LIMIT CYCLES OF PIECEWISE DIFFERENTIAL EQUATIONS ON THE CYLINDER

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Abstract. We consider the piecewise differential equations of the form

$$\begin{pmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{pmatrix} = \begin{cases} a_0(t) + a_1(t)x_1 + ... + a_n(t)x_1^n, & \text{if} \quad 0 \leqslant t \leqslant \pi, \\ b_0(t) + b_1(t)x_2 + ... + b_m(t)x_2^m, & \text{if} \quad \pi \leqslant t \leqslant 2\pi, \end{cases}$$

where $a_0(t), a_1(t), ..., a_n(t)$ and $b_0(t), b_1(t), ..., b_m(t)$ are 2π-periodic functions in the variable t, and we study the number of limit cycles of such equations on the cylinder. In this way we give exact bounds for the maximum number of limit cycles that the piecewise differential equations have in function of n and m.

1. Introduction and statements of the main results

Pugh proposed the following problem (see [7]). Let $a_0, a_1, ..., a_n : [0, 2\pi] \to \mathbb{R}$ be analytic functions and consider the differential equation

$$\frac{dx}{dt} = a_0(t) + a_1(t)x + ... + a_n(t)x^n, \qquad 0 \leqslant t \leqslant 2\pi. \tag{1}$$

A solution $x(t)$ of (1) is called a *closed solution* or a *periodic solution* if it is defined in the interval $[0, 2\pi]$ and $x(0) = x(2\pi)$. The adjectives closed and periodic are motivated by the case where $a_0, ..., a_n$ are 2π-periodic, in which (1) can be considered in the cylinder and the closed solutions really correspond to closed orbits in the cylinder. Closed orbits in polynomial planar differential systems can be isolated or belong to an annulus of periodic orbits. In the isolated case they are called *limit cycles*. So the problem is this: *Is there a bound on the number of limit cycles of* (1)*?*

We note that the differential equation (1) with $n = 1$ (resp. $n = 2$) is a *linear equation* (resp. a *Riccati equation*), and when $n = 3$, (1) is called an *Abel equation*. It is well known that linear (resp. Riccati) equations have either a continuum of periodic solutions or at most 1 (resp. 2) periodic solutions. For the Abel equation it was proved that for any k there exist equations (1) with $a_i(t)$ trigonometric 2π-periodic polynomials, having at least k limit cycles. A similar result holds for $n > 3$, see for more details [7, 9, 1].

In this paper we deal with the piecewise differential equation

$$\begin{pmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{pmatrix} = \begin{cases} a_0(t) + a_1(t)x_1 + ... + a_n(t)x_1^n, & \text{if} \quad 0 \leqslant t \leqslant \pi, \\ b_0(t) + b_1(t)x_2 + ... + b_m(t)x_2^m, & \text{if} \quad \pi \leqslant t \leqslant 2\pi, \end{cases} \tag{2}$$

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where $a_0(t), a_1(t), ..., a_n(t)$ and $b_0(t), b_1(t), ..., b_m(t)$ are 2π-periodic functions in the variable t. If we consider that $x_1(t) = \phi_1(t, \rho)$ is the solution of

$$\frac{dx_1}{dt} = a_0(t) + a_1(t)x_1 + ... + a_n(t)x_1^n,$$

such that $\phi_1(0, \rho) = \rho$, and let $x_2(t) = \phi_2(t, \rho)$ be the solution of

$$\frac{dx_2}{dt} = b_0(t) + b_1(t)x_2 + ... + b_m(t)x_2^m,$$

such that $\phi_2(2\pi, \rho) = \rho$, then a solution of the piecewise differential equation (2) satisfying $\phi_1(\pi, \rho) = \phi_2(\pi, \rho)$ is called a *closed solution* or a *periodic solution*. Like in the smooth case the adjectives closed and periodic are motivated by the case where $a_0, a_1, ..., a_n$ and $b_0, b_1, ..., b_m$ are 2π-periodic, in which (2) can be considered in the cylinder and the closed solutions really correspond to periodic orbits in the cylinder. Then the function $\Pi(\rho) := \phi_1(\pi, \rho) - \phi_2(\pi, \rho)$ plays a great role to find the number of closed orbits. Thus the zeros of $\Pi(\rho)$ correspond to closed orbits of (2), and their isolated zeros provide initial conditions for the *limit cycles* of equation (2). Then the problem is this: *Is there a bound on the number of limit cycles of* (2)*?*

We note that the problem here studied is an extension of the Pugh's problem to piecewise differential equations on the cylinder.

