LINEAR TYPE GLOBAL CENTERS OF CUBIC HAMILTONIAN SYSTEMS SYMMETRIC WITH RESPECT TO THE x-AXIS

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Abstract. A polynomial differential system of degree 2 has no global centers (that is, centers defined in all the plane except the fixed point). In this paper we characterize the global centers of cubic Hamiltonian systems symmetric with respect to the x-axis, and such that the center has purely imaginary eigenvalues.

1. Introduction and statement of the results

The notion of center goes back to Poincaré and Dulac, see [10, 4]. They defined a *center* for a vector field on the real plane as a singular point having a neighborhood filled of periodic orbits with the exception of the singular point. The problem of distinguishing when a monodromic singular point is a focus or a center, known as the focus-center problem started precisely with Poincaré and Dulac and is still active nowadays with many questions still unsolved.

If an analytic system has a center, then it is known that after an affine change of variables and a rescaling of the time variable, it can be written in one of the following three forms:

$$\dot{x} = -y + P(x, y), \quad \dot{y} = x + Q(x, y),$$

called *linear type center*, which has a pair of purely imaginary eigenvalues,

$$\dot{x} = y + P(x, y), \quad \dot{y} = Q(x, y)$$

called *nilpotent center*

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y)$$

called *degenerated center*, where $P(x, y)$ and $Q(x, y)$ are real analytic functions without constant and linear terms defined in a neighborhood of the origin.

We recall that a *global center* for a vector field on the plane is a singular point p having $\mathbb{R}^2$ filled of periodic orbits with the exception of the singular

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point. The easiest global center is the linear center $\dot{x} = -y$, $\dot{y} = x$. It is known (see [11, 1]) that quadratic polynomial differential systems have no global centers. The global degenerated (or homogeneous) centers were characterized in [3] while the global quasihomogeneous centers were studied in [5]. However the characterization of the global centers in the cases that the center is either nilpotent or a linear-type center has been done for very particular cases. In the case in which the system is of the form linear plus cubic homogeneous terms with a linear-type center at the origin was done in [8] and with a nilpotent center at the origin was done in [9]. The case in which the system is a real cubic one (in the sense that it contains also quadratic terms) has never been done because the difficulties grow brutally with the appearance of new coefficients. That is the reason for which in this paper we will focus in the case in which the system is Hamiltonian and it is symmetric with respect to the x-axis. This is the first paper in which such classification is done for these systems. We will focus in the case in which there are quadratic terms, since the case of linear+cubic was done before in [6] and we state it here as Theorem 1.

Theorem 1. *Any Hamiltonian vector field having at the origin of coordinates a singular point with purely imaginary eigenvalues of the form linear plus cubic homogeneous terms has a global center at the origin if and only if, after a linear change of variables and a rescaling of its independent variables, can be written as one of the following systems:*

(i) $\dot{x} = -dx - \frac{d^2+\omega^2}{c}y - 3\alpha\mu x^2 y - \alpha y^3$, $\dot{y} = cx + dy + \alpha x^3 + 3\alpha\mu x y^2$;
(ii) $\dot{x} = -dx - \frac{d^2+\omega^2}{c}y$, $\dot{y} = cx + dy + \alpha x^3$;
(iii) $\dot{x} = -dx - \frac{d^2+\omega^2}{c}y - 3\alpha x^2 y$, $\dot{y} = cx + dy + 3\alpha x y^2$;
(iv) $\dot{x} = -dx - \frac{d^2+\omega^2}{c}y - 3\alpha x^2 y - \alpha y^3$, $\dot{y} = cx + dy + 3\alpha x^2$,

with $\alpha = \pm 1$, $c, d \in \mathbb{R}$, $c \neq 0$, $\omega > 0$, $\mu > -1/3$ *and* $c\alpha > 0$.

We also focus in the case in which there is a linear-type center at the origin since the case in which there is a nilpotent center at the origin was done in [7] (see Theorems 1, 2 and the global phase portraits in Figure 1 of that paper) and we state it here as Theorem 2.

Theorem 2. *Any Hamiltonian planar polynomial vector field of degree three with a global nilpotent center at the origin, symmetric with respect to the x-axis and with all infinite singular points being non-degenerated hyperbolic sectors, after a linear change of variables and a rescaling of its independent variables, can be written as one of the following systems*

(i) $x' = y$, $y' = -x^3$,
(ii) $x' = y + y^3$, $y' = -x^3$,
(iii) $x' = y + x^2 y + a y^3$, $y' = -x^3 - x y^2$ *with* $a \geq 0$,
(iv) $x' = y - x^2 y + a y^3$, $y' = -x^3 + x y^2$ *with* $a \geq 1$,

(v) $x' = y + 2xy + ax^2y + by^3,\ y' = -x^3 - y^2 - axy^2$ *with* $b > 0$ *and either* $a \geq 1$, *or* $a < 1$ *with* $4(a-1)^2(a^3 - a^2 - ab - 8b) - 27b^2 > 0$.

We will now introduce the main results of the paper. For this, we recall that a polynomial differential system can be extended in a unique analytic way to infinity using the Poincaré compactification, for more details see Chapter 5 of [2]. We also introduce some notation.

$$
\begin{aligned}
&a_1 = \frac{-\tilde{a}_1}{A_0}, \quad a_0 = \frac{\tilde{a}_0}{A_0}, \quad b_2 = -\frac{\tilde{b}_2}{B_0}, \quad b_1 = -\frac{\tilde{b}_1}{B_0}, \quad b_0 = \frac{\tilde{b}_0}{B_0},\\
&A_0 = 9c^2(a_{12} - a_{30})^2, \quad B_0 = c^3(3\mu - 1)^2(3\mu + 1)^2\alpha^4,\\
&\tilde{a}_1 = 8a_{12}^4c^2 - 12a_{12}^3a_{30}c^2 - 2a_{12}^2c^3\alpha - 10a_{12}^2c\alpha\omega^2 + 6a_{12}a_{30}c^3\alpha\\
&\qquad + 24a_{12}a_{30}c\alpha\omega^2 - 18a_{30}^2c\alpha\omega^2 - (c^2 - \omega^2)^2,\\
&a_0 = \omega^2\big(4a_{12}^2c^3 - 6a_{12}a_{30}c^3 - 6a_{12}a_{30}c\omega^2 + 9a_{30}^2c\omega^2 + \alpha(c^2 - \omega^2)^2\big),\\
&\tilde{b}_2 = 3\big(9a_{12}^2c\mu^3 - 10a_{12}^2c\mu + 36a_{12}a_{30}c\mu^2 + 2a_{12}a_{30}c - 9a_{30}^2c\mu\\
&\qquad - 18c^2\mu^3\alpha + 2c^2\mu\alpha - 27\mu^4\alpha\omega^2 + 12\mu^2\alpha\omega^2 - \alpha\omega^2\big),\\
&b_1 = 8a_{12}^4c^2 - 36a_{12}^3a_{30}c^2\mu + 18a_{12}^2c^3\mu^2\alpha - 4a_{12}^2c^3\alpha - 30a_{12}^2c\mu\alpha\omega^2\\
&\qquad + 18a_{12}a_{30}c^3\mu\alpha + 108a_{12}a_{30}c\mu^2\alpha\omega^2 + 12a_{12}a_{30}c\alpha\omega^2\\
&\qquad - 54a_{30}^2c\mu\alpha\omega^2 - 9c^4\mu^2 - 54c^2\mu^3\omega^2 + 12c^2\mu\omega^2 + 18\mu^2\alpha^2\omega^4\\
&\qquad - 3\alpha^2\omega^4,\\
&b_0 = \omega^2\big(4a_{12}^2c^3 - 18a_{12}a_{30}c^3\mu - 6a_{12}a_{30}c\omega^2 + 27a_{30}^2c\mu\omega^2 + 9c^4\mu^2\alpha\\
&\qquad - 6c^2\mu\alpha\omega^2 + \alpha\omega^4\big)
\end{aligned} \tag{1}
$$

