TIPS OF TONGUES IN THE DOUBLE STANDARD FAMILY

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Abstract. We answer a question raised by Misiurewicz and Rodrigues concerning the family of degree 2 circle maps $F_\lambda : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ defined by

$$F_\lambda(x) := 2x + a + \frac{b}{\pi}\sin(2\pi x) \quad \text{with} \quad \lambda := (a,b) \in \mathbb{R}/\mathbb{Z} \times (0,1).$$

We prove that if $F_\lambda^{\circ n} - \mathrm{id}$ has a zero of multiplicity 3 in $\mathbb{R}/\mathbb{Z}$, then there is a system of local coordinates $(\alpha,\beta) : W \to \mathbb{R}^2$ defined in a neighborhood W of λ, such that $\alpha(\lambda) = \beta(\lambda) = 0$ and $F_\mu^{\circ n} - \mathrm{id}$ has a multiple zero with $\mu \in W$ if and only if $\beta^3(\mu) = \alpha^2(\mu)$. This shows that the tips of tongues are regular cusps.

Introduction

Following Misiurewicz and Rodrigues [MR07], we consider the family of double standard maps of the circle $F_\lambda : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ defined by

$$F_\lambda(x) := 2x + a + \frac{b}{\pi}\sin(2\pi x) \quad \text{with} \quad \lambda := (a,b) \in \mathbb{R}/\mathbb{Z} \times [0,1].$$

If $b \in [0,1/2)$, then $F_\lambda : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is expanding and all periodic cycles of F_λ in $\mathbb{R}/\mathbb{Z}$ are repelling. If $b \in [1/2,1]$, it may happen that $F_\lambda : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ has a non-repelling cycle. The multiplier of such a cycle belongs to $[0,1]$. There is at most one such cycle. Connected components of the open sets of parameters $\lambda \in (a,b) \in \mathbb{R}/\mathbb{Z} \times [0,1]$ for which F_λ has an attracting cycle are called *tongues* (see [MR07] and [D10]). The period of the attracting cycle remains constant in each tongue, and is called the period of the tongue

These tongues can be understood as an analogue to Arnold tongues of the standard maps of the circle. The family of standard maps $A_{a,b} : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is given by

$$A_{a,b}(x) := x + a + \frac{b}{2\pi}\sin(2\pi x) \quad \text{with} \quad (a,b) \in \mathbb{R}/\mathbb{Z} \times [0,1]$$

and was introduced by Arnold in [A65] as a family of perturbations of rotation maps of the circle. If $b \in [0,1]$, each map $A_{a,b}$ is a homeomorphism of the circle and has a rotation number associated to it. In this case, tongues are connected sets of parameters for which the corresponding map has a given rational rotation number $p/q \in \mathbb{Q}/\mathbb{Z}$. Notice that double standard maps do not have a rotation number associated to them since they are not homeomorphisms of the circle. Instead,

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tongues in the double standard family can be classified in terms of the *type* of the corresponding attracting cycle (see [MR07]).

Let T be a tongue of period $p \geq 1$ of the double standard family. The boundary of T consists of two smooth curves which are graphs with respect to b and intersect tangentially at the tip $\lambda_T \in \mathbb{R}/\mathbb{Z} \times (0,1)$ (see [MR07, MR08] and Figure 1). If $\lambda \in \partial T$, then F_λ has a cycle of period p and multiplier 1. On the one hand, if $\lambda \in \partial T \setminus \{\lambda_T\}$, then the points of the cycle are double zeros of $F_\lambda^{\circ p} - \mathrm{id}$. On the other hand, for the tip parameter λ_T the points of the cycle are triple zeros of $F_{\lambda_T}^{\circ p} - \mathrm{id}$. Moreover, there is a cusp bifurcation which takes place around λ_T (see [HK91] for an introduction to cusp bifurcations).

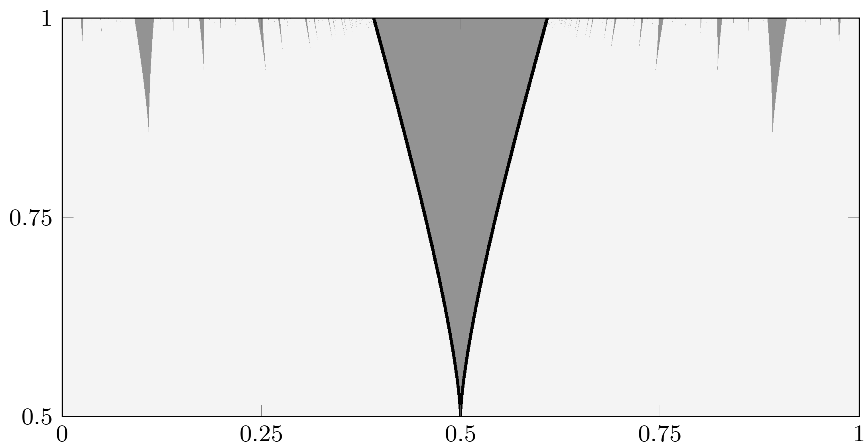

FIGURE 1. The tongues of the family F_λ. The horizontal axis corresponds to the parameter a and the vertical axis to b. We draw in black the boundary of the tongue of period 1.

It arises as a relevant topic to understand the shape of a given tongue T near its tip λ_T. This can be studied in terms of the order of contact of its boundary curves. Let $B_1(b)$ and $B_2(b)$ be the parametrizations of the boundary curves of T with respect to b. If $\lambda_T = (a_0, b_0)$, then we say that the order of contact of the two curves at the tip λ_T is r if the limit

$$\lim_{b \to b_0} \frac{|B_1(b) - B_2(b)|}{|b - b_0|^{r+1}}$$

is positive and finite. In the case of the standard family, the order of contact of the boundary curves at the tip of Arnold tongues depends on the rotation number. If a tongue has rotation number p/q, with p and q coprime, the order of contact is $q-1$ (see [A65]). However, this situation is very different for the double standard family due to the cusp bifurcation at the tip of its tongues. Misiurewicz and Rodrigues [MR07] proved that the order of contact of the boundary curves is $1/2$ for the unique tongue of period 1. This is equivalent to saying that the cusp bifurcation which takes place around λ_T for the tongue of period 1 is generic (see [MR11]). In [MR08] they asked whether this property holds for all tongues of the family F_λ. In this article, we answer positively to this question. More precisely, we prove that near the tip of any tongue of the double standard family, the two boundary curves form an ordinary cusp.

Theorem 1. *Assume* $F_\lambda^{\circ n} - \mathrm{id}$ *has a zero of multiplicity* 3 *in* $\mathbb{R}/\mathbb{Z}$*. Then there is a system of local coordinates* $(\alpha,\beta) : W \to \mathbb{R}^2$ *defined a neighborhood* W *of* λ *in* $\mathbb{R}/\mathbb{Z}\times(0,1)$*, such that* $\alpha(\lambda)=\beta(\lambda)=0$ *and* $F_\mu^{\circ n} - \mathrm{id}$ *has a multiple zero with* $\mu\in W$ *if and only if* $\beta^3(\mu)=\alpha^2(\mu)$*.*

Our proof relies on a transversality result due to Adam Epstein for families of finite type analytic maps, which itself relies on an injectivity result of a linear map acting on an appropriate space of quadratic differentials. Even though the proof we present is done specifically for the tongues of the family of double standard maps F_λ, it may be adapted to study cusp bifurcations of other families of holomorphic maps. Cusp bifurcations are a common phenomenon in the parameter planes of real-analytically parametrized families of holomorphic maps (see for instance [Mi92, CFG15, CFG16, NS03]). However, the non-holomorphic dependence on the parameter hinders the study of the parameter planes of such families. In this respect, this paper aims to provide a strategy to analyze the genericity of cusp bifurcations.

The proof of Theorem 1 is structured as follows. In §1, we prove that the maps F_λ are finite type analytic maps. In §2, we define the functions α and β. In §3, we identify the derivatives of those functions at λ. In §4, we state and prove the injectivity result. In §5, we prove that (α,β) is a system of local coordinates.

Some classical results on quadratic differentials are collected in Appendix A.

Notation

If U is a complex manifold, we denote by $\mathrm{T}U$ the tangent bundle of U and for $z\in U$, we denote by T_zU the tangent space to U at z. If $\phi : U\to\mathbb{C}$ is a holomorphic function, we denote by $\mathrm{d}\phi : \mathrm{T}U\to\mathbb{C}$ the exterior derivative of ϕ (this is a holomorphic 1-form on U). If $F : U\to V$ is a holomorphic map between complex manifolds U and V, we denote by $\mathrm{D}F : \mathrm{T}U\to\mathrm{T}V$ the bundle map $\mathrm{T}_zU\ni v\mapsto \mathrm{D}_zF(v)\in\mathrm{T}_{F(z)}V$.

Assume $f : U\to V$ is a holomorphic map between Riemann surfaces. If ω is a holomorphic 1-form on V, then $f^*\omega := \omega\circ\mathrm{D}f$ is a holomorphic 1-form on U. If ϑ is a holomorphic vector field on V, then there is a meromorphic vector field $f^*\vartheta$ on U satisfying $\mathrm{D}f\circ f^*\vartheta = \vartheta\circ f$.