Then $H(n, m)$ denotes the maximum number of limit cycles that some piecewise differential equations (2) can exhibit. The number $H(n, m)$ is usually called the *Hilbert number* of piecewise differential equation (2). The Hilbert number for polynomial differential systems in the plane is the main objective of the well known 16th Hilbert problem, see [4, 5, 6]. The Hilber number also has been considered for piecesise differential linear centers in the plane, see for instance [8].

The main results of this paper are stated in the following theorems.

Theorem 1. *The piecewise differential equation*

$$\begin{pmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{pmatrix} = \begin{cases} a_0(t) + a_1(t)x_1, & \text{if} \quad 0 \leqslant t \leqslant \pi, \\ b_0(t) + b_1(t)x_2, & \text{if} \quad \pi \leqslant t \leqslant 2\pi, \end{cases} \tag{3}$$

where $a_0(t)$, $a_1(t)$, $b_0(t)$ and $b_1(t)$ are 2π-periodic functions in the variable t, has at most one limit cycle on the cylinder. In particular, $H(1, 1) = 1$.

Theorem 2. *The following piecewise differential equations*

$$\begin{pmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{pmatrix} = \begin{cases} a_0(t) + a_1(t)x_1, & \text{if} \quad 0 \leqslant t \leqslant \pi, \\ b_0(t) + b_1(t)x_2 + b_2(t)x_2^2, & \text{if} \quad \pi \leqslant t \leqslant 2\pi, \end{cases} \tag{4}$$

where $a_i(t)$ and $b_i(t)$ are 2π-periodic functions in the variable t, have at most two limit cycles on the cylinder. In particular, $H(1, 2) = 2$.

Theorem 3. *There are piecewise differential equations of the form*

$$\begin{pmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{pmatrix} = \begin{cases} a_1(t)x_1, & \text{if} \quad 0 \leqslant t \leqslant \pi, \\ b_2(t)x_2^2 + \varepsilon b_3(t)x_2^3, & \text{if} \quad \pi \leqslant t \leqslant 2\pi, \end{cases} \tag{5}$$

where $a_i(t)$ and $b_i(t)$ are 2π-periodic functions in the variable t, having at least k limit cycles on the cylinder for all positive integer k. In particular, $H(1,3)=\infty$.

We present a corollary of Theorem 3 in the following.

Corollary 4. *The piecewise differential equation* (2) *with* $\max\{n,m\}\geq 3$ *has* $H(n,m)=\infty$.

2. Proof of the main results

Here we will prove the main results of the paper that are stated in Theorems 1-3 and Corollary 4.

Proof of Theorem 1. Consider the piecewise differential equation (3). The solution of $dx_1/dt = a_0(t)+a_1(t)x_1$ satisfying $x_1(0)=\rho$ is

$$x_1=\phi_1(t,\rho)=\left(\int_0^t a_0(s)e^{-K_1(s)}ds+\rho\right)e^{K_1(t)}, \qquad K_1(s)=\int_0^s a_1(w)dw,$$

and the solution of $dx_2/dt=b_0(t)+b_1(t)x_2$ satisfying $x_2(2\pi)=\rho$ is

$$x_2=\phi_2(t,\rho)=\left(\int_{2\pi}^t b_0(s)e^{-K_2(s)}ds+\rho\right)e^{K_2(t)}, \qquad K_2(s)=\int_{2\pi}^s b_1(w)dw.$$

Therefore we have the function

$$\Pi(\rho)=\phi_1(\pi,\rho)-\phi_2(\pi,\rho)=m_1-m_2+(n_1-n_2)\rho,$$

where

$$\begin{aligned} m_1=&e^{K_1(\pi)}\int_0^\pi a_0(s)e^{-K_1(s)}ds, \qquad n_1=e^{K_1(\pi)},\\ m_2=&e^{K_2(\pi)}\int_{2\pi}^\pi b_0(s)e^{-K_2(s)}ds, \qquad n_2=e^{K_2(\pi)}. \end{aligned}$$

Then the initial conditions for periodic orbits of equation (3) correspond to the zeros ρ of the equation $\Pi(\rho)=0$. This equation has 0, 1 or a continuum of solutions according to the values of m_1, n_1, m_2 and n_2. We conclude that piecewise differential equations (3) have at most 1 limit cycle.

