdot \mathcal{X}(x) =$$
$$\frac{2F(x)}{F(x)+1} \cdot \mathcal{X} \circ \varphi(x) = \frac{2}{F(\varphi(x))+1} \cdot \mathcal{X} \circ \varphi(x) = h(\varphi(x)) \cdot \mathcal{X}(\varphi(x)) = \mathcal{Y} \circ \varphi(x).$$

Analogously, in a connected component where $F < 0$ we take the reparametrization of time $dt/ds = h(x)$ where $h(x) = 2/(F(x) - 1)$. We have that $\mathcal{Y}(x) = h(x) \cdot \mathcal{X}(x)$ is φ-reversible.

Conversely, assume that $\mathcal{X}$ is orbitally φ-equivariant. So, there exist a diffeomorphism $y = \phi(x)$ and a reparametrization of time $dt/ds = h(x)$ such that the change $(x, t) \mapsto (\phi(x), t/h(x))$ transforms $\mathcal{X}$ to $\mathcal{Y} = \mathcal{Y}(y)$ and $\mathcal{Y}$ is φ-equivariant. We consider the involution $\psi = \phi^{-1} \circ \varphi \circ \phi$ and the continuous function $F = (h \circ \psi)/h$. We will see that $D\psi \cdot \mathcal{X} = F \cdot \mathcal{X} \circ \psi$. First of all we observe that $\mathcal{Y}(y) = h \circ \phi^{-1}(y) \cdot D\phi_{\phi^{-1}(y)} \cdot \mathcal{X} \circ \phi^{-1}(y)$. Now we use the fact that $D\varphi_y \cdot \mathcal{Y}(y) = \mathcal{Y} \circ \varphi(y)$ we have that

$$D\varphi_y \cdot (h \circ \phi^{-1}(y) \cdot D\phi_{\phi^{-1}(y)} \cdot \mathcal{X} \circ \phi^{-1}(y)) = h \circ \phi^{-1}(\varphi(y)) \cdot D\phi_{\phi^{-1}(\varphi(y))} \cdot (\mathcal{X} \circ \phi^{-1}(\varphi(y))),$$

so

$$D\varphi_y \cdot D\phi_{\phi^{-1}(y)} \cdot \mathcal{X} \circ \phi^{-1}(y) = \frac{h \circ \phi^{-1} \circ \varphi(y)}{h \circ \phi^{-1}(y)} \cdot D\phi_{\phi^{-1}(\varphi(y))} \cdot \mathcal{X} \circ \phi^{-1}(\varphi(y)).$$

Now we change $y = \phi(x)$ and obtain

$$D\varphi_{\phi(x)} \cdot D\phi_x \cdot \mathcal{X}(x) = \frac{h \circ \phi^{-1} \circ \varphi \circ \phi(x)}{h(x)} \cdot D\phi_{\phi^{-1}(\varphi(\phi(x)))} \cdot \mathcal{X} \circ \phi^{-1} \circ \varphi \circ \phi(x),$$

that implies

$$D\phi^{-1}_{\varphi(\phi(x))} \cdot D\varphi_{\phi(x)} \cdot D\phi_x \cdot \mathcal{X}(x) = \frac{h \circ \phi^{-1} \circ \varphi \circ \phi(x)}{h(x)} \cdot \mathcal{X} \circ \phi^{-1} \circ \varphi \circ \phi(x),$$

Now we change $\phi^{-1} \circ \varphi \circ \phi$ to ψ and we obtain

$$D\psi_x \cdot \mathcal{X}(x) = \frac{h \circ \psi(x)}{h(x)} \cdot \mathcal{X} \circ \psi(x) = F \cdot \mathcal{X} \circ \psi(x).$$

The same idea works for the case that $\mathcal{X}$ is orbitally φ-reversible just considering $F = -h \circ \psi/h$. □

Proof of Theorem 1.2. (i) From Lemma 2.7 we have $\mathrm{trace}(D\mathcal{X}(p)) = 0$. It is clear that if p is an equilibrium point of $\mathcal{X}$ satisfying $\mathrm{trace}(D\mathcal{X}(p)) = 0$ and $\det(D\mathcal{X}(p)) > 0$, then p is an equilibrium point of center-focus type. Considering $\mathrm{Fix}\,\varphi$ as a cross section we have that, in a neighborhood of p, for each $q \in \mathrm{Fix}\,\varphi$ the orbit γ_q that passes to q intersects $\mathrm{Fix}\,\varphi$ in another point $\widetilde{q} \in \mathrm{Fix}\,\varphi$. According to Lemma 2.4 we have that γ_q is a periodic orbit. So, p is a center of $\mathcal{X}$. The case $F(p) = -1$ and $\det(D\mathcal{X}(p)) < 0$ is easier because Lemma 2.7 ensures that $\mathrm{trace}(D\mathcal{X}(p)) = 0$, and so p is a saddle of $\mathcal{X}$.

(ii) For the case $F(p) = 1$ consider $G : \mathbb{R}^2 \to \mathbb{R}^2$ given by $G(x) = \varphi(x) - x$. Then $\mathrm{Fix}\,\varphi = G^{-1}(0)$. The tangent space of $\mathrm{Fix}\,\varphi$ at p is the set of all vectors of $\mathbb{R}^2$ such that $DG_p \cdot v = 0$. For each $p \in \mathrm{Fix}\,\varphi$ we have that $D\varphi_p \cdot \mathcal{X}(p) = F(p)\,\mathcal{X}(\varphi(p)) = \mathcal{X}(p)$. So, $DG_p \cdot \mathcal{X}(p) = 0$ and it implies $\mathcal{X}(p) \in T_p\,\mathrm{Fix}\,\varphi$. Thus, $\mathrm{Fix}\,\varphi$ is invariant under the flow of $\mathcal{X}$. □

3. Involutions

In this section we list some involutions that are necessary for the studies developed in this paper. The simple cases are given by reflexions with respect to a straight line. For example the reflexion with respect to the line $y = 0$ is $\varphi_1(x, y) = (x, -y)$, and with respect to the line $y = x$ is $\varphi_2(x, y) = (y, x)$. In [26] there is a classification of all rational involutions of degree one. Two examples of them are $\varphi_3(x, y) = \left(\frac{a_1x+a_2y+a_3}{a_4x-a_1}, y\right)$ and $\varphi_4(x, y) = \left(\frac{a_1y+a_2}{a_3y+a_4}, \frac{a_4x-a_2}{-a_3x+a_1}\right)$.

As we have commented in the previous section, in [40], Żołądek defines *Rationally Reversible Systems* and describes a classification mechanism of reversible cubic systems. But in [41] he clarifies that his methodology does not provide a complete classification. In the second paper he provides two lists of different cubic systems, some of them having equilibrium points exhibiting real centers. First, some *reversible* are listed, after a long collection of interesting new cubic systems with rational first integrals are presented.

The classification, in [40], is based on the existence of pairs $(\Phi, \mathcal{V})$ where $\Phi(x, y) = (X, Y)$ is a rational map and $\mathcal{V}$ is a vector field. In one of these pairs, we have $\Phi(x, y) = (x^2, y)$ and the cubic systems belonging to this class are reversible, in a classic sense, with the involution $\varphi(x, y) = (-x, y)$. We can see that, in this case

$$\Phi(\varphi(x, y)) = \Phi(x, y). \tag{6}$$

Using this idea we have found some other involutions associated to all rational maps Φ presented in Żołądek. In Table 1 we present the involutions φ corresponding to each transformation Φ given in [40], satisfying (6). We notice that not all of them are explicit, some are implicitly defined by a given polynomial. As we will see in the next sections, the explicit involutions are better to be used with our reversibility property, but also the implicit involutions can be useful. We have worked with both concepts to provide new interesting center families.

The functions appearing in Table 1 are

$$T_1 = x + y + c, \quad T_2 = ax^2 + bxy + cy^2 + dx + ey + 1, \tag{7}$$

$CR_m^{(n)}$	$\Phi(x,y)$	$\varphi(x,y)$
$CR_1^{(7)}$	(x^2, y)	$(-x, y)$
$CR_{2^*}^{(10)}$	$(x, y^2/(h(x)+y))$	$(x, -h(x)y/(h(x)+y))$
$CR_3^{(10)}$	$\left(x, \dfrac{y^2}{xy+ax^2+bx+1}\right)$	$\left(x, \dfrac{-y(ax^2+bx+1)}{ax^2+bx+xy+1}\right)$
$CR_4^{(8)}$ $CR_5^{(8)}$ $CR_6^{(7)}$ $CR_7^{(9)}$	$(T_1 x, T_1/y)$	$(-c-x, -xy/(x+c))$
$CR_8^{(10)}$ $CR_9^{(10)}$	$(T_1 x, T_1^2/y)$	$(\alpha, x(\alpha+c+x)/\alpha)$
$CR_{10}^{(10)}$	$(T_1 x, T_1^3/y)$	$(\beta, -(\beta^2+\beta c-cx-x^2-xy)/\beta)$
$CR_{11}^{(7)}$ $CR_{12}^{(7)}$	$(T_1^2 x, T_1/y)$	$(c^2/x, yc/x)$
$CR_{13}^{(10)}$ $CR_{14}^{(9)}$	$(T_1^3/x, T_1^2/y)$	$\left(\frac{c^2y-\beta cx^2-2\beta cxy-\beta x^2y-2\beta xy^2-\beta y^3}{x(\beta x-c-x-y)}, \beta y\right)$
$CR_{15}^{(10)}$	$(T_1^4/x, T_1^2/y)$	$(\gamma^2 x, \gamma y)$
$CR_{16}^{(5)}$	$(T_2/x, T_2/y)$	$\left(\dfrac{x}{ax^2+bxy+cy^2}, \dfrac{y}{ax^2+bxy+cy^2}\right)$
$CR_{17}^{(12)}$	$\left(\frac{x^3}{y}, \frac{x^2}{xy-ay^2+2x+2(1+a)y+1-a}\right)$	$(\delta x, \delta^3 y)$