and $\Delta = c^7\Delta_1^2\Delta_2$, where

$$
\begin{aligned}
\Delta_1 = {} & 36a_{12}^3c\mu^2 + 8a_{12}^3c - 162a_{12}^2a_{30}c\mu^3 - 72a_{12}^2a_{30}c\mu + 162a_{12}a_{30}^2c\mu^2\\
& + 81a_{12}c^2\mu^4\alpha + 18a_{12}c^2\mu^2\alpha - 81a_{12}\mu^3\alpha\omega^2 - 81a_{30}c^2\mu^3\alpha\\
& + 243a_{30}\mu^4\alpha\omega^2,
\end{aligned} \tag{2}
$$

and

$$
\begin{aligned}
\Delta_2 = {} & 36a_{12}^6c^3\mu^2 + 32a_{12}^6c^3 - 216a_{12}^5a_{30}c^3\mu + 36a_{12}^4a_{30}^2c^3 + 108a_{12}^4c^4\mu^2\alpha\\
& - 48a_{12}^4c^4\alpha - 324a_{12}^4c^2\mu^3\alpha\omega^2 - 180a_{12}^4c^2\mu\alpha\omega^2 + 216a_{12}^3a_{30}c^4\mu\alpha\\
& + 1296a_{12}^3a_{30}c^2\mu^2\alpha\omega^2 + 108a_{12}^3a_{30}c^2\alpha\omega^2 - 36a_{12}^2a_{30}^2c^4\alpha + 24a_{12}^2c^5\\
& - 864a_{12}^2a_{30}^2c^2\mu\alpha\omega^2 - 135a_{12}^2c^5\mu^2 - 648a_{12}^2c^3\mu^3\omega^2 + 18a_{12}^2c^3\mu\omega^2\\
& + 972a_{12}^2c\mu^4\omega^4 + 216a_{12}^2c\mu^2\omega^4 - 27a_{12}^2c\omega^4 + 108a_{12}a_{30}^3c^2\alpha\omega^2\\
& - 54a_{12}a_{30}c^5\mu + 810a_{12}a_{30}c^3\mu^2\omega^2 - 54a_{12}a_{30}c^3\omega^2\\
& - 1944a_{12}a_{30}c\mu^3\omega^4 + 162a_{12}a_{30}c\mu\omega^4 + 9a_{30}^2c^5 - 54a_{30}^2c^3\mu\omega^2\\
& + 81a_{30}^2c\mu^2\omega^4 + 36c^6\mu^2\alpha - 4c^6\alpha - 324c^4\mu^3\alpha\omega^2 + 36c^4\mu\alpha\omega^2
\end{aligned} \tag{3}
$$

$$+ 972c^2\mu^4\alpha\omega^4 - 108c^2\mu^2\alpha\omega^4 - 972\mu^5\alpha\omega^6 + 108\mu^3\alpha\omega^6.$$

Theorem 3. *Any Hamiltonian planar polynomial vector field of degree three with a global linear type center at the origin, symmetric with respect to the x-axis and with no infinite singular points in the Poincaré disc, after a linear change of variables and a rescaling of its independent variables, can be written as:*

$$\text{(I)} \qquad \begin{aligned} \dot{x} &= -\frac{\omega^2}{c} y - 2a_{12}xy - 3\mu\alpha x^2 y - \alpha y^3 \\ \dot{y} &= cx + 3a_{30}x^2 + a_{12}y^2 + \alpha x^3 + 3\mu\alpha x y^2, \end{aligned}$$

where $\alpha = \pm 1$ *and* $c, \omega, \mu \in \mathbb{R}$ *with* $c \neq 0$, $\omega > 0$, $\mu > -1/3$, $a_{12}^2 + a_{30}^2 \neq 0$, *and:*

- (a) *either* $\mu = 1/3$, $a_{12} = a_{30}$, $a_{30}^2(3c\omega^2 - 2c^3) + \alpha(c^2 - \omega^2)^2 < 0$ *and* $0 < 9a_{30}^2 < 4c\alpha$*; or*
- (b) $\mu = 1/3$, $a_{12} \neq a_{30}$, $a_1^2 - 4a_0 < 0$ *and* $9a_{30}^2 - 4c\alpha < 0$*, or*
- (c) $\mu = 1/3$, $a_{12} \neq a_{30}$, $a_1^2 - 4a_0 \geq 0$ *with* $a_1 > 0$, $a_0 > 0$ *and* $9a_{30}^2 - 4c\alpha < 0$*, or*
- (d) $\mu \neq 1/3$, $\Delta < 0$, $b_0 > 0$, *and* $9a_{30}^2 - 4c\alpha < 0$*, or*
- (e) $\mu \neq 1/3$, $\Delta \geq 0$, $b_2 > 0$, $b_0 > 0$, $b_2 b_1 > b_0$ *and* $9a_{30}^2 - 4c\alpha < 0$.

The proof of Theorem 3 is given in section 2.

We note that if $a_{12} = a_{30} = 0$ then our system becomes of the form linear plus cubic homogeneous terms and these systems are a particular case of the ones studied in [6] (see Theorem 1 above).