It is easy to construct examples of equations (3) with 1 limit cycle. For example, the piecewise differential equation

$$\begin{pmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{pmatrix} = \begin{cases} 2\,\sin t+\sin t\, x_1, & \text{if} \quad 0\leqslant t\leqslant \pi,\\ 2\,\sin t - x_2, & \text{if} \quad \pi\leqslant t\leqslant 2\pi, \end{cases} \tag{6}$$

has a unique limit cycle. In this case the solution of $\dot{x}_1=2\,\sin t+\sin t\, x_1$ satisfying $x_1(0)=\rho$ is

$$x_1=\phi_1(t,\rho)=-2+(2+\rho)\mathrm{e}^{1-\cos t},$$

and the solution of $\dot{x}_2=2\,\sin t-x_2$ satisfying $x_2(2\pi)=\rho$ is

$$x_2=\phi_2(t,\rho)=\sin t-\cos t+(1+\rho)\mathrm{e}^{2\pi-t}.$$

Then the solution of $\Pi(\rho)=x_1(\pi)-x_2(\pi)=0$ gives us

$$\rho=-\frac{\mathrm{e}^{\pi-2}+3\,\mathrm{e}^{-2}-2}{\mathrm{e}^{\pi-2}-1}.$$

We know that the number of simple zeros of $\Pi(\rho)$ is equivalent with the number of limit cycles of the piecewise differential equation (6). $\square$

Proof of Theorem 2. Consider the piecewise differential equation (4). On the half cylinder with $t \in [0, \pi]$ we have a linear differential system and on the half cylinder with $t \in [\pi, 2\pi]$ we have a Riccati differential system.

Suppose that we have a periodic solution $x(t) = x_{1q}(t)|_{t\in[0,\pi]} \cup x_{2q}(t)|_{t\in[\pi,2\pi]}$. Then doing the change of variable $x_2 \to y$ where

$$y = \frac{1}{x_2 - x_{2q}(t)},$$

the differential equation (4) with $t \in [\pi, 2\pi]$ is written as

$$\frac{dy}{dt} = -b_2(t) - (2b_2(t)x_{2q}(t) + b_1(t))y,$$

which is a linear differential equation. By computing its solution and undoing the change of variables we see that the solution of (4) with $t \in [\pi, 2\pi]$ satisfying $x_2(2\pi) = \rho$ is written as

$$x_2(t) = \phi_2(t, \rho) = \frac{A(t) + B(t)\rho}{C(t) + D(t)\rho},$$

where

$$A(t) = N(t)\, x_{2q}(t)\, x_{2q}(2\pi) + \mathrm{e}^{-M(2\pi)} x_{2q}(t) - \mathrm{e}^{-M(t)} x_{2q}(2\pi), \quad B(t) = -N(t)\, x_{2q}(t) + \mathrm{e}^{-M(t)},$$

$$C(t) = N(t)\, x_{2q}(2\pi) + \mathrm{e}^{-M(2\pi)}, \quad D(t) = -N(t), \quad N(t) = \int_t^{2\pi} -b_2(s)\mathrm{e}^{-M(s)} ds,$$

$$M(s) = \int_0^s -(2b_2(z)x_{2q}(z) + b_1(z))dz.$$

The solution of (4) with $t \in [0, \pi]$ satisfying $x_1(0) = \rho$ is

$$x_1(t) = \phi_1(t, \rho) = H(t) + \rho L(t),$$

where

$$H(t) = \int_0^t a_0(s)e^{-K_1(s)} ds, \quad L(t) = e^{K_1(t)}, \quad K_1(s) = \int_0^s a_1(w)dw.$$