TABLE 1. Functions Φ and φ corresponding to the reversible families $CR_m^{(n)}$, for $m = 1,\ldots,17$, in [40]. The functions T_1, T_2 are defined in (7) and α, β, γ, and δ in (8)

and the functions α, β, γ, and δ are implicitly defined respectively by

$$\begin{aligned} \alpha^2 + (x+c)\alpha - xy &= 0, \\ x^2\beta^2 - ((x+y)^2 + 2cx)\beta + c^2 &= 0, \\ x^2\gamma^3 + x(x+2y)\gamma^2 + ((x+y)^2+2cx)\gamma - c^2 &= 0, \\ ay^2\delta^5 + ay^2\delta^4 - y(x-ay)\delta^3 - y(x-ay+2a+2)\delta^2 + (2x-a+1)\delta - a + 1 &= 0. \end{aligned} \tag{8}$$

We recall that the classical Möbius involutions are rational functions of degree one. See more details for example in [26]. Just as a sample of how this involutions can be used to get new reversible systems with our definition, we present the three families CR_{18}, CR_{19}, and CR_{20} in the following sections. They are obtained using the rational involutions indicated in Table 2.

CR_m	$\varphi(x,y)$
CR_{18} CR_{19}	$(x,(2x+y-1/(y-1))$
CR_{20}	$(x,(x^2-xy+2x-y)/(x+1))$

TABLE 2. Rational involutions φ corresponding to new reversible families CR_m, for $m=18,19,20$

Finally, we can also have involutions $\widetilde{\varphi}$ obtained by composition of a simple involution φ with a diffeomorphism ϕ, i.e. $\widetilde{\varphi}=\phi\circ\varphi\circ\phi^{-1}$. Observe that if $\varphi^2=I$, then $\widetilde{\varphi}^2=I$. In our case, as we are interested in polynomial vector fields, it is convenient to choose a birational transformation ϕ. A special class of them are the TAME diffeomorphisms which are functions of the form $\phi(x,y)=(x+g(y),y)$. For more details we refer the reader to [34]. Clearly, for our purpose we take a rational function g. In Table 3 we present the ones that we have used to present new systems CR_{21}, CR_{22}, and CR_{23}.

CR_m	$\phi(x,y)$	$\varphi(x,y)$
CR_{21}	$(x+ay^2,y)$	$(-x-2ay^2,y)$
CR_{22}	$(x+ay^2+by^3,y)$	$(-x-2ay^2-2by^3,y)$
CR_{23}	$(x+ay^2/(y^2+1),y)$	$(-(2ay^2+xy^2+x)/(y^2+1),y)$

TABLE 3. Rational diffeomorphisms ϕ and involutions φ corresponding to new reversible families CR_m, for $m=21,22,23$

4. About the definition of orbital φ-symmetry

In definition (3) we have restricted the study to the existence of a factor defined globally for a given involution φ. Clearly, the existence of a factor only on the fixed points curve is not enough to guarantee the symmetry property. We will show it with a simple example: The vector field $(x',y')=(-y,x-y)$, has a stable focus at the origin. With respect to the classical involution, $\varphi(x,y)=(x,-y)$, it satisfies the property (3) only on $\operatorname{Fix}\varphi$. But it is not orbitally φ-symmetric for any φ because, using Lemma 2.7, if it exists the trace should be zero at the equilibrium point and this is not the case.

Next two simple examples show that there exist differential systems with equilibrium points of center type satisfying property (3) globally for a given involution and with a rational factor, but for which Theorem 1.2 does not apply. This is because the equilibrium points are out of the fixed points curve of the corresponding involution. Consequently, we can not use Lemma 2.5. In the first example $\operatorname{Fix}\varphi$ is an invariant curve. In the second one, Theorem 1.2 applies but not at the equilibrium point of center type, only on the one which is of saddle type. The phase portraits of both systems together with the fixed set points (in red) of the involutions are depicted in Figure 1.

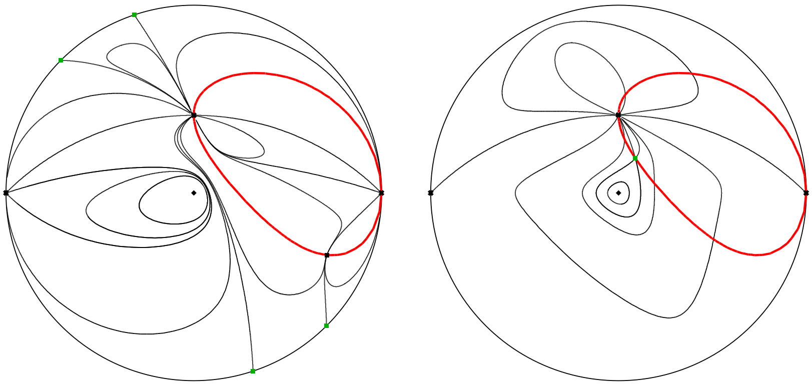

Figure 1. Phase portraits of systems (9) and (10). The fixed points set of the involution are depicted in red

Proposition 4.1. *The quartic polynomial vector field*

$$\begin{aligned} x' &= 6x^2y^2 + 6xy^3 + y^4 - 10xy^2 - 3y^3 - 2x^2 + 4xy + 3y^2 - y, \\ y' &= (y-1)(3xy^2 + 2y^3 + 2xy - 2y^2 - x), \end{aligned} \tag{9}$$

is orbitally φ-symmetric with $\varphi(x,y) = (x, (2x+y-1)/(y-1))$ and $F(x,y) = (y-1)^4/(4x^2)$. Moreover, $\operatorname{Fix}\varphi = E_\varphi^{\mathcal{X}} = \{(x,y) \in \mathbb{R}^2 : 2x - (y-1)^2 = 0, y \neq 1\}$ and $R_\varphi^{\mathcal{X}} = \emptyset$. Consequently, the set $\operatorname{Fix}\varphi$ is invariant by the flow. The equilibrium points are located at $(0,0)$, $(0,1)$, and $((z-1)^2/2, z)$ where z is the unique real root of $3z^3 + 3z^2 - 3z + 1$. The origin, which is not in $\operatorname{Fix}\varphi$, is of center type because system (9) has the rational first integral

$$H(x,y) = \frac{(y-1)^2(-3y^4 + 18xy^2 + 12y^3 + 12xy - 6y^2 - 6x - 4y + 1)}{((y-1)^2 - 2x)^3}.$$

The proof of the above result is straightforward. We notice that $y - 1 = 0$ and $2x - (y-1)^2 = 0$ are invariant curves and, consequently, there exist solutions arriving to the degenerate equilibrium $(0,1)$.

Proposition 4.2. *The quartic polynomial vector field*

$$\begin{aligned} x' &= 2xy^3 + y^4 - 4x^2y - 6xy^2 - 3y^3 + 4xy + 3y^2 - y, \\ y' &= -5xy^3 - 2y^4 + 7xy^2 + 4y^3 - 3xy - 2y^2 + x, \end{aligned} \tag{10}$$

is orbitally φ-symmetric with $\varphi(x,y) = (x, (2x+y-1)/(y-1))$ and $F(x,y) = -(y-1)^4/(4x^2)$. Moreover, $\operatorname{Fix}\varphi = R_\varphi^{\mathcal{X}} = \{(x,y) \in \mathbb{R}^2 : 2x - (y-1)^2 = 0, y \neq 1\}$, $E_\varphi^{\mathcal{X}} = \emptyset$, and there is an equilibrium point on $R_\varphi^{\mathcal{X}}$ which is of saddle type. The origin, which is an equilibrium point out of $\operatorname{Fix}\varphi$, is of center type because system (10) has the rational first integral

$$H(x,y) = \frac{5x^2y^2 + 4xy^3 + y^4 - 2x^2y - 4xy^2 - 2y^3 + x^2 + y^2}{(y-1)^2}.$$

Proof. The main parts of the proof are straightforward. We only remark that the saddle equilibrium point writes as $((z-1)^2/2, z)$, where z is the unique real root of $5z^3 - 3z^2 + 3z - 1$. Hence, Theorem 1.2 applies. □

In the last example, the system also has a degenerate equilibrium point at $(0,1)$ that has solutions arriving to it, because the straight line $y - 1 = 0$ is invariant.

We notice that with the classical time-reversibility notion, the number of equilibrium points out of $\operatorname{Fix}\varphi$ is even. This is not the case in the above families because, in both systems, the involution is not well defined in the full plane. For example, on the invariant straight line of system (9). In particular, for this involution, as the straight line $x = 0$ moves to $y = 1$, it makes no sense to look for the symmetric equilibrium points corresponding to the ones on $x = 0$. See these properties in Figure 1.

5. φ-SYMMETRIC SYSTEMS WITH EXPLICIT INVOLUTIONS

This section is devoted to show that most of systems, denoted by $CR_m^{(n)}$, in [40] or [41] satisfy the property of φ-symmetry (3) and we can prove the existence of an equilibrium point of center type using Theorem 1.2. Moreover, we will add some new systems, denoted by CR_m, having centers also satisfying the new symmetry property. We describe in detail only the first one. For the others we just list the fixed points sets $\operatorname{Fix}\varphi$, the factors F and their values on $\operatorname{Fix}\varphi$. We notice that the corresponding involutions are detailed in the Tables 1, 2, and 3 in Section 3. From now on we will denote by $\mathcal{X}$ the vector field corresponding to the differential equations in the statements.