We recall that the conditions provided in Theorem 3 are not empty. For example, for condition (a) it is sufficient to take $\mu = 1/3$, $\alpha = c = 1$, $\omega = 2$ and $a_{12} = a_{30} < 2/3$. For condition (b) it is sufficient to take $\mu = 1/3$, $\alpha = 1$, $c = 7/13120$, $\omega = 1/16$, $a_{12} = 1$, $a_{30} = 0$. For condition (c) it is sufficient to take $\mu = 1/3$, $\alpha = c = 1$, $\omega = 1/2$, $a_{12} = 5/32$ and $a_{30} = 0$. For condition (d) it is sufficient to take $\alpha = c = \mu = 1$, $\omega = 1/16$, $a_{12} = -13/16$ and $a_{30} = 5/8$. Finally, for condition (e) it suffices to take $\alpha = c = \mu = 1$, $\omega = 7/184$, $a_{12} = -21/32$ and $a_{30} = -1/2$.

A singular point p of a planar system is called *hyperbolic* if both eigenvalues of the Jacobian matrix at p have real part different from zero. It is called *semi-hyperbolic* if only one of the eigenvalues of the Jacobian matrix at p is zero, and if both eigenvalues of the Jacobian matrix at p are zero but this matrix is not identically zero it is called *nilpotent*. Finally, if the Jacobian matrix at p is identically zero then p is said to be *linearly zero*.

Let q be an infinite singular point and let h be a hyperbolic sector of q. We say that h is *degenerated* if its two separatrices are contained at infinity, that is, are contained in the equator of the Poincaré sphere.

It follows from Theorem 2.15 (for hyperbolic singular points), Theorem 2.19 (for semi-hyperbolic singular points) and Theorem 3.5 (for nilpotent singular points) in [2] that a singular point which is either hyperbolic, semi-hyperbolic or nilpotent cannot be formed by two degenerated hyperbolic sectors. So, in order that an infinite singular point q can be formed by two degenerated hyperbolic sectors it must be linearly zero.

To state the last main theorem in the paper we introduce some notation.

$$\begin{aligned}
\Delta_3 &= 9a_{12}^2c^3(4a_{12}^6c^3 - 15a_{12}^2c^5 + 12a_{12}^4c^4\alpha + 4c^6\alpha - 72a_{12}^2c^3\omega^2 \\
&\quad - 36a_{12}^4c^2\alpha\omega^2 - 36c^4\alpha\omega^2 + 108a_{12}^2c\omega^4 + 108c^2\alpha\omega^4 - 108\alpha\omega^6), \\
\tilde{b}_2 &= -\frac{a_{12}^2c - 2c^2\alpha - 3\alpha\omega^2}{3c}, \quad \tilde{b}_1 = -\frac{2a_{12}^2c - c^2\alpha - 6\alpha\omega^2}{9\alpha}, \\
\tilde{b}_0 &= \frac{c\omega^2}{9\alpha}.
\end{aligned} \tag{4}$$

Theorem 4. *Any Hamiltonian planar polynomial vector field of degree three with a global linear type center at the origin, symmetric with respect to the x-axis and with all infinite singular points formed by two degenerated hyperbolic sectors, after a linear change of variables and a rescaling of its independent variables, can be written as one of the following systems*

(ii) $x' = -\omega^2 y/c$, $y' = cx + 3a_{30}x^2 + \alpha x^3$ *with* $\omega > 0$, $c\alpha > 0$, $\alpha = \pm 1$ *and* $9a_{30}^2 - 4c\alpha < 0$;
(iii) $x' = -\omega^2 y/c - 2a_{12}xy - 3\alpha x^2 y$, $y' = cx + a_{12}y^2 + 3\alpha xy^2$ *with* $\omega > 0$, $c\alpha > 0$, $\alpha = \pm 1$ *and* $c(ca_{12} - 3\alpha\omega^2) < 0$;
(iv) $\dot{x} = -\frac{\omega^2}{c}y - 2a_{12}xy - 3\alpha x^2 y - \alpha y^3$, $\dot{y} = cx + a_{12}y^2 + 3\alpha xy^2$, *with* $\omega > 0$, $\alpha = \pm 1$, $c\alpha > 0$ *and either:*
  (iv.1) $\Delta_3 < 0$ *and* $\tilde{b}_0 > 0$ *(see (4))*;
  (iv.2) $\Delta_3 \geq 0$, $\tilde{b}_2 > 0$, $\tilde{b}_0 > 0$ *and* $\tilde{b}_2\tilde{b}_1 > \tilde{b}_0$ *(see (4))*.

Theorem 4 is proved in Section 3. Note that the conditions in statement (iv.1) are fulfilled for example when $\alpha = 1$, $c = 1/17$, $\omega = 1/8$ and $a_{12} = -1$, and the conditions in statement (iv.2) are fulfilled for example when $\alpha = 1$, $c = 2$, $\omega = 1$ and $a_{12} = 1/16$.

2. Proof of Theorem 3

In order to prove Theorem 3 we will state and prove several propositions.

Proposition 5. *Any Hamiltonian planar polynomial vector field of degree three with a linear type center at the origin, symmetric with respect to the x-axis and with no infinite singular points, after a linear change of variables and a rescaling of its independent variables it can be written as the following* *system*

$$\text{(i)} \qquad \begin{aligned} \dot{x} &= -\frac{\omega^2}{c}y - 2a_{12}xy - 3\alpha\mu x^2y - \alpha y^3, \\ \dot{y} &= cx + 3a_{30}x^2 + a_{12}y^2 + 3\alpha\mu xy^2, \end{aligned}$$

where $\alpha = \pm 1$ *and* $c, \omega, a_{12}, a_{30}, \mu \in \mathbb{R}$ *with* $c \neq 0$, $\mu > -1/3$ *and* $\omega > 0$.

Proof. Doing a linear change of variables and a rescaling of the independent variable, planar cubic homogeneous differential systems which has no infinite singular points can be classified in the following class, see [3, Theorem 3.2]:

$$\text{(i')} \qquad \begin{aligned} \dot{x} &= p_1x^3 + (p_2 - 3\alpha\mu)x^2y + p_3xy^2 - \alpha y^3, \\ \dot{y} &= \alpha x^3 + p_1x^2y + (p_2 + 3\alpha\mu)xy^2 + p_3y^3, \end{aligned}$$

with $\mu > -1/3$ and $\alpha = \pm 1$.

It was proved in [6] that in the case in which the vector field is Hamiltonian then $p_i = 0$ for $i = 1, 2, 3$, that is we get the system (i') with $p_i = 0$ for $i = 1, 2, 3$.