Therefore the zeros of ρ that are obtained from the equation

$$\Pi(\rho) = \phi_1(\pi, \rho) - \phi_2(\pi, \rho) = h + \rho\ell - \frac{a + b\rho}{c + d\rho} = 0,$$

correspond to the periodic orbits of the piecewise differential equation (4), where

$$h = H(\pi), \quad \ell = L(\pi), \quad a = A(\pi), \quad b = B(\pi), \quad c = C(\pi), \quad d = D(\pi).$$

This equation has 0, 1, 2 or a continuum of solutions according to the values of the parameters h, ℓ, a, b, c, d. We conclude that the piecewise differential equation (4) has at most 2 limit cycles.

Now we construct an example with two limit cycles for this case. Consider the following piecewise differential equation

$$\begin{pmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{pmatrix} = \begin{cases} \cos t, & \text{if } \ 0 \leqslant t \leqslant \pi, \\ 1 + \sin t + \cos t - \cos^2 t - (1 + 2\sin t)x_2 + x_2^2, & \text{if } \ \pi \leqslant t \leqslant 2\pi. \end{cases} \tag{7}$$

The solution of equation $\dot{x}_1 = \cos t$ with $t \in [0, \pi]$ satisfying $x_1(0) = \rho$ is $x_1(t) = \rho + \sin t$. The solution of

$$\dot{x}_2 = 1 + \sin t + \cos t - \cos^2 t - (1 + 2\sin t)x_2 + x_2^2,$$

with $x_2(2\pi) = \rho$ is

$$x_2(t) = \frac{-\mathrm{e}^t \sin t + (\mathrm{e}^t \sin t - \mathrm{e}^{2\pi} \sin t - \mathrm{e}^{2\pi})\,\rho}{-\mathrm{e}^t + (\mathrm{e}^t - \mathrm{e}^{2\pi})\,\rho}.$$

Therefore the zeros of equation

$$x_1(\pi) - x_2(\pi) = \rho + \frac{\rho\, \mathrm{e}^{2\pi}}{-\mathrm{e}^{\pi} + (\mathrm{e}^{\pi} - \mathrm{e}^{2\pi})\,\rho} = 0,$$

are $\rho = 0, 1$. It shows that the piecewise differential equation (7) has two limit cycles. In fact the limit cycles of piecewise differential equation (7) are $x(t) = \sin t|_{t\in[0,2\pi]}$ and $x(t) = 1 + \sin t|_{t\in[0,2\pi]}$.

□

Before starting to prove Theorem 3 we give some notations that will be used in the paper, for more details, [2, 3] is referred.

Definition 5. *Let $f_0, f_1, ..., f_{n-1}$ be real analytic functions on some open interval $\mathbb{I}$ of $\mathbb{R}$. Then*

(i) *$(f_0, f_1, ..., f_{n-1})$ is called a Chebyshev system (T-system) on $\mathbb{I}$ if any nontrivial linear combination*

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + ... + \alpha_{n-1} f_{n-1}(x)$$

has at most $n-1$ isolated zeros on $\mathbb{I}$.

(ii) *$(f_0, f_1, ..., f_{n-1})$ is called a complete Chebyshev system (CT-system) on $\mathbb{I}$ if $(f_0, f_1, ..., f_{k-1})$ is a T-system for all $k = 1, 2, ..., n$.*

(iii) *$(f_0, f_1, ..., f_{n-1})$ is called an extended complete Chebyshev system (ECT-system) on $\mathbb{I}$ if for all $k = 1, 2, ..., n$, any nontrivial linear combination*

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + ... + \alpha_{k-1} f_{k-1}(x)$$

has at most $k-1$ isolated zeros on $\mathbb{I}$ counting multiplicity.