The vector field in next proposition is denoted by $CR_2^{(10)}$ in [40]. The corresponding involution φ can be obtained in Table 1 taking $h(x) = x$ in the family denoted by $CR_{2^*}^{(10)}$ and $\operatorname{Fix}\varphi = \{y = -2x, x \neq 0\} \cup \{y = 0, x \neq 0\}$.

Proposition 5.1. *The system*

$$\begin{aligned} x' &= (mx^2 + lx + k)(2x + y), \\ y' &= ((m + p)x^2 + pxy + qy^2 + (l + n)x + ny + k)y, \end{aligned} \tag{11}$$

is orbitally φ-symmetric with $F(x, y) = (x+y)/x$. If $-4km+4kp-16kq+l^2-2ln+n^2 > 0$ and $m - p + 4q \neq 0$, then system (11) has

$$z_\pm = ((\pm w - l + n)/(2(m - p + 4q)), (\mp w + l - n)/(m - p + 4q))$$

as equilibrium points on the fixed points curve $\{y = -2x, x \neq 0\}$, where $k = -(w^2 - l^2 + 2ln - n^2)/(4(m - p + 4q))$. The trace of the differential matrix of (11), $D\mathcal{X}$, on $z_\pm$ is zero and the determinant takes positive or negative values depending on the choice of the parameters $\lambda = (k, l, m, n, p, q)$. Moreover, there are parameter values such that both equilibria are centers, both are saddles, and one is a saddle and the other is a center.

Proof. The property (3) is satisfied with the factor $(x + y)/x$, which takes the value -1 on the curve $R_\varphi^{\mathcal{X}} = \{y = -2x, x \neq 0\}$. The proof follows using Theorem 1.2 just checking that there are values for the parameters such that the determinants are both positive, both negative, or both with opposite sign.

For example, if $\lambda = \lambda_a = (5/24, 2, 5/2, 1/2, 5, 1)$ system (11) has two center equilibrium points at $(-1/6, 1/3)$ and $(-5/6, 5/3)$, the determinants of $D\mathcal{X}(z_\pm)$ are both positive. If $\lambda = \lambda_b = (-3/64, 0, 1, 1/2, 1, 1)$ system (11) has two saddle equilibrium points at $(3/16, -3/8)$ and $(-1/16, 1/8)$ because the determinants of $D\mathcal{X}(z_\pm)$ are both negative. And, finally, $\lambda = \lambda_c = (-9/400, 0, 1, 4/5, 1, 1)$ system (11) has a center at $(9/40 - 9/20)$ and a saddle $(-1/40, 1/20)$ because the determinants of $D\mathcal{X}(z_\pm)$ are positive and negative, respectively. In all cases, the extra parameter w takes the values ± 1. □

We notice that on the curve $E_\varphi^{\mathcal{X}} = \{y = 0\}$ the factor is 1, so it is invariant. Moreover, when $\lambda = \lambda_a$, system (11) has only three equilibrium points out of $\operatorname{Fix}\varphi$. Two of saddle type which are orbitally φ-symmetric and one of node type for which φ is not well defined. The other equilibrium points are on $\operatorname{Fix}\varphi$. See Figure 2 left. For $\lambda = \lambda_b$, system (11) has also only three equilibrium points out of $\operatorname{Fix}\varphi$ which are of node type. Two unstable which are orbitally φ-symmetric and one stable for which φ is not well defined. The stability of the two orbitally φ-symmetric equilibrium points is the same because they are symmetric with respect to a curve in which the factor F in (3) takes the constant value $+1$. The other equilibrium points are on $\operatorname{Fix}\varphi$. See Figure 2 middle. The symmetry lines are depicted in red.

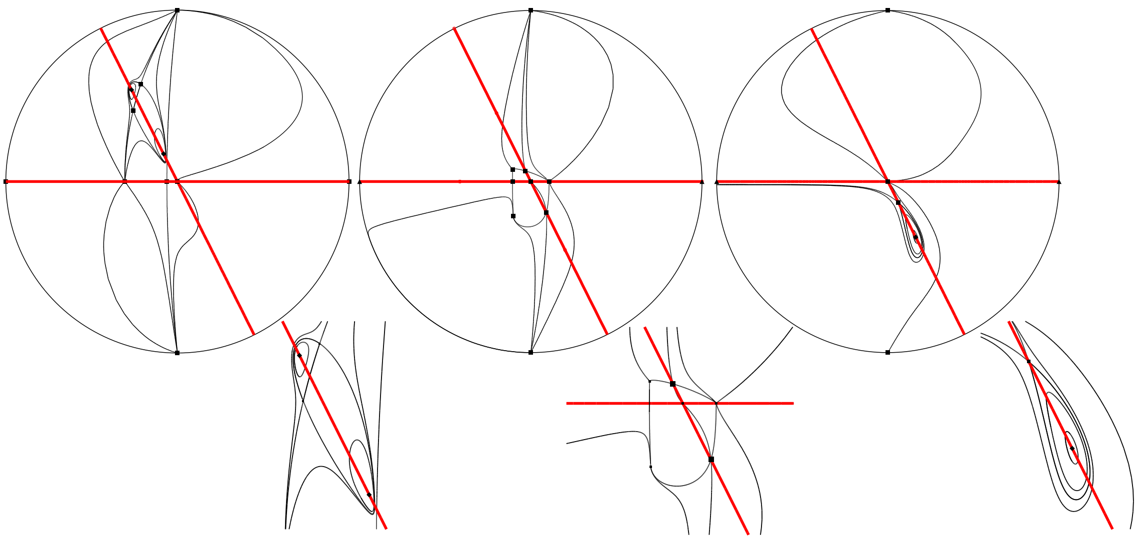

FIGURE 2. Phase portraits of (11) for λ equal to λ_a, λ_b, and λ_c, respectively

Next result uses the φ involution detailed in Table 1 as $CR_{2*}^{(10)}$ but with $h(x) = x^2$. The corresponding fixed points set is defined by the union of a straight line and a parabola, $\operatorname{Fix}\varphi = \{2x^2 + y = 0\} \cup \{y = 0\} \setminus \{(0,0)\}$.

Proposition 5.2. *The system*

$$x' = (lx + k)(2x^2 + y),$$
$$y' = y(mx^2 + ny^2 + 2kx - 2ly + my),$$

is orbitally φ-symmetric with $F(x,y) = (x^2 + y)/x^2$. There exist values of $\lambda = (k, l, m, n)$ such that the equilibrium point of the above system $(z, -2z^2)$, with $k = -2nz^3 - 2lz + mz/2$, located on $R_\varphi^{\mathcal{X}} = \{2x^2 + y = 0\} \setminus \{(0,0)\}$ is of center or saddle type depending on λ.

Proof. The proof of the first part follows as the previous ones just checking that the factor takes the constant value -1 on the parabola. It is easy to see that the equilibrium points have the form given in the statement. The last step shows the existence of center equilibrium points in this family. Because the trace of the Jacobian matrix at them is zero and the determinant is $-z^4(4nz^2 + 2l - m)(12nz^2 + 4l - m)$. Fixing $z = 1$, we have a center when $\lambda = (3, -2, 2, 1)$ and a saddle when $\lambda = (-5, 2, 2, 1)$. □

The families of the next two results are labeled, respectively, by $CR_4^{(8)}$, $CR_6^{(7)}$, $CR_5^{(8)}$, and $CR_7^{(9)}$ in [41]. They share the same involution φ in Table 1. The fixed points curve is $\operatorname{Fix}\varphi = \{2x + c = 0, x \neq -c\}$. For this involution, we consider only the case $c \neq 0$, otherwise the fixed points set degenerates to a point. We notice that we have corrected system (14) to be symmetric with respect to this involution, instead of the original one in [41] that it was not.

Proposition 5.3. *Systems*

$$\begin{aligned} x' &= -(c + x)(T_1 mx + kxy + l) - x(qT_1^2 + pyT_1 + y^2 n), \\ y' &= (2x + y + c)(qT_1^2 + pyT_1 + y^2 n) - y(mxT_1 + kxy + l), \end{aligned} \tag{12}$$

and

$$\begin{aligned} x' &= ny + (l-k-p)xy^2 + (-n+(m-l-q)xy)T_1 - (m+r)xT_1^2, \\ y' &= -ny + (p-l)xy^2 + ky^3 + ((q-m)xy + py^2)T_1 + (rx+qy)T_1^2 + rT_1^3, \end{aligned} \tag{13}$$

with $T_1 = x+y+c$, are orbitally φ-symmetric with $F(x,y) = (c+x)/x$ which takes the value -1 on $\operatorname{Fix}\varphi = R_\varphi^{\mathcal{X}} = \{2x+c=0, x \neq -c\}$. Moreover, both systems have equilibrium point located at $(-c/2, z)$ with $l = (n+p+q)z^2 + c(k+m+p+2q)z/2 + c^2(m+q)/4$ for system (12) and $n = (k+p+q+r)z^2 + c(l+m+q+2r)z/2 + c^2(m+r)/4$ for system (13). In both cases, there exist parameter values such that $(-c/2, z)$ is of center or saddle type.

