

# Tips of tongues in the double standard family\*

Kuntal Banerjee<sup>1</sup> , Xavier Buff<sup>2</sup> , Jordi Canela<sup>3,\*\*</sup>  and Adam Epstein<sup>4</sup>

<sup>1</sup> Presidency University, 86/1 College Street, Kolkata - 700073, West Bengal, India

<sup>2</sup> Institut de Mathématiques de Toulouse, UMR5219, Université de Toulouse, CNRS, UPS, F-31062 Toulouse Cedex 9, France

<sup>3</sup> Centre de Recerca Matemàtica; Barcelona Graduate School of Mathematics (BGSMath), Campus de Bellaterra, Edifici C, 08193 Bellaterra (Barcelona), Spain

<sup>4</sup> Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom

E-mail: [kbanerjee.maths@presiuniv.ac.in](mailto:kbanerjee.maths@presiuniv.ac.in), [xavier.buff@math.univ-toulouse.fr](mailto:xavier.buff@math.univ-toulouse.fr), [canela@uji.es](mailto:canela@uji.es) and [a.l.epstein@warwick.ac.uk](mailto:a.l.epstein@warwick.ac.uk)

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## Abstract

We answer a question raised by Misiurewicz and Rodrigues concerning the family of degree two circle maps  $F_\lambda : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$  defined by

$$F_\lambda(x) := 2x + a + \frac{b}{\pi} \sin(2\pi x) \quad \text{with} \quad \lambda := (a, b) \in \mathbb{R}/\mathbb{Z} \times (0, 1).$$

We prove that if  $F_\lambda^{\circ n} - \text{id}$  has a zero of multiplicity three in  $\mathbb{R}/\mathbb{Z}$ , then there is a system of local coordinates  $(\alpha, \beta) : W \rightarrow \mathbb{R}^2$  defined in a neighborhood  $W$  of  $\lambda$ , such that  $\alpha(\lambda) = \beta(\lambda) = 0$  and  $F_\mu^{\circ n} - \text{id}$  has a multiple zero with  $\mu \in W$  if and only if  $\beta^3(\mu) = \alpha^2(\mu)$ . This shows that the tips of tongues are regular cusps.

Keywords: tongues, circle maps, cusp bifurcations, transversality, double standard maps

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\*\*Author to whom any correspondence should be addressed.

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