Definition 6. *Let $f_0, f_1, ..., f_{k-1}$ be real analytic functions on some open interval $\mathbb{I}$ of $\mathbb{R}$. The Wronskian of $(f_0, f_1, ..., f_{k-1})$ at $x \in \mathbb{I}$ is*

$$W[f_0, f_1, ..., f_{k-1}](x) = \begin{vmatrix} f_0 & f_1 & \cdots & f_{k-1} \\ f_0' & f_1' & \cdots & f_{k-1}' \\ \vdots & \vdots & \vdots & \vdots \\ f_0^{(k-1)} & f_1^{(k-1)} & \cdots & f_{k-1}^{(k-1)} \end{vmatrix}.$$

The following relation between an extended complete Chebyshev space and their continuous Wronskians is well known.

Lemma 7. *The set $\{f_0, f_1, ..., f_{n-1}\}$ is an extended complete Chebyshev space on $\mathbb{I}$ if and only if, for each $k = 1, 2, ..., n$,*

$$W[f_0, f_1, ..., f_{k-1}](x) \neq 0 \quad \text{for all} \quad x \in \mathbb{I}.$$

The next result can be found in Theorem A in [2] that we will use this theorem to prove our results.

Theorem 8. *For any $n \in \mathbb{N}$ and any $\alpha \in \mathbb{R}\backslash\mathbb{Z}^-$, the ordered set of functions $(I_{0,\alpha}, I_{1,\alpha}, ..., I_{n,\alpha})$, defined by*

$$I_{k,\alpha}(y) := \int_a^b \frac{g^k(t)}{(1-yg(t))^\alpha} dt,$$

is an ECT-system on J. When $\alpha \in \mathbb{Z}^-$ it is an ECT-system on J if and only if $n \leqslant -\alpha$. In particular, when this set of functions is an ECT-system, any non-trivial function of the form

$$\Phi_\alpha(y) := \sum_{k=0}^{n} a_k I_{k,\alpha}(y), \tag{8}$$

with $a_k \in \mathbb{R}$, has at most n zeros in J counting multiplicities.

Proof of Theorem 3. Consider the piecewise differential equation

$$\begin{pmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{pmatrix} = \begin{cases} a_1(t)x_1, & \text{if} \quad 0 \leqslant t \leqslant \pi, \\ b_2(t)x_2^2, & \text{if} \quad \pi \leqslant t \leqslant 2\pi. \end{cases} \tag{9}$$

We compute the solution of

$$\frac{dx_1}{dt} = a_1(t)x_1,$$

satisfying $x_1(0) = \rho$ and we get

$$x_1 = \phi_1(t,\rho) = \rho\, e^{A_2(t)}, \quad \text{where} \quad A_2(t) = \int_0^t a_1(s)ds.$$

On the other hand the solution of

$$\frac{dx_2}{dt} = b_2(t)x_2^2, \tag{10}$$

satisfying $x_2(2\pi) = \rho$ is

$$x_2 = \phi_{20}(t,\rho) = \frac{\rho}{1-\rho B_2(t)}, \quad \text{where} \quad B_2(t) = \int_{2\pi}^t b_2(s)ds.$$

Imposing that $A_2(\pi) = B_2(\pi) = 0$ we obtain that the piecewise differential equation (9) has a continuum of periodic solutions in the neighborhood of $\rho = 0$.

In order to find an appropriate equation with at least k limit cycles we perturb (9) as follows

$$\begin{pmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{pmatrix} = \begin{cases} a_1(t)x_1, & \text{if} \quad 0 \leqslant t \leqslant \pi, \\ b_2(t)x_2^2 + \varepsilon\, b_3(t)x_2^3, & \text{if} \quad \pi \leqslant t \leqslant 2\pi, \end{cases} \tag{11}$$

where ε is small parameter. Let $\phi_2(t,\rho,\varepsilon)$ be the solution of

$$\frac{dx_2}{dt} = b_2(t)x_2^2 + \varepsilon b_3(t)x_2^3, \tag{12}$$

with initial condition $\phi_2(2\pi,\rho,\varepsilon) = \rho$. Then the solution (12) can be expressed for small $\varepsilon > 0$ in the form

$$\phi_2(t,\rho,\varepsilon) = \phi_{20}(t,\rho) + \varepsilon\psi(t,\rho) + O(\varepsilon^2),$$

where $\psi(t,\rho) = \partial\phi_2(t,\rho,\varepsilon)/\partial\varepsilon|_{\varepsilon=0}$. For simplicity we write

$$\phi_2(t,\rho,\varepsilon) = \phi_2 = \phi_{20} + \varepsilon\psi + O(\varepsilon^2), \quad b_2 = b_2(t), \quad \text{and} \quad b_3 = b_3(t).$$