Proof. The existence of the equilibrium point $(-c/2, z)$ and that the φ-symmetric property holds is straightforward using Theorem 1.2. The determinant of the Jacobian matrix of the vector fields (12) and (13) at this equilibrium point is $-(c+2z)((4n+4p+4q)z^2 + 2c(p+2q)z + c^2q)((4n+4p+4q)z + c(k+m+p+2q))/8$ and $-((8k+8p+8q+8r)z^3 + 4c(p+2q+3r)z^2 + 2c^2(q+3r)z + c^3r)((4k+4p+4q+4r)z + c(l+m+q+2r))/8$, respectively. The proof finishes just providing parameter values such that the second part of the statement holds, taking for example, $c=1$, $z=1$. The first system has a center for $(k,l,m,n,p,q) = (1, 11/4, 6, 0, 0, -1)$ and a saddle for $(k,l,m,n,p,q) = (1, 29/4, 6, 0, 0, 1)$. Finally, the second system has a center for $(k,l,m,n,p,q,r) = (1, 0, 6, 19/4, 0, 1, -1)$ and a saddle for $(k,l,m,n,p,q,r) = (1, 0, 6, 0, 0, 1, 1)$. $\square$

Proposition 5.4. *Let $c \neq 0$ be a real number. Systems*

$$\begin{aligned} x' &= x(l+p+cmx+(k+n)xy+mx^2+qxT_1), \\ y' &= -ly + mx^2y - kxy^2 - (qxT_1 + nxy + p)(x+T_1), \end{aligned} \tag{14}$$

and

$$\begin{aligned} x' &= x(-(n+k) + (l-m)xy - (l+p)xT_1), \\ y' &= nx + ky + (m-l)x^2y + (n+px^2+mxy+pxT_1)T_1, \end{aligned}$$

with $T_1 = x+y+c$, are orbitally φ-symmetric with $F(x,y) = x/(c+x)$, which takes the value -1 on $\operatorname{Fix}\varphi = R_\varphi^{\mathcal{X}} = \{2x+c=0, x \neq -c\}$. Moreover, the systems have an equilibrium point at $(-c/2, z)$ with $l = (2c(k+n+q)z + (m+q)c^2 - 4p)/4$ and $l = (-2c(m+p)z + c^2p - 4(k+n))/c^2$, respectively. In both cases, there exist parameter values such that it is of center or saddle type.

Proof. The proof follows as the previous ones. Here we only detail the explicit value for the corresponding determinants: $-c^2(-2c(m-n-q)z + qc^2 - 4p)(k+n+q)/8$ and $-c(m+p)(4c(m+p)z^2 + (2c^2m + 4c^2p - 8k - 8n)z + c(c^2p - 4n))/8$. $\square$

The fixed points set $\operatorname{Fix}\varphi = \{x - c = 0\}$ corresponds to the involution φ in Table 1 for families $CR_{11}^{(7)}$ and $CR_{12}^{(7)}$. Clearly, $c=0$ makes no sense. The existence of centers for these families is detailed in the next result. We notice that we also have corrected system (15) to be symmetric with respect this involution, instead of the original one in [41] that it was not.

Proposition 5.5. *Let $c \neq 0$ real number. Systems*

$$\begin{aligned} x' &= 2x(-(p+k)x + (l-n)y^2 + (m-l-q)yT_1 - (m+r)T_1^2), \\ y' &= (2px - ky + 2(n-l)y^2)x - (px - 2(q-m)xy + ny^2)T_1 + (2rx - qy)T_1^2 - rT_1^3, \end{aligned}$$

and

$$\begin{aligned} x' &= 2(m-k-q)xy + 2ly^3 + (2(n-l) - 2(m+r)xy^2)T_1 + 2(p-n)yT_1^2 - 2pT_1^3, \\ y' &= -ky^2 + 2(q-m)xy - 2ly^3 + (2rx - pq - 2ny^2)T_1 - (r+2py)T_1^2, \end{aligned} \tag{15}$$

with $T_1 = x + y + c$ *are orbitally* φ*-symmetric with* $F(x,y) = -x^2/c^2$ *and* $F(x,y) = -x/c$*, respectively. The factors take the value* -1 *on* $\operatorname{Fix}\varphi = R_\varphi^{\mathcal{X}}$*. Both systems have an equilibrium point at* (c,z) *taking* $k = -((-l-m+n+q+r)z^2 + 2c(-m+q+2r)z + c(4cr+p))/c$ *and* $m = -(2(l+n+p)z^2 + (4c(n+2p)+k+q+r)z + 2c(4cp+r))/(2c)$ *in the first and second systems, respectively. Moreover, depending on the parameter values, they exhibit center or saddle at this point.*

Proof. Straightforward computations show that the determinant of the Jacobian matrix at the equilibrium point detailed in the statement is $4c((n+q+r)z + c(l+m+q+2r))((-l-m+n+q+r)z^2 + 2c(-m+q+2r)z + c(4cr+p)$ and $(4(l+n+p)z + 4cn + 8cp + k + q + r)((k+q+r)z^2 + 2c(q+2r)z + 4c^2r)$, respectively. The proof follows as the previous ones. ☐

Next result provides the existence of centers for family $CR_3^{(10)}$. The corresponding involution φ is detailed in Table 1 and the fixed points set $\operatorname{Fix}\varphi = \{y=0\} \cup \{2x^2a + xy + 2bx + 2 = 0\}$ has two components.

Proposition 5.6. *System*

$$
\begin{aligned}
x' &= (2(mx^2 + lx + k))y,\\
y' &= 4a^2(-b^2k + 2ak + bl - 2m)x^3 + 6a(-b^2k + 2ak + bl - 2m)x^2y\\
&\quad + 4a(-2b^3k + 5abk + 2b^2l - 2al - 3bm)x^2 + (-2b^2k + 6ak + 2bl - 2m)xy^2\\
&\quad + (-6b^3k + 18abk + 6b^2l - 12al - 6bm)xy\\
&\quad + (-4b^4k + 8ab^2k + 4b^3l + 8a^2k - 4abl - 4b^2m - 8am)x\\
&\quad + ky^3 + (4bk - 2l)y^2 - 4b^3k + 12abk + 4b^2l - 8al - 4bm,
\end{aligned}
$$

is orbitally φ*-symmetric with* $F(x,y) = -(ax^2+bx+xy+1)/(ax^2+bx+1)$*. The factor takes the value* -1 *and* 1 *on the algebraic curves* $R_\varphi^{\mathcal{X}} = \{y=0\}$ *and* $E_\varphi^{\mathcal{X}} = \{2x^2a+xy+2bx+2 = 0\}$*, respectively. Moreover, when* $a(-b^2k+2ak+bl-2m) \neq 0$ *the vector field has an equilibrium point at* $(z,0)$ *with* $z = -(-b^3k+3abk+b^2l-2al-bm)/(a(-b^2k+2ak+bl-2m))$ *such that it is of center or saddle type depending on the values of the parameters.*

Proof. The proof follows straightforward as the previous ones. As above, only the existence of values of the parameters showing that the determinant of the Jacobian matrix at the equilibrium point change sign is necessary to be checked. If $m=1, l=0$, the determinant is $8((b^2-2a)^2k + b^2)(a^2k^2 + (b^2-2a)k + 1)^2(4a-b^2)/(a^2((-b^2+2a)k-2)^3)$. It takes positive or negative values for fixed (a,b) parameters in the involution. Because it vanishes at $k = k_0$ when $k_0 = -b^2/(-b^2+2a)^2$, and its Taylor series writes as $8(2b^4 - 5ab^2 + 4a^2)^2/(-b^2+2a)^3\,\varepsilon + O_2(\varepsilon)$ for $k = k_0 + \varepsilon$. ☐

The existence of centers and saddles for the vector field labeled as $CR_{16}^{(5)}$ in [41] is done in the next result. The involution is presented in Table 1 and the corresponding fixed points set is the conic $\operatorname{Fix}\varphi = \{ax^2 + bxy + cy^2 - 1 = 0\}$.

Proposition 5.7. *System*

$$
\begin{aligned}
x' &= -x(ky+lx) - (my+nx)T_2 - (bx+2cy+e)(qx^2 - (n-p)xy - my^2),\\
y' &= -y(ky+lx) - (py+qx)T_2 + (2ax+by+d)(qx^2 - (n-p)xy - my^2),
\end{aligned}
\tag{16}
$$

is orbitally φ*-symmetric with* $F(x,y) = -(ax^2+bxy+cy^2)$*. Moreover, there exist parameter values such that the above system has equilibrium points on* $\operatorname{Fix}\varphi = R_\varphi^{\mathcal{X}}$ *of center or saddle type.*

The proof follows exactly as the previous ones using Theorem 1.2. Clearly, the factor takes the value -1 at the fixed points set.

In most of the presented systems, the symmetry curve is a straight line. In few of them they are conics. In the last family, the symmetry curve is chosen as the unit circle to simplify computations. This is the case of system (16), taking $a = c = 1$ and $b = e = d = 0$. Fixing the other parameter values we get

$$\begin{aligned} x' &= x^3 + 3x^2y + xy^2 - y^3 - \frac{173}{20}x^2 + 5xy + x + y, \\ y' &= -x^3 + x^2y + 3xy^2 + y^3 - \frac{173}{20}xy + 5y^2 + x + y. \end{aligned} \tag{17}$$

It has an equilibrium point at $(4/5, 3/5)$ such that its Jacobian matrix has zero trace and positive determinant. The center property follows from the above result and Theorem 1.2. Moreover, the function

$$H(x,y) = \left(\frac{x+y}{x-y}\right)^{\frac{273}{80}} \exp\left(\frac{x^2+y^2-\frac{73}{40}x+1}{y-x}\right)$$

is a first integral of (17) having closed level curves around the equilibrium point. The phase portrait is depicted in Figure 3.