We will consider various holomorphic families $t\mapsto\gamma_t$ defined near 0 in $\mathbb{C}$. We will employ the notation

$$\gamma := \gamma_0 \quad\text{and}\quad \dot\gamma := \left.\frac{d\gamma_t}{dt}\right|_{t=0}.$$

1. Finite type analytic maps

The notion of finite type analytic maps originates in [E1]. Let $f:\mathbb{X}\to\mathbb{Y}$ be an analytic map of complex 1-manifolds, possibly disconnected. An open set $V\subseteq\mathbb{Y}$ is *evenly covered* by f if $f_{|U} : U\to V$ is a homeomorphism for each component U of $f^{-1}(V)$; we say that $y\in\mathbb{Y}$ is a *regular value* for f if some neighborhood $V\ni y$ is evenly covered, and a *singular value* for f otherwise. Note that the set $\mathcal{S}_f$ of singular values is closed. Recall that $x\in\mathbb{X}$ is a *critical point* if the derivative of f at x vanishes, and then $f(x)\in\mathbb{Y}$ is a *critical value*. We say that $y\in\mathbb{Y}$ is an *asymptotic value* if f approaches y along some path tending to infinity relative to $\mathbb{X}$. It follows from elementary covering space theory that the critical values together with the asymptotic values form a dense subset of $\mathcal{S}_f$. In particular, every isolated point of $\mathcal{S}_f$ is a critical or asymptotic value.

An analytic map $f : \mathbb{X} \to \mathbb{Y}$ of complex 1-manifolds is of *finite type* if

- f is nowhere locally constant,
- f has no isolated removable singularities,
- $\mathbb{Y}$ is a finite union of compact Riemann surfaces, and
- $\mathcal{S}_f$ is finite.

If $\mathbb{Y}$ is connected, we define $\deg f$ as the finite or infinite number $\mathrm{card}\big(f^{-1}(y)\big)$ which is independent of $y \in \mathbb{Y}\smallsetminus\mathcal{S}_f$. When $f : \mathbb{X} \to \mathbb{Y}$ is a finite type analytic map with $\mathbb{X} \subseteq \mathbb{Y}$, we say that f is a finite type analytic map on $\mathbb{Y}$.

We first prove that the maps F_λ extend to finite type analytic maps.

1.1. **Preliminaries.** Set $\mathbb{T} := \mathbb{C}/\mathbb{Z}$ and $\Lambda := \mathbb{T} \times \mathbb{C}^*$. Let $F : \Lambda \times \mathbb{T} \to \mathbb{T}$ be the holomorphic map defined by

$$F(\lambda, z) = 2z + a + \frac{b}{\pi}\sin(2\pi z) \quad \text{with} \quad \lambda := (a, b) \in \Lambda.$$

For $\lambda \in \Lambda$, let $F_\lambda : \mathbb{T} \to \mathbb{T}$ be the holomorphic map defined by

$$F_\lambda(z) := F(\lambda, z).$$

It will be convenient to consider the global coordinate $w : \mathbb{T} \to \mathbb{C}^*$ defined by $w(z) = \mathrm{e}^{2\pi \mathrm{i} z}$. Note that it is an isomorphism. Thus, adding two points denoted $z = +\mathrm{i}\infty$ (or $w = 0$) and $z = -\mathrm{i}\infty$ (or $w = \infty$), $\mathbb{T}$ may be compactified into a Riemann surface $\widehat{\mathbb{T}}$ isomorphic to the Riemann sphere.

We will prove that for $\lambda \in \Lambda$, the map $F_\lambda : \mathbb{T} \to \widehat{\mathbb{T}}$ is a finite type analytic map on $\widehat{\mathbb{T}}$.

1.2. **The singular set.** Fix $\lambda := (a, b) \in \Lambda$ and set $f := F_{a,b} : \mathbb{T} \to \widehat{\mathbb{T}}$. Note that

$$w \circ f = \mathrm{e}^{2\pi \mathrm{i} a} w^2 \mathrm{e}^{b(w-1/w)}$$

and

$$f^*(\mathrm{d}w) = \mathrm{e}^{2\pi \mathrm{i} a}\mathrm{e}^{b(w-1/w)}(bw^2 + 2w + b)\ \mathrm{d}w.$$

In particular, f has two critical points counting multiplicities: the solutions of $bw^2 + 2w + b = 0$, i.e., the points $c^\pm \in \mathrm{T}$ such that

$$w(c^\pm) = \frac{-1 \pm \sqrt{1 - b^2}}{b}.$$

If $b \neq 1$, those are simple critical points of f. We denote by $\mathcal{C}_f = \{c^+, c^-\} \subset \mathbb{T}$ the set of critical points of f and by $\mathcal{V}_f := f(\mathcal{C}_f) \subset \mathbb{T}$ the set of critical values of f.

Lemma 2. *The singular set $\mathcal{S}_f$ is equal to $\mathcal{V}_f \cup \{\pm \mathrm{i}\infty\}$.*

Proof. We already identified the set of critical values of f. Note that $\pm\mathrm{i}\infty$ are singular values since those points are omitted values. It is therefore enough to show that f does not have any asymptotic value in $\mathbb{T}$.

If $v \in \mathbb{T}$ is an asymptotic value, then there exists a curve $\gamma : [0, 1) \to \mathbb{T}$, such that $\gamma(t) \to \pm\mathrm{i}\infty$ and $f \circ \gamma(t) \to v$ as $t \to 1$. We assume that $\gamma(t) \to +\mathrm{i}\infty$. The proof for the case $\gamma(t) \to -\mathrm{i}\infty$ is analogous.

It is convenient to lift via the canonical covering $\boldsymbol{\pi} : \mathbb{C} \to \mathbb{T} := \mathbb{C}/\mathbb{Z}$. Choose $A \in \mathbb{C}$ such that $\boldsymbol{\pi}(A) = a$. Let $\tilde{f} : \mathbb{C} \to \mathbb{C}$ be defined by

$$\tilde{f}(Z) = 2Z + A + \frac{b}{\pi}\sin(2\pi Z) \quad \text{so that} \quad \boldsymbol{\pi} \circ \tilde{f} = f \circ \boldsymbol{\pi}.$$

Let $\Gamma : [0,1) \to \mathbb{C}$ be a lift of $\gamma : [0,1) \to \mathbb{T}$, i.e., satisfying $\boldsymbol{\pi} \circ \Gamma = \gamma$. Then, $\tilde{f} \circ \Gamma$ is a lift of $f \circ \gamma$, thus $\tilde{f} \circ \Gamma(t)$ converges in $\mathbb{C}$ as $t \to 1$.

Set $X := \mathrm{Re}(\Gamma) : [0,1) \to \mathbb{R}$ and $Y := \mathrm{Im}(\Gamma) : [0,1) \to \mathbb{R}$. Then,

$$\tilde{f} \circ \Gamma = 2(X + \mathrm{i}Y) + A + \frac{b}{\pi}\sin(2\pi\Gamma), \quad \sin(2\pi\Gamma) = \frac{\mathrm{e}^{-2\pi Y}\mathrm{e}^{2\pi \mathrm{i} X} - \mathrm{e}^{2\pi Y}\mathrm{e}^{-2\pi \mathrm{i} X}}{2\mathrm{i}}$$

and $Y(t) \to +\infty$ as $t \to 1$. It follows that as $t \to 1$,

$$\tilde{f} \circ \Gamma(t) \sim 2X(t) - \frac{b}{4\pi \mathrm{i}}\mathrm{e}^{2\pi Y(t)}\mathrm{e}^{-2\pi \mathrm{i} X(t)}.$$

We can distinguish 2 cases. If there exists a sequence $\{t_k\}_{k\in\mathbb{N}}$ converging to 1 with $\big\{X(t_k)\big\}_{k\in\mathbb{N}}$ bounded, then

$$\big|\tilde{f} \circ \Gamma(t_k)\big| \sim \frac{b}{4\pi}\mathrm{e}^{2\pi Y(t_k)} \underset{k\to+\infty}{\longrightarrow} +\infty.$$

Otherwise, $X(t) \to \pm\infty$ as $t \to 1$ and there exists a sequence $\{t_k\}_{k\in\mathbb{N}}$ converging to 1 with $X(t_k) \in \mathbb{Z}$ for all $k \in \mathbb{N}$, so that

$$\tilde{f} \circ \Gamma(t_k) \sim 2X(t_k) + \mathrm{i}\frac{b}{4\pi}\mathrm{e}^{2\pi Y(t_k)} \underset{k\to+\infty}{\longrightarrow} \infty.$$

In both cases, the sequence $\big\{\tilde{f} \circ \Gamma(t_k)\big\}_{k\in\mathbb{N}}$ cannot converge in $\mathbb{C}$. □

Corollary 3. *The map $f : \mathbb{T} \to \widehat{\mathbb{T}}$ is a finite type analytic map on $\widehat{\mathbb{T}}$. More precisely, $f : \mathbb{T} \setminus f^{-1}(\mathcal{V}_f) \to \mathbb{T} \setminus \mathcal{V}_f$ is a covering map.*

2. Splitting triple zeros

In the remainder of the article, we fix a parameter $\lambda := (a,b) \in \mathbb{R}/\mathbb{Z} \times (0,1)$ such that $F_\lambda^{\circ n} - \mathrm{id}$ has a triple zero $x \in \mathbb{R}/\mathbb{Z}$. We set $f := F_\lambda : \mathbb{T} \to \mathbb{T}$. The point x is periodic for f with period p dividing n. For $k \geq 0$, we set $x_k := f^{\circ k}(x)$ and we denote by $\langle x \rangle := \{x_0, x_1, \ldots, x_{p-1}\}$ the cycle of x.