Then we have

$$\begin{aligned}\frac{\partial}{\partial t}\left(\phi_{20}+\varepsilon\psi+O(\varepsilon^2)\right) =&b_2\left(\phi_{20}+\varepsilon\psi+O(\varepsilon^2)\right)^2+\varepsilon b_3\left(\phi_{20}+\varepsilon\psi+O(\varepsilon^2)\right)^3\\ =&b_2\left(\phi_{20}^2+2\,\varepsilon\phi_{20}\psi+O(\varepsilon^2)\right)+\varepsilon b_3(\phi_{20}^3+O(\varepsilon))\\ =&b_2\phi_{20}^2+\varepsilon(2b_2\phi_{20}\psi+b_3\phi_{20}^3)+O(\varepsilon^2).\end{aligned}$$

Therefore $\psi' = 2b_2\phi_{20}\psi + b_3\phi_{20}^3$, where $\psi' = \partial\psi(t,\rho)/\partial t$. Using that ϕ_{20} is a solution of (10), this differential equation is written as $\psi' = 2\phi_{20}'\psi/\phi_{20}+b_3\phi_{20}^3$, or equivalent $(\psi/\phi_{20}^2)' = b_3\phi_{20}$. Solving this differential equation we have

$$\psi(t,\rho)=\phi_{20}(t,\rho)^2\int_{2\pi}^{t} b_3(s)\phi_{20}(s,p)ds=\phi_{20}(t,\rho)^2\int_{2\pi}^{t}\frac{\rho b_3(s)}{1-\rho B_2(s)}ds.$$

Recall that the solution starting at ρ is a limit cycle of the perturbed differential equation if it is an isolated zero of function $D(\rho,\varepsilon) := \phi_1(\pi,\rho) - \phi_2(\pi,\rho,\varepsilon)$. This equation is

$$D(\rho,\varepsilon):=\varepsilon\rho^3\int_{2\pi}^{\pi}\frac{b_3(t)}{1-\rho B_2(t)}dt+O(\varepsilon^2)=0.$$

The function

$$M(\rho):=\int_{2\pi}^{\pi}\frac{b_3(t)}{1-\rho B_2(t)}dt,$$

is known as a *Melnikov function* associated to the problem. By the Implicit Function Theorem applied to $D(\rho,\varepsilon)/\varepsilon$ it follows that the simple zeros of $M(\rho)$ lead to simple zeros of the function $D(\rho,\varepsilon)$. More specifically if $\rho=\bar\rho$ satisfies $M(\bar\rho)=0$, $M'(\bar\rho)\neq 0$ then there exists a differentiable function g such that $g(0)=\bar\rho$ and for ε small enough $D(g(\varepsilon),\varepsilon)\equiv 0$.

In other words what we have seen is that each of the simple non-zero roots of $M(\rho)$ gives a limit cycle of the Abel differential equation (11). Therefore we reduced the proof of the theorem to find functions b_2 and b_3 such that the corresponding function $M(\rho)$ has at least k simple zeros.

For any $k\in\mathbb{N}$ we take $b_2(t)=\cos t$ and $b_3(t)=P(\sin t)$, where P is a polynomial of degree k in the variable $\sin t$. Since

$$M(\rho):=\int_{2\pi}^{\pi}\frac{P(\sin t)}{1-\rho\sin t}dt,$$

can be written into the form (8), with

$$I_{j,1}(\rho)=\int_{2\pi}^{\pi}\frac{\sin^j t}{1-\rho\sin t}dt,\qquad \text{for}\qquad j=0,1,...,k.$$

Then from Theorem 8 the maximum number of zeros of $M(\rho)$ is k, and k can be reached taking conveniently the coefficients of the polynomial $P(\sin t)$. □