For studying the Hamiltonian cubic planar polynomial vector fields having linear, quadratic and cubic terms, it is sufficient to add to the above family (with $p_i = 0$ for $i = 1, 2, 3$) a linear and quadratic parts being Hamiltonian. This is due to the fact that the linear changes of variables that are done to obtain the class (i') are not affine, they are strictly linear.

For the linear part, we add the linear terms $ax + by$ in $\dot{x}$ and the linear terms $cx + dy$ in $\dot{y}$. Doing so, taking into account that we must have a linear type center at the origin, it is not easy to see that we can add $-dx - (d^2 + \omega^2)/cy$ in $\dot{x}$ and $cx + dy$ in $\dot{y}$ with $c \neq 0$ and $\omega > 0$.

For the quadratic part, we add the linear terms $-\partial H_3/\partial y$ in $\dot{x}$ and $\partial H_3/\partial x$ in $\dot{y}$ with

$$H_3 = a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3.$$

Doing so, we get the systems

$$\begin{aligned} \dot{x} &= -dx - \frac{d^2 + \omega^2}{c}y - a_{21}x^2 - 2a_{12}xy - 3a_{03}y^3 - 3\alpha\mu x^2y - \alpha y^3, \\ \dot{y} &= cx + dy + 3a_{30}x^2 + 2a_{21}xy + a_{12}y^2 + 3\alpha\mu xy^2. \end{aligned}$$

Since this system must be invariant under the transformation $(x, y, t) \mapsto (x, -y, -t)$ we must have $d = a_{21} = a_{03} = 0$ and then we get the system (i) in the statement of the proposition. This concludes the proof. $\square$

Now we continue with the proof of the theorem. To do so, we compute the finite singular points of system (i). Note that on $y = 0$ we have the solution

$$x = \frac{-3a_{30} \pm \sqrt{9a_{30}^2 - 4c\alpha}}{2\alpha}.$$

Since this solution cannot exist we must have $9a_{30}^2 - 4c\alpha < 0$.

Now we consider $y \neq 0$ and we compute the Groëbner basis for the polynomials x' and y'. We get a set of six polynomials. The first polynomial is a polynomial of degree six in the variable y while the polynomials p_3 and p_4 are linear in the variable x. More precisely,

$$\begin{aligned}
p_1 = &-c^3(3\mu-1)^2(3\mu+1)^2y^6 + 3c^2(2a_{12}a_{30}c - 10c\mu a_{12}^2 - 9c\mu a_{30}^2 + 2c^2\alpha\mu \\
&+ 36a_{12}a_{30}c\mu^2 + 9a_{12}^2c\mu^3 - 18c^2\alpha\mu^3 - \alpha\omega^2 + 12\alpha\mu^2\omega^2 - 27\alpha\mu^4\omega^2)y^4 \\
&+ c(8a_{12}^4c^2 - 4a_{12}^2c^3\alpha - 36a_{12}^3a_{30}c^2\mu + 18a_{12}a_{30}c^3\alpha\mu + 18a_{12}^2c^3\alpha\mu^2 \\
&- 9c^4\mu^2 + 12a_{12}a_{30}c\alpha\omega^2 - 30a_{12}^2c\alpha\mu\omega^2 - 54a_{30}^2c\alpha\mu\omega^2 + 12c^2\mu\omega^2 \\
&+ 108a_{12}a_{30}c\alpha\mu^2\omega^2 - 54c^2\mu^3\omega^2 - 3\omega^4 + 18\mu^2\omega^4)y^2 + \omega^2(4a_{12}^2c^3 \\
&- 18a_{12}a_{30}c^3\mu + 9\alpha c^4\mu^2 - 6a_{12}a_{30}c\omega^2 + 27a_{30}^2c\mu\omega^2 - 6c^2\alpha\mu\omega^2 + \alpha\omega^4)
\end{aligned}$$

and

$$\begin{aligned}
p_3 = &\, c\alpha y^2(2a_{12} - 9a_{30}\mu + 9a_{12}\mu^2) + (2a_{12} - 9a_{30}\mu)\omega^2 \\
&+ x\big(4a_{12}^2c - 18a_{12}a_{30}c\mu + 9c^2\alpha\mu^2 - 3\alpha\mu\omega^2 + 3c\mu(3\mu-1)(3\mu+1)y^2\big), \\
p_4 = &-c^2y^4(3\mu-1)(3\mu+1) - \omega^2(3c^2\mu - \omega^2) + cy^2(6a_{12}^2c\mu - 3c^2\alpha\mu + 2\alpha\omega^2 \\
&- 9\alpha\mu^2\omega^2) + x\big(c^2\alpha(2a_{12} - 9a_{30}\mu + 9a_{12}\mu^2)y^2 + c(2a_{12} - 9a_{30}\mu)\omega^2\big).
\end{aligned}$$

Note that for any value of y which is a solution of $p_1 = 0$ we get a unique value of x which is a solution of $p_3 = 0$ or $p_4 = 0$, namely

$$x = \frac{(9a_{30}c\alpha\mu - 9a_{12}c\alpha\mu^2 - 2a_{12}c\alpha)y^2 - 2a_{12}\omega^2 + 9a_{20}\mu\omega^2}{4a_{12}^2c - 18a_{12}a_{30}c\mu - 3\alpha\mu\omega^2 + 9c^2\alpha\mu^2 + (3c\mu(3\mu+1)(3\mu-1))y^2},$$

or

$$\bar{x} = \frac{c^2(9\mu^2-1)y^4 - c(6a_{12}^2c\mu - 3\alpha\mu c^2 + 2\alpha\omega^2 - 9\alpha\mu^2\omega^2)y^2 + \omega^2(3c^2\mu - \omega^2)}{c(c\alpha(2a_{12} - 9a_{30}\mu + 9a_{12}\mu^2)y^2 + 2a_{12}\omega^2 - 9a_{30}\mu\omega^2)}.$$

Moreover, both denominators of x and $\bar{x}$ cannot be zero simultaneously and then for each value of y we always have a value of x. Moreover, setting x (or $\bar{x}$) into the Groëbner basis we get that all the polynomials have p_1 as a factor in them. In short, in order to show that there are no solutions of system (i) with $x \neq 0$ we must guarantee that there are no real solutions of $p_1 = 0$. Since p_1 depends in the variable y^2 we introduce the new variable $z = \sqrt{y}$ and we must have that $z > 0$ (note that $z = 0$ yields $y = 0$).

We consider different cases.