Since $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ preserves the orientation, the multiplier of $f^{\circ p}$ at x is necessarily 1 and there is a local coordinate $\zeta : (\mathbb{T}, x) \to (\mathbb{C}, 0)$ vanishing at x satisfying

$$\zeta(\bar{z}) = \bar{\zeta}(z) \quad \text{and} \quad \zeta \circ f^{\circ p} = \zeta + \zeta^3 + \mathcal{O}(\zeta^5). \tag{1}$$

According to the Weierstrass Preparation Theorem, there exist a neighborhood $W_1 \subset \Lambda$ of λ, a neighborhood $W_2 \subset \mathbb{C}$ of 0 and analytic functions $A : W_1 \to \mathbb{C}$, $B : W_1 \to \mathbb{C}$, $C : W_1 \to \mathbb{C}$ and $g : W_1 \times W_2 \to \mathbb{C}$ such that for $\mu \in W_1$,

$$\zeta \circ F_\mu^{\circ p} - \zeta = P_\mu(\zeta) \cdot g(\mu, \zeta) \tag{2}$$

with

$$A(\lambda) = B(\lambda) = C(\lambda) = 0, \quad g(\lambda, \zeta) = 1 + \mathcal{O}(\zeta^2) \tag{3}$$

and

$$P_\mu(\zeta) := A(\mu) + B(\mu)\zeta + C(\mu)\zeta^2 + \zeta^3. \tag{4}$$

The polynomial P_λ has a zero of multiplicity 3 at 0, and as μ varies in W_1, this zero splits in three zeros (counting multiplicities) of P_μ. When $\mu \in \mathbb{R}/\mathbb{Z} \times (0,1)$, the map $F_\mu^{\circ n} - \mathrm{id}$ commutes with $z \mapsto \bar{z}$, so that the polynomial P_μ has real coefficients. For such a parameter μ, a multiple zero of P_μ is necessarily real.

For any $\mu \in \Lambda$, the function $\zeta \circ F_\mu^{\circ p} - \zeta$ vanishes at the periodic points of F_μ of period dividing p, and so, divides $\zeta \circ F_\mu^{\circ n} - \zeta$ which vanishes at the periodic points of period dividing n. In addition, if $n = mp$, then $\zeta \circ f^{\circ n} - \zeta = m\zeta^3 + \mathcal{O}(\zeta^5)$. So, there is an analytic function $h : W_1 \times W_2 \to \mathbb{C}$ such that for $\mu \in W_1$,

$$\zeta \circ F_\mu^{\circ n} - \zeta = P_\mu(\zeta) \cdot h(\mu, \zeta) \quad \text{with} \quad h(\lambda, \zeta) = m + \mathcal{O}(\zeta^2)$$

Since f only has two critical points in $\mathbb{T}$, it has a single non-repelling cycle, that is, the cycle $\langle x \rangle$. All other cycles of f in $\mathbb{R}/\mathbb{Z}$ are repelling. Shrinking W_1 if necessary, it follows that for $\mu \in W_1$, the function $\zeta \circ F_\mu^{\circ n} - \zeta$ has a multiple zero in $\mathbb{R}/\mathbb{Z}$ if and only if the polynomial P_μ has a multiple zero in $\mathbb{R}/\mathbb{Z}$. According to the previous discussion, this is the case if and only if P_μ has a multiple zero.

Let $\alpha : W_1 \to \mathbb{C}$ and $\beta : W_1 \to \mathbb{C}$ be defined by

$$\alpha := \frac{C^3}{27} - \frac{BC}{6} + \frac{A}{2} \quad \text{and} \quad \beta := \frac{C^2}{9} - \frac{B}{3}.$$

Then,

$$\mathrm{discriminant}(P_\mu) = 108\beta^3(\mu) - 108\alpha^2(\mu).$$

So, if $\mu \in W_1$, the polynomial P_μ has a multiple zero if and only if $\beta^3(\mu) = \alpha^2(\mu)$.

In order to prove Theorem 1, it is therefore enough to show that (α, β) is a system of local coordinates near λ. For this purpose, we shall show that the restrictions of $\mathrm{d}\alpha$ and $\mathrm{d}\beta$ to $\mathrm{T}_\lambda\Lambda$ are linearly independent. Since A, B and C vanish at λ,

$$\mathrm{d}\alpha|_{\mathrm{T}_\lambda\Lambda} = \frac{1}{2}\mathrm{d}A|_{\mathrm{T}_\lambda\Lambda} \quad \text{and} \quad \mathrm{d}\beta|_{\mathrm{T}_\lambda\Lambda} = -\frac{1}{3}\mathrm{d}B|_{\mathrm{T}_\lambda\Lambda}.$$

It is therefore enough to show that the forms $\mathrm{d}A|_{\mathrm{T}_\lambda\Lambda}$ and $\mathrm{d}B|_{\mathrm{T}_\lambda\Lambda}$ are linearly independent.

3. Identifying the derivatives

Here, we identify $\mathrm{d}A(v)$ and $\mathrm{d}B(v)$ for $v \in \mathrm{T}_\lambda\Lambda$. First, to each $v \in \mathrm{T}_\lambda\Lambda$, we shall associate a meromorphic vector field ϑ_v on $\mathbb{T}$ having simple poles along $\mathcal{C}_f \cup \{\pm \mathrm{i}\infty\}$, such that for all $z \in \mathbb{T} \smallsetminus \mathcal{C}_f$,

$$\mathrm{D}f \circ \vartheta_v(z) := \mathrm{D}_{\lambda,z}F(v, 0).$$

Second, for $k \in [1,p]$, let $\zeta_k : (\mathbb{T}, x_k) \to (\mathbb{C}, 0)$ be the local coordinate vanishing at x_k defined by

$$\zeta_k := \zeta \circ f^{\circ(p-k)}.$$

Our identification goes as follows.

Proposition 4. *Let q_A and q_B be quadratic differentials, defined and meromorphic near $\langle x \rangle$, such that $q_A - (\mathrm{d}\zeta_k)^2/\zeta_k$ and $q_B - (\mathrm{d}\zeta_k)^2/\zeta_k^2$ are holomorphic at x_k for all $k \in [1,p]$. Then, for all $v \in \mathrm{T}_\lambda\Lambda$,*

$$\mathrm{d}A(v) = \sum_{k=1}^{p} \mathrm{residue}(q_A \otimes \vartheta_v, x_k) \quad \textit{and} \quad \mathrm{d}B(v) = \sum_{k=1}^{p} \mathrm{residue}(q_B \otimes \vartheta_v, x_k).$$

In the remaining parts of this section we prove Proposition 4.

3.1. **Meromorphic vector fields.** Assume $v \in \mathrm{T}_\lambda\Lambda$ and $z \in \mathbb{T}\smallsetminus\mathcal{C}_f$. Then, the derivative $\mathrm{D}_z f : \mathrm{T}_z\mathbb{T} \to \mathrm{T}_{f(z)}\mathbb{T}$ is an isomorphism and $\mathrm{D}_{\lambda,z}F(v,0) \in \mathrm{T}_{f(z)}\mathbb{T}$. Let ϑ_v be the vector field defined on $\mathbb{T}\smallsetminus\mathcal{C}_f$ by

$$\vartheta_v(z) := (\mathrm{D}_z f)^{-1}\big(\mathrm{D}_{\lambda,z}F(v,0)\big) \in \mathrm{T}_z\mathbb{T}.$$

Lemma 5. *For all $v \in \mathrm{T}_\lambda\Lambda$, the vector field ϑ_v is holomorphic on $\mathbb{T}\smallsetminus\mathcal{C}_f$, meromorphic on $\widehat{\mathbb{T}}$, vanishes at $z = \pm\mathrm{i}\infty$ and has at worst simple poles along $\mathcal{C}_f$.*

Proof. The map $v \mapsto \vartheta_v$ is linear. So, it is enough to prove the result for $v_a := \mathrm{d}/\mathrm{d}a$ and $v_b := \mathrm{d}/\mathrm{d}b$. We have

$$\vartheta_{v_a} = \frac{2\pi\mathrm{i}\mathrm{e}^{2\pi\mathrm{i}a}w^2\mathrm{e}^{b(w-1/w)}}{\mathrm{e}^{2\pi\mathrm{i}a}\mathrm{e}^{b(w-1/w)}(bw^2+2w+b)}\frac{\mathrm{d}}{\mathrm{d}w} = \frac{2\pi\mathrm{i}w^2}{bw^2+2w+b}\frac{\mathrm{d}}{\mathrm{d}w}$$

and

$$\vartheta_{v_b} = \frac{\mathrm{e}^{2\pi\mathrm{i}a}w^2(w-1/w)\mathrm{e}^{b(w-1/w)}}{\mathrm{e}^{2\pi\mathrm{i}a}\mathrm{e}^{b(w-1/w)}(bw^2+2w+b)}\frac{\mathrm{d}}{\mathrm{d}w} = \frac{w^3-w}{bw^2+2w+b}\frac{\mathrm{d}}{\mathrm{d}w}$$