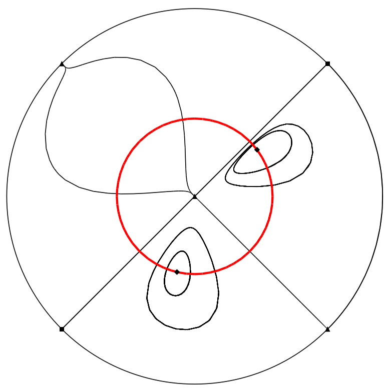

FIGURE 3. Phase portrait of vector fields (16)

Most of the orbitally φ-symmetric centers of this section, as the above one, appear in [40] but without the involutions φ. They are all listed in Table 1. The next families are new, all satisfy the definition (3) and the corresponding involution is presented in Table 2. As it is commented in [40], for some values of the parameters, the vector fields could have rational first integrals. This is the case, for example, of the last family. In Section 4 we have presented some but when the center equilibrium points are located out of the symmetry line. In the following we show that this phenomenon also occurs when the equilibrium points are located in the symmetry line. In fact, next vector fields are simultaneously orbitally φ-symmetric and also Darboux integrable.

Proposition 5.8. *Let* $\varphi = (x, (2x + y - 1)/(y - 1))$ *be an involution with the fixed points set* $\operatorname{Fix}\varphi = \{-y^2 + 2x + 2y - 1 = 0, y \neq 1\}$. *Then the next properties hold.*

(i) Consider, for $a \neq 1$, *the family* CR_{18},

$$\begin{aligned} x' &= (-y^2 + 2x + 2y - 1)(2(b + c)x - b(a - 1)^2), \\ y' &= (y - 1)(-(b + c)y^2 + 2(ba + c)y + 2(b + c)x - 2ba^2 + 2ba - b - c). \end{aligned} \tag{18}$$

It is orbitally φ-symmetric with $F(x,y) = -(y-1)^2/(2x)$. The factor F takes the value -1 on $\operatorname{Fix}\varphi = R_\varphi^{\mathcal{X}}$. *The equilibrium point* $((a-1)^2/2, a) \in R_\varphi^{\mathcal{X}}$ *is of center (saddle) type if* $bc > 0 (< 0)$. *Moreover, the function*

$$H(x,y) = \frac{((b+c)y^2 + 2(ab+c)y + 2(b+c)x + 2ab - b + c)^2}{(2(b+c)x - (a-1)^2 b)(y-1)^2}$$

is a rational first integral.

(ii) Consider, for $a, b \neq 1$, the family CR_{19},

$$\begin{aligned} x' &= 2x(-y^2 + 2x + 2y - 1), \\ y' &= (y-1)((b-1)y^2 - 2(ab-1)y + 2(b+1)x + 2ab - b - 1). \end{aligned} \tag{19}$$

It has three equilibrium points: $z_a = ((a-1)^2/2, a)$, $z_1 = (0,1)$, *and* $z_b = (0, (2ab-b-1)/(b-1))$. *Only the equilibrium point* $z_a \in \operatorname{Fix}\varphi = R_\varphi^{\mathcal{X}}$ *is an orbitally φ-symmetric point. Moreover, it is orbitally φ-symmetric with $F(x,y) = -(y-1)^2/(2x)$. The factor F takes the value -1 on* $\operatorname{Fix}\varphi = R_\varphi^{\mathcal{X}}$. *The equilibrium point z_a is of center (saddle) type if* $b > 0$ (< 0). *Additionally, the function*

$$H(x,y) = \frac{x^{b-1}((-b+1)y^2 + 2(ab-1)y - 2(b-1)x - 2ab + b + 1)^2}{(y-1)^2}$$

is a rational first integral.

Proof. (i) The property of orbital φ-symmetry follows straightforward. The proof finishes using Theorem 1.2 and the fact that the differential matrix of the vector field (18) at this equilibrium point has zero trace and a determinant $4(a-1)^2bc$.

(ii) This statement is proved as the previous one just using that the value of the determinant of the differential matrix of the vector field (19) is $b(a-1)^4$. $\square$

We notice that we have not considered $a = 1$ in family (18) because the equilibrium point would be in the invariant straight line $y = 1$. In the same way, $a \neq 1$ in family (19) because the points z_a and z_1 should be different. Moreover, we can not use Theorem 1.2 in either at z_1 or at z_b, because the involution is not well defined at those points. In fact the image by φ of the y-axis is the point z_1.

In Figures 4 and 5 we have drawn two different phase portraits in the Poincaré disk one exhibiting a center and another a saddle. In all cases, they are in the symmetry line, that is depicted in red.

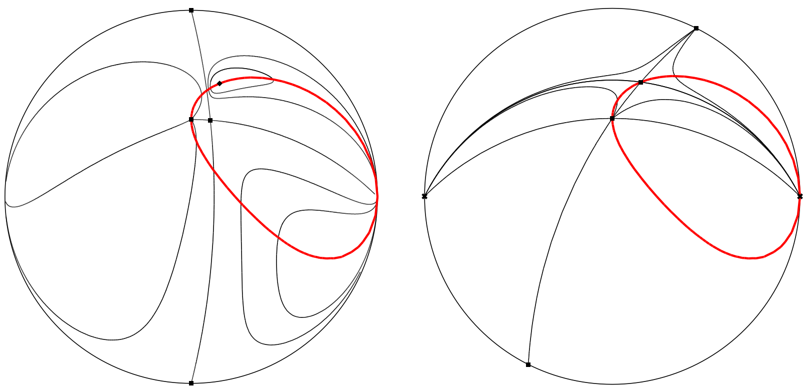

FIGURE 4. Phase portrait of (18) for $(a,b,c) = (2,1,1)$ and $(a,b,c) = (2,1,-1)$

The involution for the following family, labeled as CR_{20}, is given in Table 2. The fixed points curve is $\operatorname{Fix}\varphi = \{2(x+1)y - x(x+2) = 0\}$.

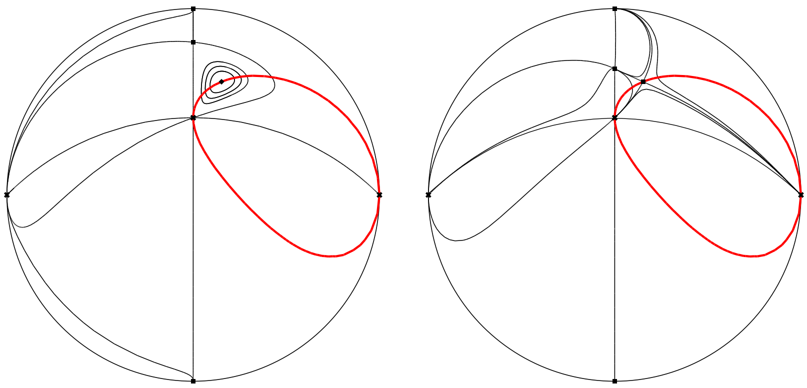

FIGURE 5. Phase portrait of (19) for $(a,b) = (2,2)$ and $(a,b) = (2,-3)$

Proposition 5.9. *Let $k \neq 3$ be a real number. System*

$$\begin{aligned} x' &= (1-x)(x^2 - 2xy + 2x - 2y), \\ y' &= -2k(xy - x + y)(x - y) + x^3 + x^2y - x^2 - 2xy + 2y^2 + 2x - 2y, \end{aligned} \tag{20}$$

is orbitally φ-symmetric with factor $F(x,y) = -1$. It has an equilibrium point at $((k-4)/(3-k), (k-4)(k-2)/(6-2k)) \in \operatorname{Fix}\varphi = R_\varphi^{\mathcal{X}}$. Moreover, it is of center type if $k \in (3, 7/2)$ and of saddle type if $k \neq 4$ and $k < 3$ or $k > 7/2$.

Proof. The proof follows straightforward as the previous ones just checking that the determinant of the Jacobian matrix of (20) at the equilibrium point is $(2k-7)(k-4)^4/(3-k)^3$ and it takes positive and negative values varying k as it is detailed in the statement. □

Phase portraits of system (20), for some values of k, are presented in Figure 6.

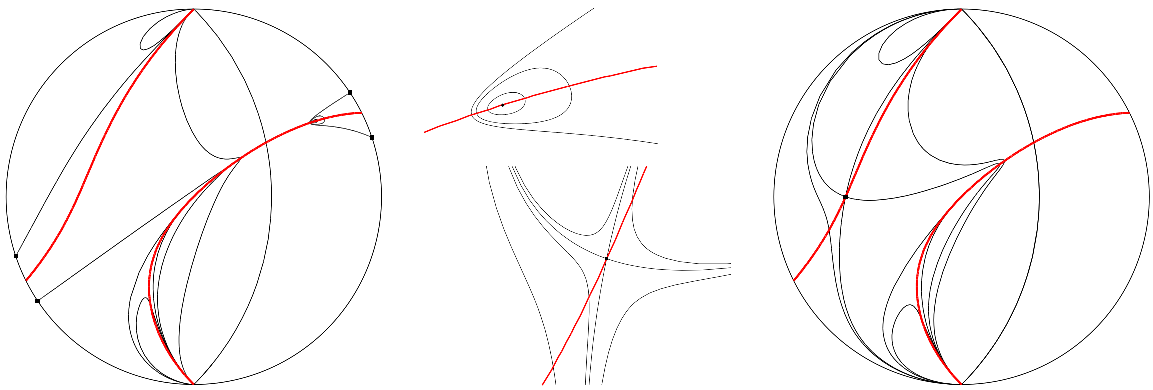

FIGURE 6. Phase portrait of (20) for $k = 13/4$ and $k = 2$ with the respective zooms near the center and the saddle symmetric equilibria

The following three families, as the above one, also satisfy the classic definition of reversibility, because the factor is constant and equal to -1. The systems corresponding to families CR_{21}, CR_{22}, and CR_{23} are presented in the next propositions. Their involutions φ, see Table 3, have as fixed points sets, the ones determined by $\operatorname{Fix}\varphi$, $\{x + ay^2 = 0\}$, $\{x + by^3 + ay^2 = 0\}$, and $\{x + ay^2/(y^2+1) = 0\}$, respectively. As all the remaining proofs in this section are similar to the previous ones we only detail the main differences. That is, the determinant of the Jacobian matrix at the equilibrium points and the explicit values for which it is positive or negative.