Case 1*: $\mu = 1/3$ and $a_{12} = a_{30}$.* In this case

$$\begin{aligned}
p_1 = &\, cz(-4a_{30}^4c^2 + 4a_{30}^2c^3\alpha - 4a_{30}^2c\alpha\omega^2 - c^4 + 2c^2\omega^2 - \omega^4) \\
&- \omega^2(-2a_{30}^2c^3 + 3a_{30}^2c\omega^2 + c^4\alpha - 2c^2\alpha\omega^2 + \alpha\omega^4).
\end{aligned}$$

Solving in the variable z we get

$$z = -\frac{\omega^2(a_{30}^2(3c\omega^2 - 2c^3) + \alpha(c^2 - \omega^2)^2)}{c(2a_{30}^2 c + \alpha(\omega^2 - c^2))^2} \tag{5}$$

whenever the denominator is not zero. Note that the denominator is equal to zero (solving in ω which is positive) whenever

$$\omega = \sqrt{\frac{c^2\alpha - 2a_{30}^2 c}{\alpha}}.$$

But then introducing ω into p_1 we get

$$p_1 = -a_{30}^2 c^3 (2a_{30}^2 - a\alpha)^2.$$

Setting it to zero, taking into account that $a\alpha > 0$ and $9a_{30}^2 - 4a\alpha < 0$ we conclude that $a_{30} = 0$. But then $a_{12} = 0$ which is not possible. Hence, z in equation (5) is well-defined. Taking into account that it must be negative we must have

$$a_{30}^2(3c\omega^2 - 2c^3) + \alpha(c^2 - \omega^2)^2 < 0$$

In short, the condition so that there are no finite singular points besides the origin in this case is

$$a_{30}^2(3c\omega^2 - 2c^3) + \alpha(c^2 - \omega^2)^2 < 0, \quad 0 < 9a_{30}^2 < 4c\alpha.$$

This concludes the proof of statement (a).

Case 2: $\mu = 1/3$ and $a_{12} \neq a_{30}$. In this case

$$\begin{aligned} p_1 = {} & -\omega^2(4a_{12}^2c^3 - 6a_{12}a_{30}c^3 - 6a_{12}a_{30}c\omega^2 + 9a_{30}^2c\omega^2 + c^4\alpha - 2c^2\alpha\omega^2 \\ & + \alpha\omega^4) + cz(8a_{12}^4c^2 - 12a_{12}^3a_{30}c^2 - 2a_{12}^2c^3\alpha - 10a_{12}^2c\alpha\omega^2 \\ & + 6a_{12}a_{30}c^3\alpha + 24a_{12}a_{30}c\alpha\omega^2 - 18a_{30}^2c\alpha\omega^2 - c^4 + 2c^2\omega^2 - \alpha^2\omega^4) \\ & - 9c^3z^2(a_{12} - a_{30})^2. \end{aligned} \tag{6}$$

Note that now p_1 is quadratic in z. We need to investigate when this quadratic equation has either no real roots or both real roots are negative. We state and prove an auxiliary result.

Proposition 6. *Any quadratic polynomial of the form*

$$P(z) = z^2 + a_1 z + a_0 \tag{7}$$

has complex roots if and only if $a_1^2 - 4a_0 < 0$. It has all roots being real and negative if and only if $a_1^2 - 4a_0 \geq 0$, $a_1 > 0$ and $a_0 > 0$.

Proof of Proposition 6. The solutions are real if and only if $a_1^2 - 4a_0 \geq 0$. In this case, it follows directly from the Routh-Hurwitz criterium that these roots are negative if and only if $a_1 > 0$and $a_0 > 0$. This concludes the proof of the proposition. □

We can write p_1 in (6) as in (7) with (note that $a_{12} \neq a_{30}$) a_1 and a_0 as in (1). So, the conditions so that there are no finite singular points are either $a_1^2 - 4a_0 < 0$ and $9a_{30}^2 - 4c\alpha < 0$, or $a_1^2 - 4a_0 \geq 0$ with $a_1 > 0$, $a_0 > 0$ and $9a_{30}^2 - 4c\alpha < 0$. This concludes the proof of statements (b) and (c).

Case 3*:* $\mu \neq 1/3$. or in other words, that all the solutions of the cubic equation

$$
\begin{aligned}
p_1 = {} & -c^3(3\mu-1)^2(3\mu+1)^2 z^3 + 3c^2(2a_{12}a_{30}c - 10c\mu a_{12}^2 - 9c\mu a_{30}^2 \\
& + 2c^2\alpha\mu + 36a_{12}a_{30}c\mu^2 + 9a_{12}^2c\mu^3 - 18c^2\alpha\mu^3 - \alpha\omega^2 + 12\alpha\mu^2\omega^2 \\
& - 27\alpha\mu^4\omega^2)z^2 + c(8a_{12}^4c^2 - 4a_{12}^2c^3\alpha - 36a_{12}^3a_{30}c^2\mu + 18a_{12}^2c^3\alpha\mu^2 \\
& + 18a_{12}a_{30}c^3\alpha\mu - 9c^4\mu^2 + 12a_{12}a_{30}c\alpha\omega^2 - 30a_{12}^2c\alpha\mu\omega^2 + 12c^2\mu\omega^2 \\
& - 54a_{30}^2c\alpha\mu\omega^2 + 108a_{12}a_{30}c\alpha\mu^2\omega^2 - 54c^2\mu^3\omega^2 - 3\omega^4 + 18\mu^2\omega^4)z \\
& + \omega^2(4a_{12}^2c^3 - 18a_{12}a_{30}c^3\mu + 9\alpha c^4\mu^2 - 6a_{12}a_{30}c\omega^2 + 27a_{30}^2c\mu\omega^2 \\
& - 6c^2\alpha\mu\omega^2 + \alpha\omega^4)
\end{aligned} \tag{8}
$$

are negative. For that we state and prove an auxiliary result that characterize when this happens for a general cubic equation.

Proposition 7. *Any cubic polynomial of the form*

$$P(z) = z^3 + b_2z^2 + b_1z + b_0 \tag{9}$$

has only negative real roots if and only if:

(1) *either* $\Delta := b_1^2b_2^2 - 4b_2^3 - 4b_1^3b_0 + 18b_1b_2b_0 - 27b_0^2 < 0$ *and* $b_0 > 0$*, or*
(2) $\Delta \geq 0$*,* $b_2 > 0$*,* $b_0 > 0$ *and* $b_2b_1 > a_0$*.*

Proof. Note that $\Delta < 0$ if and only if the cubic equation $P(z)$ has one real root and two complex conjugate roots. Since $P(z) \to -\infty$ as $z \to -\infty$, the unique real root is negative if and only if $P(0) = b_0 > 0$.