Those two vector fields have the required properties. □

Denote by $\mathcal{T}_f$ the space of meromorphic vector fields on $\widehat{\mathbb{T}}$ which are holomorphic on $\mathbb{T}\smallsetminus\mathcal{C}_f$, vanish at $\pm\mathrm{i}\infty$ and have at worst simple poles along $\mathcal{C}_f$. In other words,

$$\mathcal{T}_f := \left\{\frac{c_3w^3+c_2w^2+c_1w}{bw^2+2w+b}\frac{\mathrm{d}}{\mathrm{d}w}\ ;\ (c_1,c_2,c_3)\in\mathbb{C}^3\right\}.$$

Let $\Theta_f : \mathrm{T}_\lambda\Lambda \to \mathcal{T}_f$ be the linear map defined by

$$\Theta_f(v) := \vartheta_v.$$

Let $\tau \in \mathcal{T}_f$ be the radial vector field

$$\tau := w\frac{\mathrm{d}}{\mathrm{d}w}.$$

Note that $\tau - f^*\tau$ belongs to $\mathcal{T}_f$. Indeed,

$$\tau - f^*\tau = \frac{bw^3+w^2+bw}{bw^2+2w+b}\frac{\mathrm{d}}{\mathrm{d}w} \in \mathcal{T}_f.$$

Lemma 6. *The space $\mathcal{T}_f$ is the direct sum of the image of Θ_f and the line spanned by $\tau - f^*\tau$:*

$$\mathcal{T}_f = \mathrm{Im}(\Theta_f) \oplus \mathrm{Vect}(\tau - f^*\tau).$$

Proof. The dimension of $\mathcal{T}_f$ is 3. Thus, it is enough to show that the three vector fields ϑ_{v_a}, ϑ_{v_b} and $\tau - f^*\tau$ are linearly independent. Equivalently, it is enough to show that the three functions

$$w^2,\quad w^3-w\quad\text{and}\quad bw^3+w^2+bw$$

are linearly independent. This is true since $b \neq 0$. □

Assume now $v \in \mathrm{T}_\lambda\Lambda$ and let $t \mapsto \lambda_t \in \Lambda$ be a curve such that $\dot\lambda = v$. Let $t \mapsto f_t$ be the family of maps defined by

$$f_t := F_{\lambda_t} : \mathbb{T} \to \mathbb{T}.$$

Then, for each $z \in \mathbb{T}$,

$$\dot f(z) = D_{\lambda,z}F(v,0) = \mathrm{D}_z f \circ \vartheta_v(z)\quad\text{with}\quad \vartheta_v := \Theta_f(v) \in \mathcal{T}_f.$$

Lemma 7. *For all* $k \geq 1$,

$$\frac{\mathrm{d}f_t^{\circ k}}{\mathrm{d}t}\Big|_{t=0} = \mathrm{D}f^{\circ k} \circ \vartheta_v^k \quad \textit{with} \quad \vartheta_v^k := \vartheta_v + f^*\vartheta_v + \cdots + \left(f^{\circ(k-1)}\right)^*\vartheta_v.$$

Proof. The proof follows from an elementary induction on $k \geq 1$ using the following fact: if $h_t = g_t \circ f_t$ with $\dot{f} = \mathrm{D}f \circ \vartheta$ and $\dot{g} = \mathrm{D}g \circ \tau$, then

$$\dot{h} = \dot{g} \circ f + \mathrm{D}g \circ \dot{f} = \mathrm{D}g \circ \tau \circ f + \mathrm{D}g \circ \mathrm{D}f \circ \vartheta = \mathrm{D}h \circ (f^*\tau + \vartheta). \qquad \square$$

Note that the poles of ϑ_v^n are the critical points of f and their iterated preimages (up to order $n-1$). The two critical points of f are in $\mathbb{T} \smallsetminus \mathbb{R}/\mathbb{Z}$, and so are all their preimages. Therefore, ϑ_v^n is holomorphic in a neighborhood of $\mathbb{R}/\mathbb{Z}$. In particular, it is holomorphic near the parabolic periodic point $x \in \mathbb{R}/\mathbb{Z}$.

3.2. **Polar parts of quadratic differentials.** Our identification of the derivatives $\mathrm{d}A|_{\mathrm{T}_\lambda\Lambda}$ and $\mathrm{d}B|_{\mathrm{T}_\lambda\Lambda}$ relies on the use of quadratic differentials (see Appendix A for basics regarding quadratic differentials). Recall that $\zeta : (\mathbb{T}, x) \to (\mathbb{C}, 0)$ is a local coordinate vanishing at x such that

$$\zeta \circ f^{\circ p} = \zeta + \zeta^3 + \mathcal{O}(\zeta^5).$$

We shall use the quadratic differential $(\mathrm{d}\zeta)^2/\zeta$ and $(\mathrm{d}\zeta)^2/\zeta^2$ which are defined and meromorphic near x in $\mathbb{T}$.

Following §A.7, if $Z \subset \mathrm{T}$ is a finite set, if q is a quadratic differential, defined and meromorphic near Z, and if ϑ is a vector field, defined and meromorphic near Z, we shall use the notation

$$\langle q, \vartheta \rangle_Z := \sum_{z \in Z} \mathrm{residue}(q \otimes \vartheta, z).$$

If q has at worst simple poles along Z and if θ is defined on Z with $\theta(z) \in \mathrm{T}_z\mathbb{T}$ for $z \in Z$, we shall use the notation

$$\langle q, \theta \rangle_Z := \langle q, \vartheta \rangle_Z$$

where ϑ is any vector field, defined and holomorphic near Z, with $\vartheta(z) = \theta(z)$ for $z \in Z$. The result does not depend on the choice of extension.

Lemma 8. *For all* $v \in \mathrm{T}_\lambda\Lambda$,

$$\mathrm{d}A(v) = \left\langle \frac{(\mathrm{d}\zeta)^2}{\zeta}, \vartheta_v^p \right\rangle_x \quad \textit{and} \quad \mathrm{d}B(v) = \left\langle \frac{(\mathrm{d}\zeta)^2}{\zeta^2}, \vartheta_v^p \right\rangle_x.$$

Proof. According to Equations (2), (3) and (4),

$$\zeta \circ f_t^{\circ p} - \zeta = \left(A(\lambda_t) + B(\lambda_t)\zeta + \mathcal{O}(\zeta^2)\right) \cdot \left(1 + \mathcal{O}(\zeta^2)\right).$$

Taking the derivative with respect to t and evaluating at $t = 0$ yields

$$\mathrm{d}\zeta \circ \mathrm{D}f^{\circ p} \circ \vartheta_v^p = \mathrm{d}A(v) + \mathrm{d}B(v)\zeta + \mathcal{O}(\zeta^2).$$

According to Equation (1),

$$\zeta \circ f^{\circ p} = \zeta + \mathcal{O}(\zeta^3) \quad \text{so that} \quad \mathrm{d}\zeta \circ \mathrm{D}f^{\circ p} = \left(1 + \mathcal{O}(\zeta^2)\right)\mathrm{d}\zeta.$$

As a consequence

$$\mathrm{d}\zeta(\vartheta_v^p) = \mathrm{d}\zeta \circ \mathrm{D}f^{\circ p} \circ \vartheta_v^p + \mathcal{O}(\zeta^2) = \mathrm{d}A(v) + \mathrm{d}B(v)\zeta + \mathcal{O}(\zeta^2).$$

Thus,

$$\mathrm{d}A(v) = \text{residue}\left(\frac{\mathrm{d}\zeta(\vartheta_v^p)}{\zeta}\mathrm{d}\zeta, x\right) = \left\langle \frac{(\mathrm{d}\zeta)^2}{\zeta}, \vartheta_v^p \right\rangle_x$$

and similarly

$$\mathrm{d}B(v) = \text{residue}\left(\frac{\mathrm{d}\zeta(\vartheta_v^p)}{\zeta^2}\mathrm{d}\zeta, x\right) = \left\langle \frac{(\mathrm{d}\zeta)^2}{\zeta^2}, \vartheta_v^p \right\rangle_x. \qquad \square$$

Rather than working near x with the vector field ϑ_v^p, it will be convenient to work along the cycle $\langle x \rangle$ with the vector field ϑ_v. Recall that for $k \in [1, p]$, the local coordinate $\zeta_k : (\mathbb{T}, x_k) \to (\mathbb{C}, 0)$ vanishes at x_k and is defined by

$$\zeta_k := \zeta \circ f^{\circ(p-k)}.$$

Lemma 9. *For all $k \in \mathbb{Z}/p\mathbb{Z}$,*

$$f^*\left(\frac{(\mathrm{d}\zeta_{k+1})^2}{\zeta_{k+1}}\right) - \frac{(\mathrm{d}\zeta_k)^2}{\zeta_k} \quad \textit{and} \quad f^*\left(\frac{(\mathrm{d}\zeta_{k+1})^2}{\zeta_{k+1}^2}\right) - \frac{(\mathrm{d}\zeta_k)^2}{\zeta_k^2}$$

are holomorphic near x_k.