Proposition 5.10. *Let $a \neq 0$ be a real number. System*

$$x' = -2anxy + ly^3 - 2mxy^2 + ky^2,$$
$$y' = my^3 + mxy/a + any^2 + nx,$$

is orbitally φ-symmetric with $F(x,y) = -1$. It has an equilibrium point at $(-az^2, z) \in \operatorname{Fix}\varphi = R_\varphi^{\mathcal{X}}$ with $k = -(2amz + 2a^2n + l)z$. Moreover, there exist parameter values such that the equilibrium point is of center or saddle type.

Proof. The determinant of the Jacobian matrix is $-z^2(an+mz)(4amz+2a^2n+l)/a$ and, for $z = n = 1$ and $m = 0$, it is $-2a^2 - l$. The proof finishes choosing $l = 0$ and $l = -3a^2$ to get saddle and center type equilibrium points, respectively. □

Proposition 5.11. *Let $b \neq 0$ be a real number. System*

$$x' = -3bpxy^2 + ny^3 - 2apxy + my^2 + ly + k,$$
$$y' = bpy^3 + apy^2 + px,$$

is orbitally φ-symmetric with $F(x,y) = -1$. It has an equilibrium point at $(-az^2-bz^3, z) \in \operatorname{Fix}\varphi = R_\varphi^{\mathcal{X}}$ with $k = -3b^2pz^5 - 5abpz^4 - (2a^2p+n)z^3 - mz^2 - lz$. Moreover, there exist parameter values such that the equilibrium point is of center or saddle type.

Proof. The determinant of the Jacobian matrix is $-p(15b^2pz^4 + 20abpz^3 + (6a^2p+3n)z^2 + 2mz + l)$ and, for $z = p = 1$ and $m = n = 0$, it is $l_{ab} - l$, with $l_{ab} = -6a^2 - 20ab - 15b^2$. The proof finishes choosing $l = l_{ab} + 1$ and $l = l_{ab} - 1$ to get saddle and center type equilibrium points, respectively. □

Proposition 5.12. *Let $a \neq 0$ be a real number. System*

$$x' = ny^3 + px^2y + 2apxy + my^2 + ly + k,$$
$$y' = -p(xy^2 + ay^2 + x),$$

is orbitally φ-symmetric with $F(x,y) = -1$. It has an equilibrium point at $(-az^2/(z^2+1), z) \in \operatorname{Fix}\varphi = R_\varphi^{\mathcal{X}}$ with $k = z(-nz^6 - mz^5 + (a^2p - l - 2n)z^4 - 2mz^3 + (2a^2p - 2l - n)z^2 - mz - l)/(z^2+1)^2$. Moreover, there exist parameter values such that the equilibrium point is of center or saddle type.

Proof. The determinant of the Jacobian matrix is $p(-3nz^8 - 2mz^7 + (a^2p - l - 9n)z^6 - 6mz^5 + (3a^2p - 3l - 9n)z^4 - 6mz^3 + (6a^2p - 3l - 3n)z^2 - 2mz - l)/(z^2+1)^2$ and, for $z = p = 1$ and $m = n = 0$, it is $5a^2/2 + 2l$. The proof finishes choosing, for example, $l = 0$ and $l = 3a^2$ to get center and saddle type equilibrium points, respectively. □

6. φ-SYMMETRIC SYSTEMS WITH IMPLICIT INVOLUTIONS

The involutions described in Section 3 can be considered as explicit or implicit type. The orbitally φ-symmetric vector fields with respect to the first ones have been studied in the previous section. The involutions $CR_8^{(10)}$, $CR_9^{(10)}$, $CR_{10}^{(10)}$, $CR_{13}^{(10)}$, $CR_{14}^{(9)}$, $CR_{15}^{(10)}$, and $CR_{17}^{(12)}$ in Table 1 are given in an implicit form. Degree three orbitally φ-symmetric vector fields corresponding to them are detailed in [40, 41]. The cyclicity of the cubic vector field (21) labeled by $CR_{17}^{(12)}$ in [41] is studied in [21]. Here we prove that the equilibrium point is of center type because it satisfies (3). Additionally, we provide vector fields of degrees 4 and 6, also orbitally φ-symmetric with respect to this implicit involution, and their cyclicities are studied in the following sections. These new polynomial vector fields have been obtained using similar transformations as in [40] to get cubic vector fields. They provide good lower bounds for the local Hilbert number $M(n)$, for $n = 4, 6$. Similarly, other vector fields of degrees 5 and 7 can be given, but as the corresponding cyclicity are lower than the one previously known in the literature we have omitted.

Proposition 6.1. *Let $a \notin \{0, -1/6\}$ be a real number. The cubic system*

$$\begin{aligned} x' &= -6x^3 + (15a-1)x^2y + 5axy^2 - 4a^2y^3 + (3a-11)x^2 + (5a-4)xy \\ &\quad + 2a(3a+5)y^2 + 8(a-1)x + 4(a-1)y - 2(a-1)^2, \\ y' &= 3y(y-1)((1+3a)x - ay - 2a + 2), \end{aligned} \tag{21}$$

satisfies the property (3) *taking the involution $\varphi(x,y) = (\delta x, \delta^3 y)$ with δ defined implicitly in* (8)*. Clearly, $\delta = 1$ on the fixed points of φ and $\operatorname{Fix}\varphi = R_\varphi^{\mathcal{X}} \cup E_\varphi^{\mathcal{X}}$ with $R_\varphi^{\mathcal{X}} = \{-x + 2ay + a - 1 = 0, x \neq 3a-1\}$ and $E_\varphi^{\mathcal{X}} = \{y - 1 = 0, x \neq 3a-1\}$. Obviously, $E_\varphi^{\mathcal{X}}$ is invariant. Moreover, it has an equilibrium point at $((a-1)/(6a+1), -(3a^2-4a+1)/(6a^2+a))$ which is of center type when $-1/6 < a < 0$ or $1/3 < a < 1$ or $a > 1$ and is of saddle type when $0 < a < 1/3$ or $a < -1/6$.*

Proof. The symmetry property can be easily checked. The second part of the statement follows, applying Theorem 1.2, because the Jacobian matrix of the vector field at the equilibrium point has zero trace and a determinant $54(a-1)^2(9a^2-3a+1)^2(3a-1)/(a(6a+1)^3)$. □

In Figure 7 we have depicted the phase portrait of (21) in the Poincaré disk together with the symmetry line $-x + 2ay + a - 1 = 0$ for $a = -1, -1/12, 1/4, 1/2, 2$.

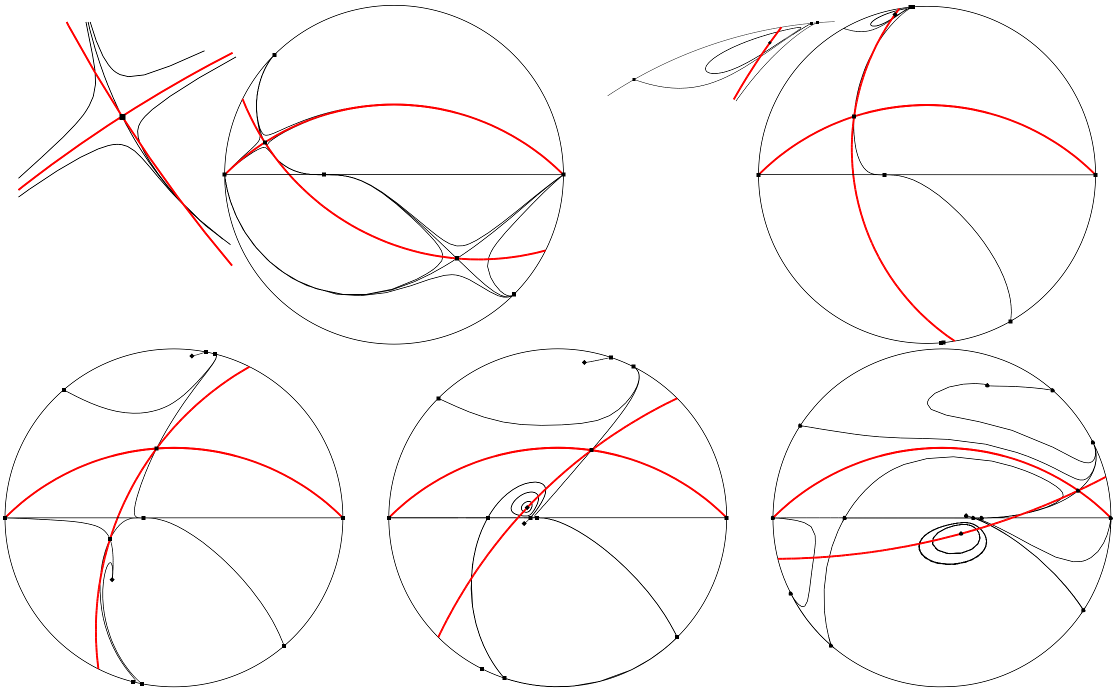

FIGURE 7. Phase portrait of (21) for $a = -1, -1/12, 1/4, 1/2, 2$

Proposition 6.2. *The quartic polynomial system*

$$\begin{aligned} x' &= -26x^4 + 70x^3y - 11x^2y^2 + 6xy^3 - 16x^3 + 9x^2y - 38xy^2 + 56xy - 56y^2, \\ y' &= y(y-1)(52x^2 - 27xy + 14y^2 - 84y), \end{aligned} \tag{22}$$

has a center at the equilibrium point $p = (1, 1/2)$.