On the other hand, $\Delta \geq 0$ if and only if the cubic equation $P(z)$ has three (counted with multiplicity) real roots. Using the Routh-Hurwitz criterium these roots are negative if and only if $b_2 > 0$, $b_0 > 0$ and $b_2b_1 > b_0$. This concludes the proof of the proposition. □

We can write p_1 in (8) as in (9) with b_2, b_1 and b_0 as in (1). So, the conditions so that there are no finite singular points are either $\Delta < 0$ (see (2)–(3)), $b_0 > 0$ and $9a_{30}^2 - 4c\alpha < 0$, or $\Delta \geq 0$, $b_2 > 0$, $b_0 > 0$, $b_2b_1 > b_0$ and $9a_{30}^2 - 4c\alpha < 0$. This concludes the proof of statements (d) and (e) and finishes the proof of the theorem.

3. Proof of Theorem 4

In order to prove Theorem 4 we will state and prove several propositions.

Proposition 8. *Any Hamiltonian planar polynomial vector field of degree three with a linear type center at the origin, symmetric with respect to the x-axis and with all infinite singular points being linearly zero, after a linear change of variables and a rescaling of its independent variables it can be written as one of the following systems*

(II) $\dot{x} = -\frac{\omega^2}{c}y - 2a_{12}xy,\ \dot{y} = cx + 3a_{30}x^2 + a_{12}y^2 + \alpha x^3;$
(III) $\dot{x} = -\frac{\omega^2}{c}y - 2a_{12}xy - 3\alpha x^2 y,\ \dot{y} = cx + 3a_{30}x^2 + a_{12}y^2 + 3\alpha xy^2;$
(IV) $\dot{x} = -\frac{\omega^2}{c}y - 2a_{12}xy - 3\alpha x^2 y - \alpha y^3,\ \dot{y} = cx + 3a_{30}x^2 + a_{12}y^2 + 3\alpha xy^2$

where $\alpha = \pm 1$ *and* $c, \omega, a_{12}, a_{30} \in \mathbb{R}$ *with* $c \neq 0$, $a_{12}^2 + a_{30}^2 \neq 0$ *and* $\omega > 0$.

Proof. The proof is very similar to the one of Proposition 5. Doing a linear change of variables and a rescaling of the independent variable, planar cubic homogeneous differential systems which have infinite singular points being linearly zero can be classified in the following three classes, see [3]:

(ii') $\dot{x} = p_1x^3 + p_2x^2y, \quad \dot{y} = \alpha x^3 + p_1x^2y + p_2xy^2;$
(iii') $\dot{x} = (p_2 - 3\alpha)x^2y, \quad \dot{y} = (p_2 + 3\alpha)xy^2;$
(iv') $\dot{x} = (p_2 - 3\alpha)x^2y + p_3xy^2 - \alpha y^3, \quad \dot{y} = (p_2 + 3\alpha)xy^2 + p_3y^3,$

where $\alpha = \pm 1$.

System (ii') comes from systems (IX') of Theorem 3.2 of [?] of taking into account that the unique pair of infinite singular points at the origins of the local chart U_2 and V_2 (see the Poincaré compactification in [2]) is linearly zero if and only if $p_3 = 0$).

System (iii') comes from system (VII') of [?] taking into account that the unique pair of infinite singular points are the origins of U_i and V_i for $i = 1, 2$, and all of them are linearly zero if and only if $p_1 = p_3 = 0$.

System (iv') comes from system (IV') of [?]. Here the unique pair of infinite singular points is the origins of U_1 and V_1, which are linearly zero if and only if $p_1 = 0$.

It was proved in [6] that if systems (ii'), (iii') and (iv') are Hamiltonian then $p_i = 0$ for $i = 1, 2, 3$. So, we get systems (ii')–(iv') with $p_i = 0$ for $i = 1, 2, 3$.

For studying the Hamiltonian cubic planar polynomial vector fields having linear, quadratic and cubic terms, it is sufficient to add to the above families (with $p_i = 0$ for $i = 1, 2, 3$) linear and quadratic parts being Hamiltonian. This is due to the fact that the linear changes of variables that are done to obtain the classes (ii'), (iii') and (iv') are strictly linear, see [?].

For the linear part we add the linear terms $ax + by$ in $\dot{x}$ and the linear terms $cx + dy$ in $\dot{y}$. Doing so, taking into account that we must have a linear type center at the origin, it is easy to see that we can add $-dx - \frac{d^2+\omega^2}{c}y$ in $\dot{x}$ and $cx + dy$ in $\dot{y}$ with $c \neq 0$ and $\omega > 0$.

For the quadratic part we add the quadratic terms $-\partial H_3/\partial y$ in $\dot{x}$ and $\partial H_3/\partial x$ in $\dot{y}$ with

$$H_3 = a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3.$$

Doing so, we get the systems

$$\begin{aligned}\dot{x} &= -dx - \frac{d^2+\omega^2}{c}y - a_{21}x^2 - 2a_{12}xy - 3a_{03}y^3 + P_3,\\ \dot{y} &= cx + dy + 3a_{30}x^2 + 2a_{21}xy + a_{12}y^2 + Q_3,\end{aligned}$$

where P_3 and Q_3 are the vector fields in each of the classes (ii')–(iv') with $p_i = 0$ for $i = 1, 2, 3$.

Since the above systems must be invariant under the transformation $(x, y, t) \mapsto (x, -y, -t)$ we must have $d = a_{21} = a_{03} = 0$ and then we get the systems (II)–(IV) in the statement of the proposition. This concludes the proof. □

Proposition 9. *Systems (II) have a global linear-type center at the origin and no more finite singular points with all the infinite singular points formed by two degenerated hyperbolic sectors if and only if they can be written as in (ii) of Theorem 4. Consequently systems (II) have a global center at the origin.*

Proof. We already know that the pair of infinite singular points of system (II) are the origins of the local charts U_2 and V_2. Thus on the local chart U_2 we get

$$\begin{aligned}u' &= -3a_{12}uv - 3a_{30}u^3v - cu^2v^2 - \alpha u^4 - \frac{\omega^2}{c}v^2,\\ v' &= -v(a_{12}v + cuv^2 + 3a_{30}u^2v + \alpha u^3).\end{aligned} \tag{10}$$

The origin of the local chart U_2 is linearly zero. We need to do blow-ups to understand the local behavior at this point. We perform the directional blow-up $(u, v) \mapsto (u, w)$ with $w = v/u$ and we get

$$u' = -u^2\Big(3a_{12}w + 3a_{30}u^2 + cu^2w^2 + \alpha u^2 + \frac{\omega^2}{c}w^2\Big), \quad v' = uw^2\Big(2a_{12} + \frac{\omega^2}{c}w\Big).$$