Proof. If $k \in [1, p-1]$, then $\zeta_k = \zeta_{k+1} \circ f$, so that

$$f^*\left(\frac{(\mathrm{d}\zeta_{k+1})^2}{\zeta_{k+1}}\right) = \frac{(\mathrm{d}\zeta_k)^2}{\zeta_k} \quad \text{and} \quad f^*\left(\frac{(\mathrm{d}\zeta_{k+1})^2}{\zeta_{k+1}^2}\right) = \frac{(\mathrm{d}\zeta_k)^2}{\zeta_k^2}.$$

If $k = p$, then $\zeta_p = \zeta$ and $\zeta_1 \circ f = \zeta \circ f^{\circ p} = \big(1 + \mathcal{O}(\zeta_p^2)\big)\zeta_p$. As a consequence, $f^*(\mathrm{d}\zeta_1) = \big(1 + \mathcal{O}(\zeta_p^2)\big)\mathrm{d}\zeta_p$,

$$f^*\left(\frac{(\mathrm{d}\zeta_1)^2}{\zeta_1}\right) = \big(1 + \mathcal{O}(\zeta_p^2)\big)\frac{(\mathrm{d}\zeta_p)^2}{\zeta_p} \quad \text{and} \quad f^*\left(\frac{(\mathrm{d}\zeta_1)^2}{\zeta_1^2}\right) = \big(1 + \mathcal{O}(\zeta_p^2)\big)\frac{(\mathrm{d}\zeta_p)^2}{\zeta_p^2}. \ \square$$

Proof of Proposition 4. Recall that $\zeta_p = \zeta$. According to the previous lemma, for all $k \in \mathbb{Z}/p\mathbb{Z}$,

$$(f^{\circ k})^* \frac{(\mathrm{d}\zeta_k)^2}{\zeta_k} - \frac{(\mathrm{d}\zeta)^2}{\zeta}$$

is holomorphic near x. By assumption, $q_A - (\mathrm{d}\zeta_k)^2/\zeta_k$ is holomorphic at x_k. It follows that $(f^{\circ k})^* q_A - (\mathrm{d}\zeta)^2/\zeta$ is holomorphic near x.

Since $(f^{\circ k})^* \vartheta_v$ is holomorphic near x, we therefore have

$$\left\langle \frac{(\mathrm{d}\zeta)^2}{\zeta}, (f^{\circ k})^* \vartheta_v \right\rangle_x = \big\langle (f^{\circ k})^* q_A, (f^{\circ k})^* \vartheta_v \big\rangle_x = \langle q_A, \vartheta_v \rangle_{x_k}.$$

As a consequence

$$\mathrm{d}A(v) = \left\langle \frac{(\mathrm{d}\zeta)^2}{\zeta}, \sum_{k=0}^{p-1} (f^{\circ k})^* \vartheta_v \right\rangle_x = \sum_{k=0}^{p-1} \langle q_A, \vartheta_v \rangle_{x_k} = \langle q_A, \vartheta_v \rangle_{\langle x \rangle}.$$

This proves Proposition 4 for $\mathrm{d}A$. The proof for $\mathrm{d}B$ is similar. $\square$

4. Injectivity of ∇_f

In order to prove Theorem 1, we need to use the global properties of the map f. Up to now, we only used the local properties near the cycle. For this purpose, it is important that the quadratic differentials q_A and q_B which appear in Proposition 4 are globally meromorphic on $\widehat{\mathbb{T}}$. Here, we define such quadratic differentials q_A and q_B and we prove that the linear map

$$\nabla_f := \mathrm{id} - f_*$$

is well defined and injective on the vector space $\mathrm{Vect}(q_A, q_B)$ spanned by q_A and q_B.

4.1. **A space of quadratic differentials.** Denote by $\mathcal{Q}(\mathbb{T})$ the space of meromorphic quadratic differentials on $\widehat{\mathbb{T}}$ which have at worst simple poles at $z = \pm \mathrm{i}\infty$. Given $Z \subset \mathbb{T}$, denote by $\mathcal{Q}(\mathbb{T}; Z) \subset \mathcal{Q}(\mathbb{T})$ the subspace of quadratic differentials which are holomorphic outside Z. Finally, we denote by $\mathcal{Q}^1(\mathbb{T}; Z) \subset \mathcal{Q}(\mathbb{T}; Z)$ the subspace of quadratic differentials having at worst simple poles.

Lemma 10. *Any polar part of quadratic differential along $\langle x\rangle$ may be realized as the polar part of a quadratic differential in $\mathcal{Q}\big(\mathbb{T}; \langle x\rangle \cup \{c^+\}\big)$ having at worst a simple pole at c^+.*

Proof. For all $k \in [0, p-1]$, the quadratic differentials

$$\frac{(\mathrm{d}w)^2}{\big(w - w(x_k)\big)\big(w - w(c^+)\big)w}, \quad \frac{(\mathrm{d}w)^2}{\big(w - w(x_k)\big)^2 w} \quad \text{and} \quad \frac{(\mathrm{d}w)^2}{\big(w - w(x_k)\big)^j} \quad \text{for } j \geq 3$$

belong to $\mathcal{Q}\big(\mathbb{T}; \langle x\rangle \cup \{c^+\}\big)$. The first has a simple pole at x_k, the second has a double pole at x_k, and the third has a pole of order $j \geq 3$ at x_k. Thus, they generate the space of polar parts at x_k. □

From now on, we assume that $q_A \in \mathcal{Q}\big(\mathbb{T}; \langle x\rangle \cup \{c^+\}\big)$ and $q_B \in \mathcal{Q}\big(\mathbb{T}; \langle x\rangle \cup \{c^+\}\big)$ have at worst simple poles at c^+ and that $q_A - (\mathrm{d}\zeta_k)^2/\zeta_k$ and $q_B - (\mathrm{d}\zeta_k)^2/\zeta_k^2$ are holomorphic at x_k for all $k \in [0, p-1]$. We set

$$\mathcal{Q}_f := \mathrm{Vect}(q_A, q_B).$$

4.2. **Pairing quadratic differentials in $\mathcal{Q}_f$ with vector fields in $\mathcal{T}_f$.** Recall that

$$\tau := w\frac{\mathrm{d}}{\mathrm{d}w} \quad \text{and} \quad \tau - f^*(\tau) \in \mathcal{T}_f.$$

Lemma 11. *For all $q \in \mathcal{Q}_f$,*

$$\langle q, \tau - f^*\tau\rangle_{\langle x\rangle} = 0.$$

Proof. Assume $q \in \mathcal{Q}_f$. According to Lemma 9, $q - f^*q$ is holomorphic near $\langle x\rangle$. Since τ is also holomorphic near $\langle x\rangle$, and since f is a local isomorphism near $\langle x\rangle$,

$$\langle q, f^*\tau\rangle_{\langle x\rangle} = \langle f^*q, f^*\tau\rangle_{\langle x\rangle} = \langle q, \tau\rangle_{\langle x\rangle}. \qquad \square$$

4.3. **Pushing forward quadratic differentials in $\mathcal{Q}_f$.** According to Corollary 3, $f : \mathbb{T}\smallsetminus f^{-1}(\mathcal{V}_f) \to \mathbb{T}\smallsetminus\mathcal{V}_f$ is a covering map. Here, we show that for all $q \in \mathcal{Q}_f$, the following series defines a meromorphic quadratic differential on $\widehat{\mathbb{T}}$:

$$f_*q = \sum_{g \text{ inverse branch of } f} g^*q. \tag{5}$$

The (minor) difficulty is that the degree of the covering map is not finite, and that q may fail to be integrable on $\widehat{\mathbb{T}}$ since it may have multiple poles along $\langle x\rangle$. So, we cannot apply directly the results presented in Appendix A. The reason why the series in Equation (5) converges is that q is locally integrable near the essential singularities of f, i.e., the points $\pm\mathrm{i}\infty$.

Lemma 12. *If $q \in \mathcal{Q}_f$, the series in Equation (5) converges locally uniformly in $\mathbb{T}\smallsetminus\big(\mathcal{V}_f \cup \langle x\rangle\big)$. Its sum f_*q is a meromorphic quadratic differential on $\widehat{\mathbb{T}}$.*

Proof. Assume $q \in \mathcal{Q}_f$. Let $V \subset \mathbb{T}\smallsetminus\big(\mathcal{V}_f \cup \langle x\rangle\big)$ is compactly contained in $\widehat{\mathbb{T}}\smallsetminus\langle x\rangle$. Then, $U := f^{-1}(V)$ is compactly contained in $\widehat{\mathbb{T}}\smallsetminus\langle x\rangle$. In particular, q is integrable on U. In addition, $f : U \to V$ is a covering map. It follows that the series in Equation (5) converges uniformly on V and that f_*q is integrable on V. This shows f_*q is holomorphic on $\mathbb{T}\smallsetminus\big(\mathcal{V}_f \cup \langle x\rangle\big)$ and has at worst simple poles at $\pm\mathrm{i}\infty$ and on $\mathcal{V}_f$.