Proof of Corollary 4. Similar to the proof of Theorem 3 we consider the piecewise differential equation

$$\begin{pmatrix}\dfrac{dx_1}{dt}\\ \dfrac{dx_2}{dt}\end{pmatrix}=\begin{cases} a_1(t)x_1, & \text{if}\quad 0\leqslant t\leqslant\pi,\\ b_2(t)x_2^2, & \text{if}\quad \pi\leqslant t\leqslant 2\pi.\end{cases}\tag{13}$$

The solution of

$$\frac{dx_1}{dt}=a_1(t)x_1,$$

satisfying $x_1(0)=\rho$ is

$$x_1=\phi_{10}(t,\rho)=\rho\, e^{A_2(t)},\quad \text{where}\quad A_2(t)=\int_0^t a_1(s)ds,$$

and the solution of

$$\frac{dx_2}{dt}=b_2(t)x_2^2, \tag{14}$$

satisfying $x_2(2\pi)=\rho$ is

$$x_2=\phi_{20}(t,\rho)=\frac{\rho}{1-\rho B_2(t)},\quad \text{where}\quad B_2(t)=\int_{2\pi}^t b_2(s)ds.$$

Imposing that $A_2(\pi)=B_2(\pi)=0$ we obtain that the piecewise differential equation (13) has a continuum of periodic solutions in the neighborhood of $\rho=0$.

Now we perturb (13) as follows

$$\begin{pmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{pmatrix} = \begin{cases} a_1(t)x_1+\varepsilon^2\left(a_0(t)+a_2(t)x_1^2+...+a_n(t)x_1^n\right), & \text{if}\quad 0\leqslant t\leqslant \pi, \\ b_2(t)x_2^2+\varepsilon\, b_3(t)x_2^3+\varepsilon^2\left(b_0(t)+b_4(t)x_2^4+...+b_m(t)x_2^m\right), & \text{if}\quad \pi\leqslant t\leqslant 2\pi, \end{cases} \tag{15}$$

where ε is small parameter and $\max\{n,m\}\geq 3$. The solution of the differential equations

$$\frac{dx_1}{dt}=a_1(t)x_1+\varepsilon^2\left(a_0(t)+a_2(t)x_1^2+...+a_n(t)x_1^n\right),$$

and

$$\frac{dx_2}{dt}=b_2(t)x_2^2+\varepsilon\, b_3(t)x_2^3+\varepsilon^2\left(b_0(t)+b_4(t)x_2^4+...+b_m(t)x_2^m\right),$$

can be expressed for small $\varepsilon>0$ in the form

$$\phi_1(t,\rho,\varepsilon)=\phi_{10}(t,\rho)+\varepsilon\psi_1(t,\rho)+O(\varepsilon^2),$$

and

$$\phi_2(t,\rho,\varepsilon)=\phi_{20}(t,\rho)+\varepsilon\psi_2(t,\rho)+O(\varepsilon^2),$$

respectively, where $\psi_i(t,\rho)=\partial\phi_i(t,\rho,\varepsilon)/\partial\varepsilon|_{\varepsilon=0}$ for $i=1,2$. Then we have

$$\psi_1(t,\rho)=0,\quad \psi_2(t,\rho)=\phi_{20}(t,\rho)^2\int_{2\pi}^t b_3(s)\phi_{20}(s,p)ds=\phi_{20}(t,\rho)^2\int_{2\pi}^t \frac{\rho b_3(s)}{1-\rho B_2(s)}ds.$$

Therefore the simple zeros of the function

$$D(\rho,\varepsilon):=\phi_1(\pi,\rho,\varepsilon)-\phi_2(\pi,\rho,\varepsilon)=\varepsilon\rho^3\int_{2\pi}^{\pi}\frac{b_3(t)}{1-\rho B_2(t)}dt+O(\varepsilon^2)=\varepsilon\rho^3M(\rho)+O(\varepsilon^2)=0,$$

correspond to limit cycles of piecewise differential equation (15). Now Taking again $b_2(t)=\cos t$ and $b_3(t)=P(\sin t)$, where P is a polynomial of degree k in the variable $\sin t$, we can apply Theorem 8 to function $M(\rho)$ and obtain that the maximum number of simple zeros of the function $M(\rho)$ is k. This completes the proof of Corollary 4. □

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