We eliminate the common factor u between u' and w' and we obtain the system

$$u' = -u\Big(3a_{12}w + 3a_{30}u^2 + cu^2w^2 + \alpha u^2 + \frac{\omega^2}{c}w^2\Big), \quad v' = w^2\Big(2a_{12} + \frac{\omega^2}{c}w\Big).$$

When $u = 0$, this system has the singular points

$$(0, 0) \quad \text{and} \quad \Big(0, -\frac{2ca_{12}}{\omega^2}\Big).$$

Computing the eigenvalues of the Jacobian matrix at the second singular point we get that it is a node. Going back through the changes of variables we see that in this case the origin of U_2 must have parabolic sectors, and so the origin of U_2 cannot be the union of two degenerated hyperbolic sectors. Hence, $a_{12} = 0$. In this case the unique singular point is the origin which is again linearly zero. Hence we need to do another blow up. We do it in the form $(u, w) \mapsto (u, z)$ with $z = w/u$. Doing so, and eliminating the common factor u^2 we get

$$u' = -u\Big(\alpha + 3a_{30}uz + cu^2z^2 + \frac{\omega^2}{c}z^2\Big), \quad z' = z\Big(\alpha + 3a_{30}uz + cu^2z^2 + \frac{2\omega^2}{c}z^2\Big).$$

When $u = 0$ the possible singular points are

$$(0,0) \quad \text{and} \quad \Big(0, \pm\frac{i}{\omega}\sqrt{\frac{c\alpha}{2}}\Big).$$

The sign of $c\alpha$ determines the existence of the last two singular points, and so we analyze both possibilities.

If $c\alpha < 0$, all three singular points are real. The origin which is a saddle and the points $\big(0, \pm ic\alpha/(\sqrt{2}\omega)\big)$ which are nodes. Again going back through the changes of variables up to systems (10) we get that the origin of U_2 contains parabolic sectors. Hence, we must have $c\alpha > 0$.

If $c\alpha > 0$ the only singular point is the origin which is a saddle. Going back to the changes of variables until systems (10) we see that locally the origin of U_2 consists of two degenerated hyperbolic sectors.

In short, in order that the origin of the local chart U_2 is formed by two degenerated hyperbolic sectors. we must have $a_{12} = 0$ and $c\alpha > 0$. Then system (II) becomes

$$\dot{x} = -\frac{\omega^2}{c}y, \quad \dot{y} = cx + 3a_{30}x^2 + \alpha x^3$$

with $c\alpha > 0$. The finite singular points are

$$(0,0), \quad \Big(\frac{-3a_{30} \pm \sqrt{9a_{30}^2 - 4c\alpha}}{2\alpha}, 0\Big).$$

Since we want that there are no finite singular points among the origin we must have $9a_{30}^2 - 4c\alpha < 0$. This completes the proof of the proposition. □

Proposition 10. *Systems (III) have a global linear-type center at the origin and no more finite singular points with all the infinite singular points formed by two degenerated hyperbolic sectors if and only if they can be written as in (iii) of Theorem 4.*

Proof. We already know that the unique pairs of infinite singular points of system (III) are the origins of U_i and V_i for $i = 1, 2$. System (III) on the

local chart U_1 becomes
(11)
$$u' = 6\alpha u^2 + 3a_{30}v + 3a_{12}u^2v + cv^2 + \frac{\omega^2}{c}u^2v^2, \quad v' = uv\Big(3\alpha + 2a_{12}v + \frac{\omega^2}{c}v^2\Big).$$
Computing the eigenvalues of the Jacobian matrix at the origin we get that it is nilpotent if $a_{30} \neq 0$ and linearly zero if $a_{30} = 0$. So, in order that it is formed by the union of two degenerated hyperbolic sectors we must have $a_{30} = 0$. We need to do blow-ups to understand the local behavior at this point. We perform the directional blow-up $(u, v) \mapsto (u, w)$ with $w = v/u$ and we eliminate the common factor u between u' and w' and we get
$$u' = -u\Big(6\alpha + 3a_{12}uw + cw^2 + \frac{\omega^2}{c}u^2w^2\Big), \quad w' = -w\Big(3\alpha + a_{12}uw + cw^2\Big).$$
When $u = 0$, there are three singular points which are
$$(0, 0) \quad \text{and} \quad \Big(0, \pm i\sqrt{\frac{3\alpha}{c}}\Big),$$
if $c\alpha < 0$. Computing the eigenvalues of the Jacobian matrix at the second and third singular points we get that they are nodes. Going back through the changes of variables we see that in this case the origin of U_1 must have parabolic sectors and so the origin of U_1 cannot be the union of two degenerated hyperbolic sectors. Hence, $c\alpha > 0$. In this case the unique singular point is the origin which is a saddle. Going back through the changes of variables until systems (11) we see that locally the origin of U_1 consists of two degenerated hyperbolic sectors.

On the local chart U_2 system (III) is
$$(12)\ \ u' = -6\alpha u^2 - 3a_{12}uv - \frac{\omega^2}{c}v^2 - cu^2v^2, \quad v' = -v(3\alpha u + a_{12}v + cuv^2).$$
The origin of the local chart U_2 is linearly zero. We perform the directional blow-up $(u, v) \mapsto (u, w)$ with $w = v/u$, and eliminating the common factor u we get
$$u' = -u\Big(6\alpha + 3a_{12}w + cu^2w^2 + \frac{\omega^2}{a}w^2\Big), \quad v' = w\Big(3\alpha + 2a_{12}w + \frac{\omega^2}{c}w^2\Big).$$
When $u = 0$ the possible singular points are
$$(0, 0) \quad \text{and} \quad \Big(0, \frac{-ca_{12} \pm \sqrt{a^2a_{12}^2 - 3c\alpha w^2}}{\omega^2}\Big).$$
If $c^2a_{12}^2 - 3c\alpha w^2 > 0$ the two last singular points exist. Computing the eigenvalues of the Jacobian matrix at these points we get that at least one of them is a node. Hence, going back through the changes of variables until system (12) we get that the origin of the local chart U_2 must have parabolic sectors and so it cannot be formed by two degenerated hyperbolic sectors.