Proof. We restrict our interested to the equilibrium point p of the statement.

The transformation corresponding to $CR_{17}^{(12)}$ in Table 1 with $a = 1$,

$$(X, Y) = (x^3/y, x^2/(xy - y^2 + 2x + 4y)),$$

applied to the vector field

$$\begin{aligned} X' &= 39X^2Y - X^2 - 42XY, \\ Y' &= 13XY^2 - 81Y^3 + 4XY - 28Y^2, \end{aligned}$$

provides the vector field (22). The Jacobian matrix of the transformed system at p has zero trace and positive determinant. Moreover, system (22) satisfies the property (3) taking the involution $\varphi(x, y) = (\delta x, \delta^3 y)$ with δ defined implicitly in (8) for $a = 1$, that has δ as a common factor. Consequently, the irreducible polynomial which defines the involution is

$$\delta^3(\delta + 1)y^2 - (\delta^2 + 1)y(x - y) - 4\delta y + 2x = 0.$$

The fixed points curve is $\operatorname{Fix}\varphi = R_\varphi^{\mathcal{X}} \cup E_\varphi^{\mathcal{X}}$ where $R_\varphi^{\mathcal{X}} = \{x - 2y = 0, x \neq 2\}$ and $E_\varphi^{\mathcal{X}} = \{y - 1 = 0, x \neq 2\}$. Then p is of center type because of Theorem 1.2 and it is in $R_\varphi^{\mathcal{X}}$. □

System (22) has two orbitally φ-symmetric equilibrium points of saddle type at $(14/13, 1)$ and $(14z, 13z^3)/13$, being z the unique real root of the polynomial $13z^3 + 26z^2 + 25z - 28$. The heteroclinic orbit that connects them is also orbitally φ-symmetric. That is, Lemmas 2.5 and 2.6 apply. We can use Theorem 1.2 for studying the qualitative local behaviors of the points $(2, 1)$, and $(1, 1/2)$. They are on $\operatorname{Fix}\varphi$ and each one is orbitally φ-symmetric of itself. Also the origin, which is a degenerate equilibrium, is its own symmetric. See these properties together with the phase portrait of (22) in Figure 8. We notice that the implicit relation (8) with $a = 1$ for the equilibrium points $(2, 8)$ and $(-8/13, 0)$ gives $\delta = 0$. Consequently, the involution φ is not well defined on them because their images by φ would go to the origin.

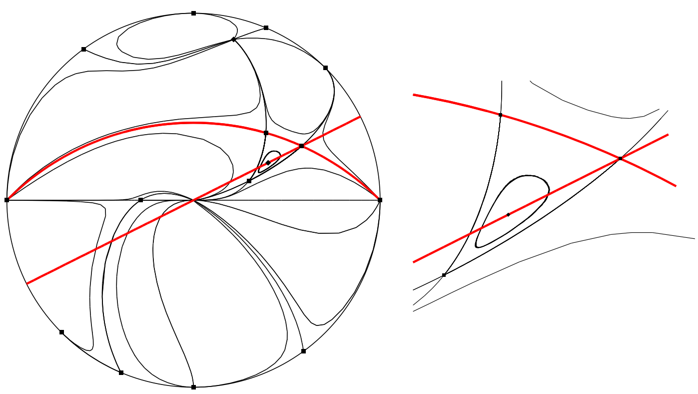

FIGURE 8. Phase portrait of (22) with a zoom near the center point

Proposition 6.3. *The sextic polynomial system*

$$\begin{aligned} x' &= -68082x^6 + 1060844x^5y - 3761510x^4y^2 + 15309875x^3y^3 - 13108500x^2y^4 \\ &\quad + 21847500xy^5 + 487720x^5 - 3970914x^4y - 23536165x^3y^2 + 23595300x^2y^3 \\ &\quad - 135454500xy^4 - 7984x^4 + 4391140x^3y + 61529220x^2y^2 + 307612800xy^3 \\ &\quad - 52434000y^4 - 6983216x^3 - 57185352x^2y - 248187600xy^2 + 104868000y^3 \\ &\quad + 16778880x^2 + 106266240xy + 8389440y^2 - 11185920x - 16778880y, \\ y' &= 3y(y-1)(181552x^4 - 784430x^3y + 5386275x^2y^2 - 13108500xy^3 + 21847500y^4 \\ &\quad - 2373680x^3 + 1697310x^2y + 10486800xy^2 - 113607000y^3 + 10158768x^2 \\ &\quad + 48239280xy + 178275600y^2 - 16778880x - 82496160y + 11185920), \end{aligned} \tag{23}$$

has a center at the equilibrium point $(3, -1/10)$.

Proof. As in the previous proof, we restrict our analysis to the equilibrium point $p = (3, -1/10)$. The proof follows as the previous one but taking $a = 5$ in the transformation and checking that the symmetry line, which is $x - 10y - 4 = 0$, contains the point p and is one of the components of $\operatorname{Fix}\varphi$. Finally, the system of the statement is obtained transforming the vector field

$$\begin{aligned} X' &= \frac{11347}{402410}X^2Y - \frac{49618}{603615}X^2 + \frac{413757}{402410}XY + \frac{17478}{40241}X, \\ Y' &= \frac{11347}{1207230}XY^2 + Y^3 - \frac{87889}{3621690}XY + \frac{312699}{201205}Y^2 - \frac{1942}{40241}X + \frac{11652}{40241}Y, \end{aligned}$$

with the change of coordinates $(X, Y) = (x^3/y, x^2/(xy - 5y^2 + 2x + 12y - 4))$. □

The phase portrait of (23) in the Poincaré disk is depicted in Figure 9. The set $\operatorname{Fix}\varphi$ is drawn in red. It can be checked that the factor F, over the invariant straight line $y - 1 = 0$, is 1. We have depicted a zoom near the equilibrium point $(3, -1/10)$ of center type where Theorem 1.2 applies.

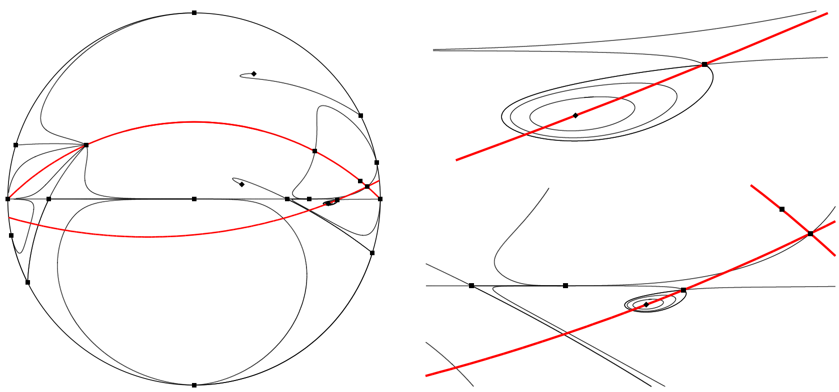

FIGURE 9. Phase portrait of (23) with two zooms near the center point

We finish this section with an orbital φ-reversible Hamiltonian vector field which the corresponding involution φ is also implicit. In [1] is proved that there exists a normal form change of variables, ψ, such that the Hamiltonian $H(x, y) = -(2x^6 + 12ax^3y^2 + 3y^4 + 12ax^5y + 4bx^2y^3)/12$ is transformed in the time-reversible Hamiltonian $H(\psi(x, y)) = -(2x^6 + 12ax^3y^2 + 3y^4 + 12cx^4y^2)/12$. This property is used in [2] to prove its orbital reversibility. The associated vector field satisfies (3) with the transformed involution, using Lemma 2.8, given by the change ψ and the classical involution $\varphi_1(x, y) = (x, -y)$.

As usual, in the normal form theory, the change ψ is not explicitly provided but to show explicitly the application of our approach, we get a simpler example. The Hamiltonian $H(x,y) = y^9 + 3xy^6 + y^6 + 3x^2y^3 + 2xy^3 + x^3 + x^2 + y^2$ is orbitally φ-reversible with respect to the involution $\varphi(x,y) = (x + 2y^3, -y)$. In particular, applying the change $\psi(x,y) = (x - y^3, y)$ we obtain $H(\psi(x,y)) = x^2 + y^2 + x^3$ which is time-reversible with respect to φ_1. As a natural consequence of Lemma 2.8, we notice that as the factor F is constant, every transformation will get orbital φ-symmetric systems with constant factor.