To see that f_*q is meromorphic near x_k, $k \in [1,p]$, let $V \subset \mathbb{T}\smallsetminus\mathcal{V}_f$ be a topological disk containing x_k. Then, $U := f^{-1}(V)$ is the disjoint union of a topological disk U' containing x_{k-1} and a open set U'' compactly contained in $\widehat{\mathbb{T}}\smallsetminus\langle x\rangle$. Then, $(f|_{U''})_*q$ is holomorphic. In addition, $f : U' \to U$ is an isomorphism so that $(f|_{U'})_*q$ – and thus $f_*q = (f|_{U'})_*q + (f|_{U'})_*q$ – is meromorphic near x_k. □

We may therefore consider the linear map

$$\nabla_f := \mathrm{id} - f_* : \mathcal{Q}_f \to \mathcal{Q}(\mathbb{T}).$$

It will be convenient to set

$$Y := \{c^+\} \cup \mathcal{V}_f.$$

Lemma 13. *We have the inclusion*

$$\nabla_f(\mathcal{Q}_f) \subseteq \mathcal{Q}^1(\mathbb{T}; Y).$$

Proof. Assume $q \in \mathcal{Q}_f$. As mentioned in the proof of the previous lemma, f_*q is holomorphic on $\mathbb{T}\smallsetminus\big(\mathcal{V}_f \cup \langle x\rangle\big)$ and has at worst simple poles at $\pm\mathrm{i}\infty$ and on $\mathcal{V}_f$. In addition, for $k \in [1,p]$, the polar part of f_*q at x_k is equal to the polar part of g^*q where g is the inverse branch of f sending x_k to x_{k-1}. According to Lemma 9, $q - f_*q$ is therefore holomorphic near $\langle x\rangle$. It follows that $q - f_*q \in \mathcal{Q}^1(\mathbb{T}; Y)$. □

4.4. **Injectivity of ∇_f.** An observation due to the fourth author [E2] is that the linear map ∇_f is injective on $\mathcal{Q}_f$, and that this is the key to the proof of Theorem 1.

Proposition 14. *The linear map $\nabla_f : \mathcal{Q}_f \to \mathcal{Q}(\mathbb{T})$ is injective.*

Proof. We must prove that $f_*q \neq q$ for $q \in \mathcal{Q}_f\smallsetminus\{0\}$. If q were integrable on $\mathbb{T}$, the result would follow immediately from Proposition 21, since we would have $\|f_*q\|_{L^1(\mathbb{T})} < \|q\|_{L^1(\mathbb{T})}$. Since q may have double poles near $\langle x\rangle$, it may fail to be integrable on $\mathbb{T}$. In that case, we may proceed as follows.

Assume $q \in \mathcal{Q}_f\smallsetminus\{0\}$. For $\varepsilon > 0$ small, let $V_\varepsilon \subset \mathrm{T}$ be the union of topological disks

$$V_\varepsilon := \bigcup_{k=1}^{p} \bigl\{|\zeta_k| < \varepsilon\bigr\}.$$

Set $U_\varepsilon := f^{-1}(V_\varepsilon) \subset \mathbb{T}$. Then, $\langle x\rangle \subset U_\varepsilon$ and so, q is integrable on $\mathbb{T}\smallsetminus U_\varepsilon$. As a consequence,

$$\|f_*q\|_{L^1(\mathbb{T}\smallsetminus V_\varepsilon)} < \|q\|_{L^1(\mathbb{T}\smallsetminus U_\varepsilon)}.$$

Similarly, for $\varepsilon' < \varepsilon$,

$$\|f_*q\|_{L^1(V_\varepsilon\smallsetminus V_{\varepsilon'})} < \|q\|_{L^1(U_\varepsilon\smallsetminus U_{\varepsilon'})}.$$

As a consequence, the function

$$\varepsilon \mapsto \|q\|_{L^1(\mathbb{T}\smallsetminus U_\varepsilon)} - \|f_*q\|_{L^1(\mathbb{T}\smallsetminus V_\varepsilon)}$$

is positive and decreasing. In particular, it has a positive limit. Note that

$$\|q\|_{L^1(\mathbb{T}\smallsetminus U_\varepsilon)} - \|q\|_{L^1(\mathbb{T}\smallsetminus V_\varepsilon)} = \|q\|_{L^1(V_\varepsilon\smallsetminus U_\varepsilon)} - \|q\|_{L^1(U_\varepsilon\smallsetminus V_\varepsilon)} \le \|q\|_{L^1(V_\varepsilon\smallsetminus U_\varepsilon)}.$$

We deduce from the following lemma that $f_*q \neq q$. □

Lemma 15. *For any $q \in \mathcal{Q}_f$,*

$$\lim_{\varepsilon\to 0} \|q\|_{L^1(V_\varepsilon\smallsetminus U_\varepsilon)} = 0.$$

Proof. Since $\zeta\circ f^{\circ p} = \zeta + \mathcal{O}(\zeta^3)$, there is a constant κ_1 such that

$$|\zeta\circ f^{\circ p}| \ge |\zeta| - \kappa_1|\zeta|^3.$$

Since q has at worst a double pole at x, there is a constant κ_2 such that for $|\zeta|$ small enough

$$|q| \le \kappa_2 \frac{|\mathrm{d}\zeta^2|}{|\zeta|^2}.$$

Note that for $\varepsilon > 0$ small enough,

$$V_\varepsilon\smallsetminus U_\varepsilon = \bigl\{|\zeta| < \varepsilon\bigr\}\smallsetminus\bigl\{|\zeta\circ f^{\circ p}| < \varepsilon\bigr\} \subset \bigl\{\varepsilon - \kappa_1\varepsilon^3 \le |\zeta| < \varepsilon\bigr\}.$$

Thus,

$$0 \le \|q\|_{L^1(V_\varepsilon\smallsetminus U_\varepsilon)} \le \int_0^{2\pi}\int_{\varepsilon-\kappa_1\varepsilon^3}^{\varepsilon} \kappa_2 \frac{r\mathrm{d}r\mathrm{d}t}{r^2} = 2\pi\kappa_2 \ln\frac{1}{1-\kappa_1\varepsilon^2} \underset{\varepsilon\to 0}{\longrightarrow} 0. \qquad □$$

5. Linear independence

We may now complete the proof that $\mathrm{d}A|_{\mathrm{T}_\lambda\Lambda}$ and $\mathrm{d}B|_{\mathrm{T}_\lambda\Lambda}$ are linearly independent. According to Proposition 4, for all $v \in \mathrm{T}_\lambda\Lambda$,

$$\mathrm{d}A(v) = \bigl\langle q_A, \Theta_f(v)\bigr\rangle_{\langle x\rangle} \quad\text{and}\quad \mathrm{d}B(v) = \bigl\langle q_B, \Theta_f(v)\bigr\rangle_{\langle x\rangle}.$$

According to Lemma 6,

$$\mathcal{T}_f = \mathrm{Im}(\Theta_f) \oplus \mathrm{Vect}(\tau - f^*\tau).$$

Showing that $\mathrm{d}A|_{\mathrm{T}_\lambda\Lambda}$ and $\mathrm{d}B|_{\mathrm{T}_\lambda\Lambda}$ are linearly independent therefore amounts to proving that for all $q \in \mathcal{Q}_f\smallsetminus\{0\}$, there exists $\vartheta \in \mathcal{T}_f$ such that $\langle q, \vartheta\rangle_{\langle x\rangle} \neq 0$.

5.1. **Guiding vector fields.** Set $Z := \mathcal{C}_f \cup \mathcal{V}_f$ and denote by $\mathrm{T}Z$ the space of maps $\xi : Z \to \mathrm{T}\mathbb{T}$ satisfying $\xi(z) \in \mathrm{T}_z\mathbb{T}$ for all $z \in \mathbb{Z}$.

Lemma 16. *For any $\vartheta \in \mathcal{T}_f$, there exists a unique $\xi_\vartheta \in \mathrm{T}Z$ such that for any vector field ξ, defined and holomorphic near Z with $\xi(z) = \xi_\vartheta(z)$ for $z \in Z$, the vector field $\vartheta + \xi - f^*\xi$ is holomorphic and vanishes along $\mathcal{C}_f$.*

Proof. Fix $\vartheta \in \mathcal{T}_f$. Let us first prove the uniqueness of $\xi_\vartheta \in \mathrm{T}Z$. Assume ξ_1 and ξ_2 are two vector fields, defined and holomorphic near Z, such that $\vartheta + \xi_1 - f^*\xi_1$ and $\vartheta + \xi_2 - f^*\xi_2$ are holomorphic near $\mathcal{C}_f$. Then, $(\xi_1 - \xi_2) - f^*(\xi_1 - \xi_2)$ is holomorphic and vanishes along $\mathcal{C}_f$. As a consequence, $\mathrm{D}f \circ (\xi_1 - \xi_2) - (\xi_1 - \xi_2) \circ f$ vanishes on $\mathcal{C}_f$. Since $\mathrm{D}f \circ (\xi_1 - \xi_2)$ vanishes on $\mathcal{C}_f$, this forces $\xi_1 - \xi_2$ to vanish on $\mathcal{V}_f$. In that case, $f^*(\xi_1 - \xi_2)$ vanishes on $\mathcal{C}_f$ and so, $\xi_1 - \xi_2$ vanishes on $\mathcal{C}_f$. This shows the uniqueness of $\xi_\vartheta \in \mathrm{T}Z$.