If $c^2a_{12} - 3c\alpha w^2 = 0$, that is $a_{12} = \sqrt{3\alpha}\omega/\sqrt{c}$, we have two finite singular points which are the origin (which is a saddle) and the point

$(0, -\sqrt{3}\sqrt{\alpha c}/\omega)$ which is linearly zero. First we translate it to the origin setting $w = -\sqrt{3c\alpha}/\omega + W$. We perform the directional blow-up $(u, W) \mapsto (u, z)$ with $z = W/u$, and eliminating the common factor u^2 we get

$$u' = -\frac{1}{c\omega^2}\Big(3c^3\alpha u - 2\sqrt{3}a^3\sqrt{c\alpha}u^2 z + c^2u^3z^2\omega^2 + \sqrt{3}\sqrt{c\alpha}z\omega^3 + u\omega^4 z^2\Big),$$
$$z' = \frac{z}{c\omega^2}\Big(3\alpha c^3 - 2\sqrt{3}c^3\sqrt{c\alpha}\omega uz + c^2\omega^2u^2z^2 + 2\omega^4z^2\Big).$$

The singular points are $(0,0)$ and $\Big(0, i\sqrt{3}c^{3/2}\sqrt{\alpha}/(\sqrt{2}\omega^2)\Big)$ which due to the fact that $c\alpha > 0$ the last two do not exist. Computing the eigenvalues of the Jacobian matrix at the origin we get that it is a saddle. Going back through the changes of variables up to system (12) we get that the origin of the local chart U_2 is not formed by two degenerated hyperbolic sectors, because there are more hyperbolic sectors not in the equator of the Poincaré sphere coming from these last saddles. Finally, if $c^2a_{12} - 3c\alpha\omega^2 < 0$ we get that the unique finite singular point is the origin which is a saddle. Hence, going back trough the changes of variables up to system (12) we get that the origin of the local chart U_2 is formed by two degenerated hyperbolic sectors.

In short, in order that the origins of the local charts U_1 and U_2 are formed by two degenerated hyperbolic sectors we must have $a_{30} = 0$, $c\alpha > 0$ and $c(ca_{12} - 3\alpha\omega^2) < 0$.

System becomes

$$x' = -\frac{\omega^2}{c}y - 2a_{12}xy - 3\alpha x^2y, \quad y' = cx + a_{12}y^2 + 3\alpha xy^2.$$

The singular points are $(0,0)$ and $(\pm x_\pm, y_\pm)$ where

$$x_\pm = \frac{-ca_{12} \mp \sqrt{c^2a_{12} - 3c\alpha\omega^2}}{3c\alpha}, \quad y_\pm = \pm\frac{\sqrt{cx_\pm}}{\sqrt{-a_{12} - 3\alpha x_\pm}}.$$

Taking into account that $c^2a_{12} - 3c\alpha w^2 < 0$ we get that the unique finite singular point is the origin. This completes the proof of the proposition. □

Proposition 11. *Systems (IV) have a global linear-type center at the origin and no more finite singular points with all the infinite singular points formed by two degenerated hyperbolic sectors if and only if they can be written as in (iv.1) or (iv.2) of Theorem 4.*

Proof. We already know that the unique pair of infinite singular points are the origins of U_1 and V_1. On the local chart U_1 system (IV) becomes

$$\begin{aligned} u' &= 3a_{30}v + 3a_{12}u^2v + cv^2 + 6\alpha u^2 + \alpha u^4 + \frac{\omega^2}{c}v^2, \\ v' &= uv\Big(3\alpha + 2a_{12}v + \frac{\omega^2}{c}v^2 + \alpha u^2\Big). \end{aligned} \tag{13}$$

Computing the eigenvalues of the Jacobian matrix at the origin we get that it is nilpotent if $a_{30} \neq 0$ and linearly zero if $a_{30} = 0$. So, $a_{30} = 0$.

We perform the directional blow-up $(u, v) \mapsto (u, w)$ with $w = v/u$, and eliminating the common factor u we get

$$\begin{aligned} u' &= u(6\alpha + 3a_{12}uw + cw^2 + \alpha u^2 + \frac{\omega^2}{c}u^2w^2), \\ w' &= -w(3\alpha + a_{12}uw + cw^2). \end{aligned}$$

The singular points are $(0, 0)$ and $(0, \pm i\sqrt{(3\alpha)/c})$. The two last points exist if and only if $c\alpha < 0$. In this case computing the eigenvalues of the Jacobian matrix at these points we get that they are nodes. So this case is not possible. Hence, we must have $c\alpha > 0$. Then the unique singular point is the origin which is a saddle. Going back through the changes of variables till system (13) we get that the origin of the local chart U_1 is formed by two degenerated hyperbolic sectors.

System (IV) becomes

$$\dot{x} = -\frac{\omega^2}{c}y - 2a_{12}xy - 3\alpha x^2y - \alpha y^3, \quad \dot{y} = cx + a_{12}y^2 + 3\alpha xy^2.$$

On $y = 0$ the unique finite singular point is the origin. With $y \neq 0$ solving $\dot{y} = 0$ we get

$$x = -\frac{a_{12}}{c + 3\alpha y^2}y^2. \tag{14}$$

Note that if $y = \pm\frac{i\sqrt{c}}{\sqrt{3\alpha}}$ then $\dot{y}$ has no solution and so we can assume that $c + 3\alpha y^2 \neq 0$.

Substituting (14) in $\dot{x}$ we get

$$\begin{aligned} &\frac{y}{c(c + 3\alpha y^2)^2}(-2c^2a_{12}^2y^2 + c^3y^2\alpha - 3ca_{12}^2y^4\alpha + 6c^2y^4\alpha^2 + 9cy^6\alpha^3 + c^2\omega^2 \\ &\quad + 6cy^2\alpha\omega^2 + 9y^4\alpha^2\omega^2). \end{aligned}$$

We must see that there is no solutions of the sextic equation

$$c\alpha y^6 + 3\alpha(-ca_{12}^2 + 2\alpha c^2 + 3\alpha\omega^2)y^4 + c(-2ca_{12}^2 + \alpha c^2 + 6\alpha\omega^2)y^2 + c^2\omega^2$$

Or in other words, that there are not positive roots of the cubic equation

$$Q = 9c\alpha z^3 + 3\alpha x(-ca_{12}^2 + 2\alpha c^2 + 3\alpha\omega^2)z^2 + c(-2ca_{12}^2 + \alpha c^2 + 6\alpha\omega^2)z + c^2\omega^2.$$

We can write Q as in (9) with $b_2 = \tilde{b}_2$, $b_1 = \tilde{b}_1$ and $b_0 = \tilde{b}_0$ given in (4). So, in view of Proposition 7, the conditions so that there are no finite singular points are either $\Delta_3 < 0$ (see (4)) and $\tilde{b}_0 > 0$, or $\Delta_3 \geq 0$, $\tilde{b}_2 > 0$, $\tilde{b}_0 > 0$ and $\tilde{b}_2\tilde{b}_1 > \tilde{b}_0$. This concludes the proof of the proposition. □

Proof of Theorem 4. The proof of Theorem 4 is an immediate consequence of Propositions 8–10. □

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