7. Cyclicity of centers

This section is devoted to study the bifurcation of limit cycles of small amplitude in some of the polynomial families presented in this work. The mechanism is the classical degenerate Hopf-bifurcation, we have closely follow the notation and results in [21]. In fact, the main bifurcation theorems were given by Christopher in [10], where he details in Theorems 2.1 and 3.1 how to use first and higher-order developments, respectively, of the Lyapunov constants to provide the complete unfolding of limit cycles bifurcating from the center equilibrium point. In the following proofs, the higher-order developments are necessary. The bifurcation mechanism has three steps. First we restrict the analysis to trace zero perturbation because then the Lyapunov constants are polynomials in the perturbation parameters. Second, after computing the linear developments, we use the Implicit Function Theorem to remove the parameters associated to the maximal rank k. This is Theorem 2.1 of [10]. Finally, we use higher-order developments of order two or three, to check the existence of a transversal straight line in the parameters space that vanishes the next $\ell - 1$ Taylor developments except the last one. This is Theorem 3.1 of [10]. Then a curve of weak foci of order $k + \ell$ emerges from the origin of the parameters space such that, using the trace parameter, the system unfolds the $k + \ell$ limit cycles.

The next results provide lower bounds for the local cyclicity for some cubic polynomial vector fields that do not improve the current one, $M(3) \geq 12$. Although the lower bound for the cyclicity of the quartic system (22), see Proposition 7.4, is not better than the current one, $M(4) \geq 21$, we have added here because it is very close to that value and there are no many quartic systems with high local cyclicity. Systems exhibiting the best lower bound cyclicity can be found in [18]. We have studied only the cyclicity problem of this system because the number of small amplitude limit cycles obtained from the ones in Propositions 4.1 and 4.2 is not very high using only first-order Taylor developments of the Lyapunov constants. Proposition 7.5 provides a new lower bound for the local Hilbert number for degree six vector fields, $M(6) \geq 48$. The previous best lower bound was 44, see [20]. For other low degree vector fields the best lower bounds for $M(n)$ are given in [21]. Up to our knowledge, these are the first studies of this problem for the presented orbitally φ-symmetric systems.

Proposition 7.1. *The system*

$$\begin{aligned} x' &= 2y(x+2)(x-1), \\ y' &= x - \frac{3}{2}xy + \frac{7}{2}y^2 - 2x^2 + x^3 + \frac{3}{2}x^2y + 3xy^2 + \frac{1}{4}y^3, \end{aligned} \tag{24}$$

has an equilibrium point of center type at the origin and, perturbing with cubic polynomials, the local cyclicity is at least 9.

Proof. The vector field (24) is orbitally φ-symmetric, according (3), with

$$\varphi(x,y) = \left(x, \frac{(x-1)y}{1-x-y}\right) \quad \text{and} \quad F(x,y) = \frac{1-x-y}{x-1}.$$

We notice that $\text{Fix}\,\varphi = \{y = 0, x \neq 1\} \cup \{2x + y - 2 = 0, x \neq 1\}$. The center property follows from the above properties and by using Theorem 1.2. In fact, the linear part at the origin is of center-focus type.

The local lower bound for the cyclicity value follows computing the Taylor series of second-order of the Lyapunov constants. As the first 7 linear developments have rank 7, using the Implicit Function Theorem, there exists a local change of coordinates in the parameters space such that they write as $L_i = u_i$ for $i = 1, \ldots, 7$. Then, the next two become $L_8 = u_8 u_9 + O_3(u_8, u_9)$ and $L_9 = u_9^2 + O_3(u_8, u_9)$ after vanishing the first ones. The proof finishes applying Theorem 3.1 of [10]. The curve L_8 has two branches near the origin, one tangent to $u_8 = 0$ and another to $u_9 = 0$. As on the second branch L_9 the order two terms vanishes (and most probably all), we should work with the first one, because L_9 is nonvanishing when u_9 is not zero but small. That is, with the change of variables of blow-up type $u_8 = \hat{v}_8 u_9$ we have that $L_8 = u_9^2(\hat{v}_8 + u_9 f(\hat{v}_8, u_9))$ where $f(0,0) = 0$. Finally, using again the Implicit Function Theorem we write $L_8 = u_9^2 v_8$ and $L_9 = u_9^2(1 + O_1(v_8, u_9))$. □

The cyclicity results showed in the next two propositions can be obtained for different values of the parameters. We have only detailed one for each family.

Proposition 7.2. *System* (18) *with* $(a, b, c) = (-3, 1, 2)$ *has a center at* $(8, -3)$ *and, perturbing with cubic polynomials, the local cyclicity is at least* 8.

Proof. The proof follows a similar scheme as the proof of Proposition 7.1 and applying again Theorem 3.1 of [10]. Straightforward computations of the Lyapunov constants with an adequate changes of variables allow us to write $L_i = u_i$ for $i = 1, \ldots, 6$, $L_7 = u_7 u_8 + O_3(u_7, u_8, u_9)$, and $L_8 = K u_8 u_9^2 + O_4(u_7, u_8, u_9)$ for some $K \neq 0$. Clearly, there is a curve in the parameters space tangent to the branch $u_7 = 0$, where L_7 vanishes but L_8 not. □

Proposition 7.3. *System* (19) *with* $(a, b) = (5, 2)$ *has a center at* $(8, 5)$ *and, perturbing with cubic polynomials, the local cyclicity is at least* 8.

Proof. Straightforward computations of the Lyapunov constants provide, doing convenient change of parameters, $L_i = u_i + O_2(u)$ for $i = 1, \ldots, 6$, $L_7 = u_7 u_8 + O_3(u)$ and $L_8 = K u_7^2 + O_3(u)$ for some $K \neq 0$. The proof finishes similarly as the proof of Proposition 7.1, because the singularity of the curve $L_7 = 0$ near the origin in the parameters space is of the same type and it also has a branch of a curve over that L_8 is not zero. □

Proposition 7.4. *System* (22) *has a center at* $(1, 1/2)$ *and the local cyclicity in the class of degree four polynomial vector fields is at least* 20.

Proof. Here we need to compute the Lyapunov constants up to second-order and, as in the previous results, as the rank of the Taylor series of first-order of the first 18 Lyapunov constants is 18 (by using the Implicit Function Theorem) there exists a local changes of coordinates in the parameters space such that $L_i = u_i$ for $i = 1, \ldots, 18$. Moreover, adding the next ones the rank does not increase. Then assuming that $u_i = 0$ for $i = 1, \ldots, 18$, we can write $L_{19} = u_{19} u_{20} + O_3(u)$, and $L_{20} = K u_{20}^2 + O_3(u)$ for some non-zero value K where u has only the remaining parameters. Then, the proof follows as the previous ones. □

Proposition 7.5. *System* (23) *perturbing in the class of polynomials of degree* 6 *has local cyclicity at least* 48.

Proof. The proof follows similarly as the previous ones but with a more detailed study because of the degeneracy of the intersection of the varieties defined by the vanishing of all Lyapunov constants near the origin in the parameters space. The Taylor series expansions up to first-order of the first Lyapunov constants are written, up to a linear change of coordinates, as $L_i = u_i + O_2(u_1, \ldots, u_{50})$ for $i = 1, \ldots, 43$, and $L_i = O_2(u_1, \ldots, u_{50})$ for $i = 44, \ldots, 50$. Then, using Theorem 2.1 of [10] and the trace parameter, we get that only 43 limit cycles of small amplitude bifurcate from the center itself, but using Taylor first-order series.

For proving the statement we need to compute the Taylor series of higher-order. But developing up to second-order no more limit cycles bifurcate from the center because $L_i = u_i + O_3(u_1, \ldots, u_{50})$ for $i = 1, \ldots, 43$, and $L_i = O_3(u_1, \ldots, u_{50})$ for $i = 44, \ldots, 50$. Then, third-order terms are necessary to be analyzed.

Finally, from the Taylor series of third-order and using again the Implicit Function Theorem to remove the first 43 variables we write $L_i = M_i(u_{44}, \ldots, u_{50}) + O_4(u_{44}, \ldots, u_{50})$, for $i = 44, \ldots, 50$, where M_i are given homogeneous polynomials of degree 3. It can be seen that there exists a straight line $u_{44} = -46551/2795\lambda, u_{45} = \lambda, u_{46} = 13782/2795\lambda, u_{47} = 0, u_{48} = 0, u_{49} = 0, u_{50} = 0$ such that, over this line, $M_i = 0$ for $i = 44, \ldots, 50$. Then, considering the perturbation $u_{44} = (-46551/2795 + \varepsilon_1)\lambda, u_{45} = \lambda, u_{46} = (13782/2795 + \varepsilon_2)\lambda, u_{47} = \varepsilon_3\lambda, u_{48} = \varepsilon_4\lambda, u_{49} = \varepsilon_5\lambda, u_{49} = \varepsilon_6\lambda$, $u_{50} = \varepsilon_7\lambda$, the Jacobian matrix of M_i, for $i = 44, \ldots, 48$, with respect to ε_j, for $j = 1, \ldots, 7$, has rank 5. Then, using Theorem 3.1 of [10], the result follows. □

We notice that in the last proof we have computed two Lyapunov constants more. But as the last rank is only 5 we can not improve more the local cyclicity. In fact, up to Taylor series of third-order we have checked that both L_{49} and L_{50} vanish. As we have commented at the beginning of the paper, we have not gone further in the computations of higher-order because of the difficulties and the fact that we have used almost all the perturbation parameters, which are 50 in this last case. Moreover, we have obtained the number of small limit cycles that we think will be the maximum for degree 6 polynomial vector fields.

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