This also proves that if $\vartheta + \xi - f^*\xi$ is holomorphic and vanishes along $\mathcal{C}_f$ for some vector field ξ, defined and holomorphic near Z with $\xi(z) = \xi_\vartheta(z)$ for $z \in Z$, then $\vartheta + \xi - f^*\xi$ is holomorphic and vanishes along $\mathcal{C}_f$ for any vector field ξ, defined and holomorphic near Z with $\xi(z) = \xi_\vartheta(z)$ for $z \in Z$.

Let us now prove the existence of $\xi_\vartheta \in \mathrm{T}Z$. Note that $\mathrm{D}f \circ \vartheta$ is a map from $\mathbb{T} \smallsetminus \mathcal{C}_f$ to the tangent bundle $\mathrm{T}\mathbb{T}$. Note that it is not a vector field since for $z \in \mathbb{T}$, the vector $\mathrm{D}f \circ \vartheta(z)$ belongs to $\mathrm{T}_{f(z)}\mathbb{T}$. However, since ϑ has at worst simple poles along $\mathcal{C}_f$ and since $\mathrm{D}f$ vanishes on $\mathcal{C}_f$, the map $\mathrm{D}f \circ \vartheta$ extends holomorphically to $\mathbb{T}$. Set

$$\xi_\vartheta\big(f(c^\pm)\big) := \mathrm{D}f \circ \vartheta(c^\pm).$$

Next, let ξ be any vector field, defined and holomorphic near $\mathcal{V}_f$, coinciding with ξ_ϑ on $\mathcal{V}_f$. Then, $\vartheta - f^*\xi$ is holomorphic near $\mathcal{C}_f$ and we may set

$$\xi_\vartheta(c^\pm) := (\vartheta - f^*\xi)(c^\pm). \qquad \square$$

Recall that $Y := \{c^+\} \cup \mathcal{V}_f \subset Z$. It will be convenient to consider the linear map $\Xi_f : \mathcal{T}_f \to \mathrm{T}Y$ defined by

$$\Xi_f(\vartheta) = \xi_\vartheta|_Y.$$

Lemma 17. *The map $\Xi_f : \mathcal{T}_f \to \mathrm{T}Y$ is an isomorphism.*

Proof. Since the dimensions of $\mathcal{T}_f$ and $\mathrm{T}Y$ are both equal to three, it is enough to show that the map is injective. Assume $\vartheta \in \mathcal{T}_f$ and ξ_ϑ vanishes on $\{c^+\} \cup \mathcal{V}_f$. Let ξ be a vector field, defined and holomorphic near Z, which coincides with ξ_ϑ on Z. We may assume that ξ identically vanishes near $\{c^+\} \cup \mathcal{V}_f$. Then, $\vartheta + f^*\xi - \xi = \vartheta - \xi$ is holomorphic and vanishes on $\mathcal{C}_f$. This shows that ϑ is holomorphic near $\mathcal{C}_f$ and vanishes at c^+. As a consequence, ϑ is globally holomorphic on $\widehat{\mathbb{T}}$, and vanishes at three points: c^+, $+\mathrm{i}\infty$ and $-\mathrm{i}\infty$. So, $\vartheta = 0$. $\square$

5.2. **From the cycle to the critical set.** We may now transfer the local computations done near the cycle $\langle x \rangle$ to local computations done near the critical set Y.

Lemma 18. *For all $\vartheta \in \mathcal{T}_f$ and all $q \in \mathcal{Q}_f$,*

$$\langle q, \vartheta \rangle_{\langle x \rangle} = \big\langle \nabla_f q, \Xi_f(\vartheta) \big\rangle_Y.$$

Proof. Let ξ be a vector field, defined and holomorphic near Z, coinciding with $\xi_\vartheta := \Xi_f(\vartheta)$ on Z. Then, $\vartheta + \xi - f^*\xi$ is holomorphic near $\mathcal{C}_f$. In addition, since $\nabla_f q$ is holomorphic near c^-,

$$\begin{aligned}\langle \nabla_f q, \xi_\vartheta\rangle_Y = \langle q - f_* q, \xi\rangle_Z &= \langle q, \xi\rangle_{\mathcal{C}_f} - \langle f_* q, \xi\rangle_{\mathcal{V}_f}\\ &= \langle q, \xi\rangle_{\mathcal{C}_f} - \langle q, f^*\xi\rangle_{\mathcal{C}_f}\\ &= \langle q, -\vartheta\rangle_{\mathcal{C}_f} = \langle q, \vartheta\rangle_{\langle x\rangle}.\end{aligned}$$

In the second line, we used the fact that the only poles of $q \otimes f^*\xi$ in $f^{-1}(\mathcal{V}_f)$ belong to $\mathcal{C}_f$. For the last equality, we used the fact that $q \otimes \vartheta$ is a globally meromorphic 1-form on $\widehat{\mathbb{T}}$, whose poles are contained in $\mathcal{C}_f \cup \langle x\rangle$, and that the sum of all residues of a globally meromorphic 1-form on a compact Riemann surface is 0. □

5.3. **Completion of the proof.** Assume by contradiction that $\mathrm{d}A|_{\mathrm{T}_\lambda\Lambda}$ and $\mathrm{d}B|_{\mathrm{T}_\lambda\Lambda}$ are not linearly independent. Then, there is a $q \in \mathcal{Q}_f \smallsetminus \{0\}$ such that for all $\vartheta \in \mathcal{T}_f$,

$$0 = \langle q, \vartheta\rangle_{\langle x\rangle} = \big\langle \nabla_f q, \Xi_f(\vartheta)\big\rangle_Y.$$

According to Lemma 17, the map $\Xi_f : \mathcal{T}_f \to \mathrm{T}Y$ is an isomorphism. In particular, it is surjective. It follows that for all $\xi \in \mathrm{T}Y$,

$$\langle \nabla_f q, \xi\rangle_Y = 0.$$

As a consequence, $\nabla_f q$ is holomorphic near Y and thus, has at most three simple poles at c^-, $+\mathrm{i}\infty$ and $-\mathrm{i}\infty$. A non zero quadratic differential on $\widehat{\mathbb{T}}$ has at least four poles, counting multiplicities. Thus, $\nabla_f q = 0$.

According to Proposition 14, the map $\nabla_f : \mathcal{Q}_f \to \mathcal{Q}^1(\mathbb{T}; Y)$ is injective. It follows that $q = 0$. Contradiction.

This completes the proof of Theorem 1.

Appendix A. Quadratic differentials

A.1. **Meromorphic quadratic differentials.** A *quadratic differential* on a Riemann surface U is a section of the square of the cotangent bundle $\mathrm{T}^*U \otimes \mathrm{T}^*U$. We shall usually think of a quadratic differential q as a field of quadratic forms. In particular, if ϑ is a vector field on U and ϕ is a function on U, then $q(\vartheta)$ is a function on U and $q(\phi\vartheta) = \phi^2 q(\vartheta)$.

If $\zeta : U \to \mathbb{C}$ is a coordinate, we shall use the notation $(\mathrm{d}\zeta)^2 = \mathrm{d}\zeta \otimes \mathrm{d}\zeta$ - not be confused with 1-form $\mathrm{d}(\zeta^2)$. Then, a quadratic differential q on U is of the form $q = \phi\ (\mathrm{d}\zeta)^2$ for some function ϕ. We say that q is meromorphic on U if ϕ is meromorphic on U. In that case, the order of q at a point $x \in U$ is $\mathrm{ord}_x q := \mathrm{ord}_x\phi$, i.e., 0 if ϕ is holomorphic and does not vanish at x, $k \geq 1$ if ϕ has a zero of multiplicity k at x, and $-k \leq -1$ if ϕ has a pole of multiplicity k at x.

A.2. **Pullback.** The derivative $\mathrm{D}f : \mathrm{T}U \to \mathrm{T}V$ of a holomorphic map $f : U \to V$ naturally induces a pullback map f^* from quadratic differentials on V to quadratic differentials on U:

$$f^*q := q \circ \mathrm{D}f.$$

Lemma 19. *If $f : (U, x) \to (V, y)$ is holomorphic at x, and q is meromorphic at $y = f(x)$, then*

$$2 + \mathrm{ord}_x(f^*q) = \deg_x f \cdot (2 + \mathrm{ord}_y q).$$

Proof. Choose local coordinates $z : (U, x) \to (\mathbb{C}, 0)$ and $w : (V, y) \to (\mathbb{C}, 0)$ such that $w \circ f = z^k$, with $k := \deg_x f$. If $q = \phi\ (\mathrm{d}w)^2$, then $f^*q = \phi \circ f \cdot (kz^{k-1}\mathrm{d}z)^2$. Thus,

$$2 + \mathrm{ord}_x(f^*q) = 2 + \mathrm{ord}_x(\phi \circ f) + (2k - 2) = 2k + k \cdot \mathrm{ord}_y\phi = k \cdot (2 + \mathrm{ord}_y q). \quad \square$$

A.3. **Pushforward for finite degree covering maps.** Assume $f : U \to V$ is a finite degree covering map. If q is a quadratic differential on U, we define a quadratic differential f_*q on V by

$$f_*q := \sum_{g \text{ inverse branch of } f} g^*q.$$

If q is holomorphic on U, then f_*q is holomorphic on V.

Lemma 20. *Assume $U := \widehat{U} \setminus \{x\}$ and $V := \widehat{V} \setminus \{y\}$ are punctured disks, $f : U \to V$ is a covering map ramifying at x with local degree $\deg_x f$ and q is meromorphic at x. Then, f_*q has at worst a pole at y and*

$$2 + \mathrm{ord}_y(f_*q) \geq \frac{2 + \mathrm{ord}_x q}{\deg_x f}.$$

Proof. The group of deck transformations of $f : U \to V$ is a cyclic group of order $\deg_x f$. Note that

$$f^*(f_*q) = \sum_{h \text{ deck transformation of } f} h^*q,$$

and $\mathrm{ord}_x h^*q = \mathrm{ord}_x q$ for all deck transformations h, so that

$$\mathrm{ord}_x f^*(f_*q) \geq \mathrm{ord}_x q.$$

Then,

$$2 + \mathrm{ord}_y(f_*q) = \frac{2 + \mathrm{ord}_x f^*(f_*q)}{\deg_x f} \geq \frac{2 + \mathrm{ord}_x q}{\deg_x f}. \quad \square$$

A.4. **Integrable quadratic differentials.** If q is a quadratic differential on U, we denote by $|q|$ the positive $(1,1)$-form on U defined by its action on pairs of vector fields $(\vartheta_1, \vartheta_2)$ as follows:

$$|q|(\vartheta_1, \vartheta_2) := \frac{1}{4}\big|q(\vartheta_1 - \mathrm{i}\vartheta_2)\big| - \frac{1}{4}\big|q(\vartheta_1 + \mathrm{i}\vartheta_2)\big|.$$

If $\zeta : U \to \mathbb{C}$ is a coordinate and $q = \phi\ (\mathrm{d}\zeta)^2$, then

$$|q| = |\phi| \cdot \frac{\mathrm{i}}{2}\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}.$$

We shall say that q is *integrable* on U if

$$\|q\|_{L^1(U)} := \int_U |q| < \infty.$$

Note that q is integrable in a neighborhood of a pole if and only if the pole is simple. If $f : U \to V$ is an isomorphism and q is an integrable quadratic differential on V, then f^*q is integrable on U and $\|f^*q\|_{L^1(U)} = \|q\|_{L^1(V)}$.

A.5. **Pushforward for infinite degree covering maps.** Assume $f : U \to V$ is an infinite degree covering map. If q is an integrable quadratic differential on U, we may still define

$$f_*q := \sum_{g \text{ inverse branch of } f} g^*q.$$

Indeed, the series converges in L^1_{loc} since if $V' \subset V$ is a topological disk, so that the inverse branches $g : V' \to U$ of f are defined on V', and if $U' := f^{-1}(V')$, then

$$\sum \|g^*q\|_{L^1(V')} = \|q\|_{L^1(U')} \leq \|q\|_{L^1(U)}.$$

The limit of a sequence of holomorphic functions converging in L^1_{loc} is itself holomorphic. It follows that if q is holomorphic on U, then f_*q is holomorphic on V.

A.6. **The Contraction Principle.**

Proposition 21. *Let $f : U \to V$ be a covering map and let q be a holomorphic integrable quadratic differential on U. Then, $\|f_*q\|_{L^1(V)} \leq \|q\|_{L^1(U)}$ and equality holds if and only if either $q = 0$, or the degree of f is finite and $f^*(f_*q) = \deg(f)\cdot q$.*

Proof. The proof is an immediate application of the triangle inequality: for any topological disk $V' \subset V$, we have

$$\int_{V'} |f_*q| = \int_{V'} \Big|\sum g^*q\Big| \leq \int_{V'} \sum |g^*q| = \sum \int_{V'} |g^*q| = \int_{f^{-1}(V')} |q|,$$

where the sums range over the inverse branches $g : V' \to U$ of f. It follows that

$$\int_V |f_*q| \leq \int_{f^{-1}(V)} |q| = \int_U |q|$$

with equality if and only if for all inverse branches g of f, we have $g^*q = \psi_g\ f_*q$ for some function $\psi_g : V' \to [0,1]$ satisfying $\sum_g \psi_g = 1$. Setting $\phi\big(g(y)\big) := \psi_g(y)$, we see that $q = \phi\ f^*(f_*q)$ for some function $\phi : U \to [0,1]$. Since q and $f^*(f_*q)$ are holomorphic, either $q = 0$, or the function ϕ is constant, let us say equal to $c \in [0,1]$. Since $\sum_g \psi_g = 1$, we have that $\deg(f)\cdot c = 1$, which forces the degree of f to be finite with $f^*(f_*q) = \deg(f)\cdot q$. □

A.7. **Pairing quadratic differentials and vector fields.** If q is a quadratic differential on U and ϑ is a vector field on U, we may consider the 1-form $q \otimes \vartheta$ defined on U by its action on vector fields τ:

$$q \otimes \vartheta(\tau) = \frac{1}{4}\big(q(\vartheta+\tau) - q(\vartheta-\tau)\big).$$

Note that if $q = \phi\ (\mathrm{d}\zeta)^2$ and $\vartheta = \psi\ \mathrm{d}/\mathrm{d}\zeta$, then $q \otimes \vartheta = \phi\psi\ \mathrm{d}\zeta$.

If $x \in U$, and if ϑ and q are meromorphic on U, we set

$$\langle q, \vartheta\rangle_x := \text{residue}(q \otimes \vartheta, x).$$

If q has at worst a simple pole at x, then $\langle q, \vartheta\rangle_x$ only depends on $\theta := \vartheta(0)$, and we use the notation

$$\langle q, \theta\rangle_x := \langle q, \vartheta\rangle_x\,.$$

Lemma 22. *Let $U := \widehat{U} \smallsetminus \{x\}$ and $V := \widehat{V} \smallsetminus \{y\}$ be punctured disks, let $f : U \to V$ be a covering map ramifying at x, let q be a meromorphic quadratic differential on $\widehat{U}$ and let ϑ be a meromorphic vector field on U. Then*

$$\langle f_* q, \vartheta \rangle_y = \langle q, f^* \vartheta \rangle_x .$$

Proof. Let $\gamma \subset V$ be a loop around y with basepoint a. Then

$$\int_\gamma (f_* q) \otimes \vartheta = \sum_g \int_{\gamma \smallsetminus \{a\}} (g^* q) \otimes \vartheta = \sum_g \int_{g(\gamma \smallsetminus \{a\})} q \otimes f^* \vartheta = \int_{f^{-1}(\gamma)} q \otimes f^* \vartheta,$$

where the sum ranges over the inverse branches g of f defined on $\gamma \smallsetminus \{a\}$. □

References

[A65] V. I. Arnold, *Small denominators I, Mappings of the Circumference onto Itself*, Amer. Math. Soc. Translations **46** (1965), 213?284.

[CFG15] J. Canela & N. Fagella& A. Garijo, *On a family of rational perturbations of the doubling map*, J. Difference Equ. Appl., **21** (2015), 715–741.

[CFG16] J. Canela & N. Fagella& A. Garijo, *Tongues in degree 4 Blaschke products*, Nonlinearity **29** (2016), 3464–3495.

[D10] A. Dezotti, *Connectedness of the Arnold tongues for double standard maps*, Proc. Amer. Math. Soc. **138** (2010), 3569–3583.

[E1] A. Epstein, *Towers of finite type complex analytic maps*, Ph. D. Thesis, CUNY, 1993.

[E2] A.L. Epstein, *Infinitesimal Thurston Rigidity and the Fatou-Shishikura Inequality*, https://arxiv.org/abs/math/9902158.

[HK91] J.K. Hale & H. Koçak, *Dynamics and bifurcations*, Texts Appl. Math. 3, Springer, New York (1991).

[Mi92] J. Milnor, *Remarks on iterated cubic maps*, Exp. Math. **1** (1992), 5–24.

[MR07] M. Misiurewicz & A. Rodrigues, *Double Standard Maps*, Commun. Math. Phys. **273** (2007), 37–65.

[MR08] M. Misiurewicz & A. Rodrigues, *On the Tip of the Tongue*, J. Fixed Point Theory Appl. **3** (2008), 131–141.

[MR11] M. Misiurewicz & A. Rodrigues, *Non-Generic Cusps*, Trans. Amer. Math. Soc. **363** (2011), 3553–3572.

[NS03] S. Nakane & D. Schleicher, *On multicorns and unicorns I: Antiholo- morphic dynamics, hyperbolic components and real cubic polynomials*, Int. J. Bif. Chaos **13** 10 (2003), 2825–2844.

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